Problem 1. For \( n \geq 1 \) let
\[
s_n = \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) \quad \text{and} \quad t_n = \left( \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n+1} \right).
\]

(a) Determine, with proof, the limit \( \lim_{n \to \infty} s_n \).
(b) Determine, with proof, the limit \( \lim_{n \to \infty} t_n \).

Hint. For part (a) use ideas related to the integral test for convergence of the series.

Problem 2. Prove that if \( f: [a, b] \to \mathbb{R} \) is continuous, then
\[
\lim_{n \to \infty} \sqrt[n]{\int_a^b |f(x)|^n \, dx} = \sup_{x \in [a, b]} |f(x)|.
\]

Problem 3. Let \( f_n: [0, 1] \to \mathbb{R} \) be a sequence of \( C^1 \) functions satisfying the conditions, for each \( n \in \mathbb{N} \):

- \( f_n \left( \frac{1}{2} \right) = 0 \),
- \( |f'_n(x)| < \frac{1}{x} \) for all \( x \in (0, 1) \).

Show that there is a continuous function \( f: (0, 1] \to \mathbb{R} \) and a subsequence \( f_{n_k}(x) \), such that for each \( x \in (0, 1] \):
\[
\lim_{k \to \infty} f_{n_k}(x) = f(x).
\]

Problem 4. Let \( \mathcal{F} \subset C^\infty[0, 1] \) be a uniformly bounded and equicontinuous family of smooth functions on \( [0, 1] \) such that \( f' \in \mathcal{F} \) whenever \( f \in \mathcal{F} \). Suppose that
\[
\sup_{x \in [0, 1]} |f'(x) - g'(x)| \leq \frac{1}{2} \sup_{x \in [0, 1]} |f(x) - g(x)| \quad \text{for all} \quad f, g \in \mathcal{F}.
\]
Show that there exists a sequence \( f_n \) of functions in \( \mathcal{F} \) that tends uniformly to \( C e^x \), for some real constant \( C \).

Hint: Use contraction principle. In order to apply the contraction principle you can use, without proof, the fact that if \( X \) is a complete metric space, \( A \subset X \) and \( T: A \to X \) is uniformly continuous, then \( T \) uniquely extends to a continuous map \( \overline{T}: \overline{A} \to \overline{X} \) defined on the closure \( \overline{A} \).

Problem 5. Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be a mapping of class \( C^1 \). Prove that there is an open and dense set \( \Omega \subset \mathbb{R}^n \) such that the function \( R(x) = \text{rank} \, Df(x) \) is locally constant on \( \Omega \), i.e. it is constant in a neighborhood of every point \( x \in \Omega \).

Problem 6. Let \( \Phi: \mathbb{R}^2 \to \Phi(\mathbb{R}^2) \subset \mathbb{R}^2 \) be a diffeomorphism. Prove that
\[
\int_{B^2(0,1)} \|D\Phi\| = \int_{\Phi(B^2(0,1))} \|D(\Phi^{-1})\|
\]
where \( \|A\| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2} \) is the Hilbert-Schmidt norm of the matrix.

Hint. Compare \( \|A\| \) and \( \|A^{-1}\| \) for a \( 2 \times 2 \) matrix.