ANALYSIS OF THE TIME FILTERS ON THE IMPLICIT METHOD INCREASED ACCURACY AND IMPROVED STABILITY

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Abstract. This report considers linear multistep methods through time filtering. The approach has several advantages. It is modular and requires the addition of only one line of additional code. Error estimation and variable timesteps is straightforward and the individual effect of each step is conceptually clear. We present its development for the backward Euler method and a curvature reducing time filter leading to a 2-step, strongly A-stable, second order linear multistep method.

Key words. time filter, linear multistep method

AMS subject classification. 1234.56

1. Introduction.

The fully implicit/backward Euler method is commonly the first method implemented when extending a code for the steady state problem and often the method of last resort for complex applications. The issue can then arise of how to increase numerical accuracy in a complex, possibly legacy code without implementing from scratch another, better method. We show herein that adding one line, a curvature reducing time filter, increases accuracy from first to second order, gives an immediate error estimator and induces a method akin to BDF2. In the 2-step combination the effect of each step is conceptually clear and immediately adapts to variable timesteps.

To begin, consider the initial value problem

$$y'(t) = f(t, y(t)), y(0) = y_0.$$

Denote the n^{th} timestep by k_n. Let $t_{n+1} = t_n + k_n$, $\tau = k_n/k_{n-1}$, ν be an algorithm parameter and y_n an approximation to $y(t_n)$. Discretize this by the standard backward Euler (fully implicit) method followed by a simple time filter (next for constant timestep)

$$\begin{aligned} \text{Step 1} \quad &: \quad \tfrac{y_{n+1}-y_n}{k} = f(t_{n+1}, y_{n+1}), \\ \text{Step 2} \quad &: \quad y_{n+1} \Leftarrow y_{n+1} - \tfrac{\nu}{2}\left\{y_{n+1} - 2y_n + y_{n-1}\right\}. \end{aligned} \tag{1.1}$$

Step 2 is the only 3−point filter for which the combination of backward Euler plus a time filter produces a consistent approximation. The combination is second order accurate for $\nu = +2/3$, Proposition 2.1. Proposition 2.2 establishes that the combination is 0−stable for $-2 \le \nu < +2$, unstable otherwise and $A-$stable for $-2/3 \le \nu \le +2/3$. Since Step 2 with $\nu = +2/3$ has greater accuracy than Step 1, the pre- and post- filter difference

$$EST = |y_{n+1}^{prefilter} - y_{n+1}^{postfilter}| \tag{1.2}$$

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can be used in a standard way to estimate the error in the method and adapt the timestep.

The variable timestep case is considered in Section 3 based on a definition of discrete curvature and a curvature reducing discrete filter, Step 2 in (1.3):

$$\begin{aligned} \text{Step 1} \quad &: \quad \tfrac{y_{n+1}-y_n}{k_n} = f(t_{n+1}, y_{n+1}), \\ \text{Step 2} \quad &: \quad y_{n+1} \Leftarrow y_{n+1} - \tfrac{\nu}{2}\left\{ \tfrac{2k_{n-1}}{k_n+k_{n-1}} y_{n+1} - 2y_n + \tfrac{2k_n}{k_n+k_{n-1}} y_{n-1} \right\}. \end{aligned} \tag{1.3}$$

For variable timestep, the choice of ν for second order accuracy depends on τ and is $\nu = \frac{\tau(1+\tau)}{(1+2\tau)}$, Proposition 3.3. The filter step reduces the discrete curvature, Definition 3.1, at the three points (t_{n+1}, y_{n+1}), (t_n, y_n), (t_{n-1}, y_{n-1}), Proposition 3.1, provided

$$0 < \nu < 1 + k_n/k_{n-1}.$$

For constant time step, the special value $\nu = 2/3$ induces a one-leg, two step method[1] that is second order accurate and strongly $A-$stable, given by

$$\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = kf(t_{n+1}, \frac{3}{2}y_{n+1} - y_n + \frac{1}{2}y_{n-1}). \tag{1.4}$$

For general ν and variable timestep the equivalent linear multistep method is (2.4). The LHS of (1.4) is the same as BDF2. The RHS differs from BDF2 by

$$\frac{3}{2}y(t_{n+1}) - y(t_n) + \frac{1}{2}y(t_{n-1}) = \ y(t_{n+1}) + \mathcal{O}(k^2),$$

as required for second order accuracy.

