Problem 1. Let \( f : \mathbb{R} \to [0, \infty) \) be continuous and assume that for any \( x \in \mathbb{R} \)
\[
g(x) := \sum_{k=1}^{\infty} (f(x))^k < \infty.
\]
Show that \( g \) is continuous.

Problem 2.
(1) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be continuous. Prove that \( f \) is uniformly continuous if and only if \( |f| \) is uniformly continuous.
(2) Show an example that the above claim is false without assuming continuity of \( f \). That is, find a counterexample to the following claim: \( f : \mathbb{R}^n \to \mathbb{R} \) is uniformly continuous if and only if \( |f| \) is uniformly continuous.

Problem 3. Show that for any \( x, y \in \mathbb{R}^2 \) and for any \( r, s, t \in [0, 1] \) with \( r + s + t = 2 \) we have
\[
|\det(x, y)| \leq |x|^s |y|^t |x - y|^r.
\]
Here \((x, y)\) represents the \(2 \times 2\) matrix with vectors \(x\) and \(y\) as columns.

Hint: If you replace \( y \) by \( y - x \)...

Problem 4. Assume that \( f \in C^\infty(-1, 1) \) is such that the \(n\)-th derivative satisfies \( |f^{(n)}(x)| \leq n \) for all \( x \in (-1, 1) \) and all \( n = 1, 2, \ldots \) Prove that there is \( F \in C^\infty(\mathbb{R}) \) such that \( F(x) = f(x) \) for all \( x \in (-1, 1) \).

Hint: Taylor.

Problem 5. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a mapping of class \( C^1 \) and let \( K \subset \mathbb{R}^n \) be a compact set. Assume that
(1) \( |f(x) - f(y)| \geq |x - y|^{2020} \) for all \( x, y \in K \).
(2) For all \( x \in K \),
\[
\frac{\partial f_i(x)}{\partial x_i} \geq i, \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and} \quad \frac{\partial f_i(x)}{\partial x_j} = 0, \quad \text{for } 1 \leq i < j \leq n.
\]
Prove that there is an open set \( U \) such that \( K \subset U \), and \( f \) is one-to-one in \( U \).

Problem 6. Let \( S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| = |y| = 1\} \) and let \( F : S \to \mathbb{R}, F(x, y) = x \cdot y \).
(a) Use the Lagrange multiplier theorem to find \( \sup_{(x,y) \in S} F(x,y) \).
(b) Use the result of part (a) to conclude the Cauchy-Schwarz inequality:
\[
\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}.
\]

Remark. You must use the Lagrange multiplier theorem in part (a) in order to get credit.