Preliminary Exam May 2018

Problem 1. Prove that every real-valued function, continuous on the real interval $[0,1]$, is the uniform limit of continuous piecewise linear functions.

Problem 2.

(a) State a theorem that says under what conditions we can differentiate a function series term by term $(\sum_{n=1}^{\infty} f_n(x))' = \sum_{n=1}^{\infty} f_n'(x)$, $f_n : (a,b) \to \mathbb{R}$.

(b) Prove that if $a > 1$ and $k \geq 1$, then
$$\sum_{k=2}^{\infty} \frac{(\log n)^k}{n^a} < \infty \,.$$

(c) Prove that the function
$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad x > 1$$
is infinitely differentiable in $(1, \infty)$.

Hint: *You can use part (b) to prove (c), even if you do not know how to prove (b).*

Problem 3. Let $f : \mathbb{Q}^n \to \mathbb{R}$ be a function defined on the set $\mathbb{Q}^n \subset \mathbb{R}^n$ consisting of points whose coordinates are rational numbers. Prove that if f satisfies the inequality $|f(x) - f(y)|^{2018} \leq 2018|x-y|$ for all $x, y \in \mathbb{Q}^n$, then there is a unique continuous function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in \mathbb{Q}^n$. Provide a direct proof without referring to any deep results.

Problem 4. Let $P = \{(x_1, x_2, x_3, x_4) : x_4 = 0\}$ be a hyperplane in $\mathbb{R}^4$. Let $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ be a C^1-diffeomorphism of $\mathbb{R}^4$ onto $\mathbb{R}^4$, and let $S = \Phi(P)$ be the image of the hyperplane P under the diffeomorphism Φ. Prove that for every $x \in S$, there is a neighborhood $B(x, \varepsilon) \subset \mathbb{R}^4$ such that the set $S \cap B(x, \varepsilon)$ is a graph of a C^1 function of one of the following forms
$$x_1 = f(x_2, x_3, x_4) \quad \text{or} \quad x_2 = f(x_1, x_3, x_4) \quad \text{or} \quad x_3 = f(x_1, x_2, x_4) \quad \text{or} \quad x_4 = f(x_1, x_2, x_3).$$
In the proof you are allowed to use the inverse function theorem or the implicit function theorem only. You are **not** allowed to use the rank theorem.

Problem 5. Let n be a positive integer. Denote by $\mathcal{M}_n$ the space of all real $n \times n$ matrices. By $A^T \in \mathcal{M}_n$ denote the transpose of a matrix $A \in \mathcal{M}_n$ and by tA, $t \in \mathbb{R}$, the matrix where all components of A are multiplied by t.
Prove that if $A \in \mathcal{M}_n$, then there is $B \in \mathcal{M}_n$ and $\varepsilon > 0$ such that $\varepsilon A = B + B^T B$.
Hint. *Differentiate the mapping $F : \mathcal{M}_n \to \mathcal{M}_n$ defined by $F(X) = X + X^T X$.*

Problem 6. Let f be a polynomial of total degree at most three in $(x,y,z) \in \mathbb{R}^3$. Prove that:
$$\int_{x^2+y^2+z^2 \leq 1} f(x,y,z)\, dx\, dy\, dz = \frac{4\pi f((0,0,0))}{3} + \frac{2\pi\, (\Delta f)\, ((0,0,0))}{15}.$$
Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian operator on $\mathbb{R}^3$.