Remark 1.1. *The filter value $\nu = 2$ in (1.1) is not good since it forces y_{n+1} to be the linear extrapolation of y_n, y_{n-1}. Thus we always assume $\nu \neq 2$. The filter can also be repeated several times (but not iterated to convergence). Filtering twice is equivalent to increasing the value of the filter parameter $\nu \to \nu(2 - \frac{\nu}{2})$ and filtering once.*

Time filters centered at t_n rather than t_{n+1}, are often used in geophysical fluid dynamics simulations with the leapfrog integrator to reduce oscillations in the computed solution, Asselin [A72], Robert [R69], Williams [W11]. As a related example, the Robert-Asselin filter is commonly used and given by

$$y_n \Leftarrow y_n + \frac{\nu}{2}\left\{y_{n+1} - 2y_n + y_{n-1}\right\}, \nu \simeq 0.1.$$

The extension of the RA filter to variable timesteps based on Section 3.1 is

$$y_n \Leftarrow y_n + \frac{\nu}{2}\left\{ \frac{2k_{n-1}}{k_n + k_{n-1}} y_{n+1} - 2y_n + \frac{2k_n}{k_n + k_{n-1}} y_{n-1} \right\}.$$

For a one step method, filters centered at t_n, like the Robert-Asselin filter, postprocess the computed solution but do not alter the evolution of the approximate solution. For that reason the filter is shifted to t_{n+1} herein.

[1] This 2−step method seems to be new in the sense that, while for the special value $\nu = 2/3$ the LHS is the same as BDF2 for both constant and variable timesteps, the RHS, as well as the approach to implementation, seems to be new.

2. Constant timestep.

We develop the properties of the method for constant time step in this section.

2.1. Derivation of the method. Denote the pre-filtered value y^*_{n+1}. Consider backward Euler plus a general, $3-$point time filter

$$\begin{aligned} \text{Step 1} &: \quad \frac{y^*_{n+1}-y_n}{k} = f(t_{n+1}, y^*_{n+1}), \\ \text{Step 2} &: \quad y_{n+1} = y^*_{n+1} + \{a y^*_{n+1} + b y_n + c y_{n-1}\}. \end{aligned} \tag{2.1}$$

Eliminating the intermediate value y^*_{n+1}, Steps 1 and 2 induce an equivalent 2-step method for the post-filtered values.

We prove the following.

PROPOSITION 2.1. *Let the time step be constant. The combination backward Euler plus time filter is consistent if and only if the filter coefficients are:*

$$a = -\frac{\nu}{2}, c = -\frac{\nu}{2}, b = \nu,$$

for some $\nu \neq 2$, and the filter is thus

$$y_{n+1} = y^*_{n+1} - \frac{\nu}{2}\left(y^*_{n+1} - 2\, y_n + y_{n-1}\right). \tag{2.2}$$

In this case the equivalent $2-$step method is

$$\begin{aligned} &\frac{1}{1-\frac{\nu}{2}} y_{n+1} - \frac{1+\frac{\nu}{2}}{1-\frac{\nu}{2}} y_n + \frac{\frac{\nu}{2}}{1-\frac{\nu}{2}} y_{n-1} = \\ &= k f(t_{n+1}, \frac{1}{1-\frac{\nu}{2}} y_{n+1} - \frac{\nu}{1-\frac{\nu}{2}} y_n + \frac{\frac{\nu}{2}}{1-\frac{\nu}{2}} y_{n-1}). \end{aligned} \tag{2.3}$$

The combination of is second order accurate if and only if

$$\nu = +\frac{2}{3}.$$

Proof. Eliminating y^*_{n+1} in Step 1 using

$$y^*_{n+1} = \frac{1}{1+a}\left(y_{n+1} - b y_n - c y_{n-1}\right)$$

yields an equivalent one-leg linear multistep method for the post-filtered values

$$\frac{1}{1+a} y_{n+1} - \frac{1+a+b}{1+a} y_n - \frac{c}{1+a} y_{n-1} = k f(t_{n+1}, \frac{1}{1+a} y_{n+1} - \frac{b}{1+a} y_n - \frac{c}{1+a} y_{n-1}).$$

In terms of the standard description of a general $2-$step method, the coefficients are

$$\begin{aligned} \alpha_2 &= \tfrac{1}{1+a} \quad , \quad \beta_2 = \tfrac{1}{1+a} \\ \alpha_1 &= -\tfrac{1+a+b}{1+a} \quad , \quad \beta_1 = -\tfrac{b}{1+a} \\ \alpha_0 &= -\tfrac{c}{1+a} \quad , \quad \beta_0 = -\tfrac{c}{1+a}. \end{aligned}$$

The method is consistent if and only if the first two terms in the method's LTE expansion are zero and second order accurate if and only if the third term vanishes. Consistency thus requires

$$consistent \Leftrightarrow \begin{cases} \alpha_2 + \alpha_1 + \alpha_0 = 0 \\ \Updownarrow \\ a + b + c = 0, \\ \\ \alpha_2 \ - \alpha_0 \ - (\beta_2 \ + \beta_1 \ + \beta_0 \ ) = 0 \\ \Updownarrow \\ a = c. \end{cases}$$

Thus for the method to be consistent

$$a + b + c = 0, a = c, b = -2a$$

and the first claim follows. The second claim follows by inserting these values for a, b, c.

The condition for second order accuracy is

$$\frac{1}{2}\alpha_2 \ + \frac{1}{2}\alpha_0 \ - \beta_2 \ + \beta_0 = 0$$
$$\Updownarrow$$
$$\frac{1}{2} \cdot 1 + \frac{1}{2}(-c) - 1 + (-c) = 0$$
$$\Updownarrow$$
$$c = -\frac{1}{3}.$$

These values correspond, as claimed, to $\nu = \frac{2}{3}$ and

$$y_{n+1} = \ y^*_{n+1} - \frac{1}{3} \ \left(y^*_{n+1} - 2 \ y_n + \ y_{n-1}\right).$$

□

2.2. Stability. We analyze stability for constant timestep. Consider

$$\frac{y^*_{n+1} - y_n}{k} = f(t_{n+1}, y^*_{n+1}),$$

$$y_{n+1} = y^*_{n+1} - \frac{\nu}{2}\left\{y^*_{n+1} - 2y_n + y_{n-1}\right\}.$$

The equivalent linear multistep method is

$$\frac{1}{1-\frac{\nu}{2}} y_{n+1} - \frac{1+\frac{\nu}{2}}{1-\frac{\nu}{2}} y_n + \frac{\frac{\nu}{2}}{1-\frac{\nu}{2}} y_{n-1} = \tag{2.4}$$
$$= kf(t_{n+1}, \frac{1}{1-\frac{\nu}{2}} y_{n+1} - \frac{\nu}{1-\frac{\nu}{2}} y_n + \frac{\frac{\nu}{2}}{1-\frac{\nu}{2}} y_{n-1}).$$

This corresponds to

$$\alpha_2 = \frac{1}{1-\frac{\nu}{2}} \quad , \quad \beta_2 = \frac{1}{1-\frac{\nu}{2}}$$

$$\alpha_1 = -\frac{1+\frac{\nu}{2}}{1-\frac{\nu}{2}} \quad , \quad \beta_1 = -\frac{\nu}{1-\frac{\nu}{2}}$$

$$\alpha_0 = \frac{\frac{\nu}{2}}{1-\frac{\nu}{2}} \quad , \quad \beta_0 = \frac{\frac{\nu}{2}}{1-\frac{\nu}{2}}.$$

There are various places where $A-$stable $2-$step methods are characterized in terms of their coefficients, e.g., Dahlquist [D78, D79], Dahlquist, Liniger and Nevanlinna [DLN83], Grigorieff [G83], Nevanlinna [N84]. We shall apply the characterization (for variable timesteps) in Dahlquist [D79], Lemma 4.1 page 3, 4 (specifically rearranging the equation on page 4 following (4.1)), which states that the method is $A-$stable if

$$\begin{cases} -\alpha_1 \geq 0, \\ 1 - 2\beta_1 \geq 0 \text{ and} \\ 2(\beta_2 - \beta_0) + \alpha_1 \geq 0. \end{cases} \tag{2.5}$$

Proposition 2.2. *The method (2.4) is* $0-$*stable for*

$$-2 \leq \nu < 2$$

and $0-$*unstable otherwise. Let* $-2 \leq \nu < 2$. *The method is* $A-$*stable for*

$$-\frac{2}{3} \leq \nu \leq +\frac{2}{3}.$$

Proof. For $0-$stability, the associated polynomial is

$$\frac{1}{1-(\nu/2)}z^2 - \frac{1+(\nu/2)}{1-(\nu/2)}z + \frac{(\nu/2)}{1-(\nu/2)} = 0$$

$$\Updownarrow$$

$$z^2 - (1+(\nu/2))\,z + (\nu/2) = 0.$$

Its roots are

$$z_{\pm} = 1, \frac{\nu}{2},$$

from which $0-$stability follows for $-2 \leq \nu < 2$.

To show $A-$stability we apply the characterization (2.5). Due to the $0-$stability result, restrict to values $-2 \leq \nu < 2$ for which the denominator

$$1 - \frac{\nu}{2} > 0.$$

The first of the three conditions is

$$
\begin{gathered}
-\alpha_1 \geq 0 \\
\Updownarrow \\
\frac{1+\frac{\nu}{2}}{1-\frac{\nu}{2}} \geq 0 \\
\Updownarrow \\
\nu \geq -2.
\end{gathered}
$$

The second is

$$
\begin{gathered}
1-2\beta_1 \geq 0 \\
\Updownarrow \\
1-2\left(-\frac{\nu}{1-\frac{\nu}{2}}\right) \geq 0 \\
\Updownarrow \\
1+\frac{3\nu}{2} \geq 0 \\
\Updownarrow \\
\nu \geq -\frac{2}{3}.
\end{gathered}
$$

The third condition is

$$
\begin{gathered}
2(\beta_2 \ -\beta_0)+\alpha_1 \geq 0 \\
\Updownarrow \\
2\left(\frac{1}{1-\frac{\nu}{2}}-\frac{\frac{\nu}{2}}{1-\frac{\nu}{2}}\right)-\frac{1+\frac{\nu}{2}}{1-\frac{\nu}{2}} \geq 0 \\
\Updownarrow \\
2\left(1-\frac{\nu}{2}\right)-1-\frac{\nu}{2} \geq 0 \\
\Updownarrow \\
\nu \leq \frac{2}{3}.
\end{gathered}
$$

□

For the interesting choice $\nu = +2/3$ we have computed the stability region of the induced 2−step method by the root locus method and present it next in Figure 2.1.

For comparison, the stability regions of BE and BDF2 follow in Figure 2.2. The boundaries of the three stability regions are presented next in Figure 2.3.

Remark 2.3. *The stability region of the new method is larger than that of BDF2 suggesting the new method is somewhat more dissipative than BDF2. This is consistent with the numerical results in Section 4.*

3. Variable Timestep.

Since the implicit method is a one step method the key is to extend time filters to variable timesteps. We begin.

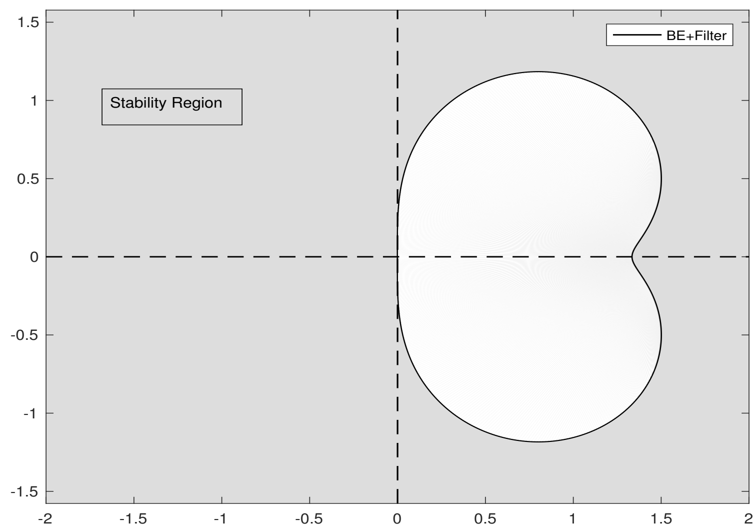

FIG. 2.1. *Stability region of Backward euler plus time filter*

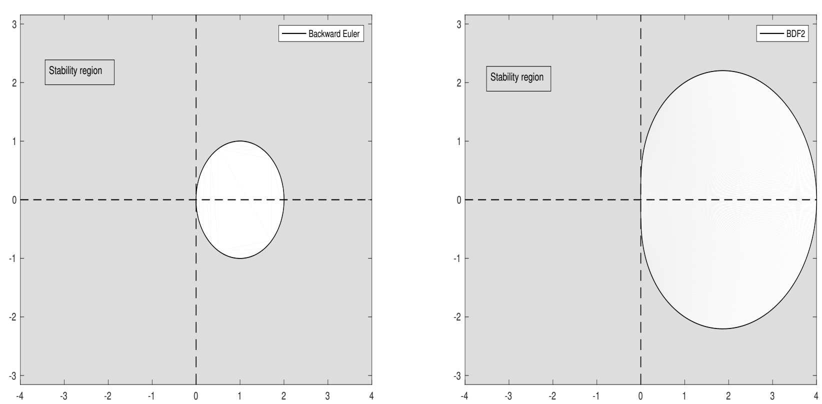

FIG. 2.2. *Stability region of Backward euler(left) and BDF2(right).*

3.1. Time Filters on Nonuniform Meshes. To extend time filters to nonuniform timesteps we must first define the discrete curvature. The extension of differential geometry to discrete settings is an active research fields with considerable work on discrete curvature, e.g., Najman [N17]. For 3 points the natural definitions are either the discrete second difference or the inverse of the radius of the interpolating circle. Consistent with work in GFD, we employ the former scaled by $k_{n-1}k_n$, e.g., Williams [W11], Kalnay [K03]. Consider the points

$$(t_{n-1}, y_{n-1}), (t_n, y_n), (t_{n+1}, y_{n+1}).$$

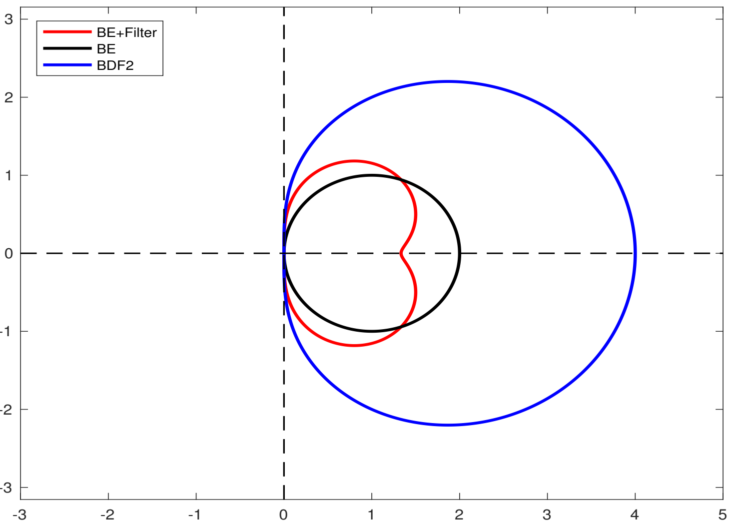

Fig. 2.3. *Boundaries of Stability Regions*

Let the Lagrange basis functions for these three points be denoted

$$\ell_{n-1}(t),\ \ell_n(t),\ \ell_{n+1}(t).$$

The quadratic interpolant at the three points is then

$$\phi(t) = y_{n+1}\ell_{n+1}(t) + y_n\ell_n(t) + y_{n-1}\ell_{n-1}(t).$$

Definition 3.1. *The discrete curvature at* $(t_{n-1}, y_{n-1}), (t_n, y_n), (t_{n+1}, y_{n+1})$ *is*

$$\begin{aligned}\kappa_n &= k_{n-1}k_n\phi'' \\ &= \frac{2k_{n-1}}{k_n + k_{n-1}}y_{n+1} - 2y_n + \frac{2k_n}{k_n + k_{n-1}}y_{n-1}.\end{aligned}$$

Equivalently, recalling $\tau = \frac{k_n}{k_{n-1}}$,

$$\kappa_n = \frac{2}{1+\tau}y_{n+1} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1}.$$

We define the extension of the filter (2.2) in (1.1) to nonuniform meshes as

$$y_{n+1} \Leftarrow y_{n+1} - \frac{\nu}{2}\left\{\frac{2}{1+\tau}y_{n+1} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1}\right\}. \tag{3.1}$$

Proposition 3.2. *The filter (3.1) alters the discrete curvature before,* κ^{old}, *and after,* κ^{new}, *filtering by*

$$\kappa^{new} = (1 - \frac{\nu}{1+\tau})\kappa^{old}.$$

The variable timestep filter reduces, without changing sign, the discrete curvature, $|\kappa^{new}| < |\kappa^{old}|$*, provided*

$$0 < \nu < 1 + \tau.$$

Proof. The first claim follows by algebraic rearrangement of the filter equation (3.1)

$$y_{n+1}^{new} = y_{n+1}^{old} - \frac{\nu}{2}\left\{\frac{2}{1+\tau}y_{n+1}^{old} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1}\right\}, or$$

$$\frac{2}{1+\tau}y_{n+1}^{new} = \frac{2}{1+\tau}y_{n+1}^{old} - \frac{\nu}{2}\frac{2}{1+\tau}\left\{\frac{2}{1+\tau}y_{n+1}^{old} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1}\right\}$$

$$\frac{2}{1+\tau}y_{n+1}^{new} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1} = \frac{2}{1+\tau}y_{n+1}^{old} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1}$$

$$-\frac{\nu}{2}\frac{2}{1+\tau}\left\{\frac{2}{1+\tau}y_{n+1}^{old} - 2y_n + \frac{2\tau}{1+\tau}y_{n-1}\right\},$$

$$\kappa^{new} = (1 - \frac{\nu}{1+\tau})\kappa^{old}.$$

Curvature reduction thus holds provided

$$0 < \nu\frac{1}{1+\tau} < 1,$$

as claimed. □

In the next figure the three points (t_{n-1}, y_{n-1}), (t_n, y_n), (t_{n+1}, y_{n+1}) and their quadratic interpolant are depicted. The discrete curvature is the second derivative of the interpolating quadratic scaled by $k_{n-1}k_n$. For $0 < \nu < 1+\tau$ the filter would move the value y_{n+1} down slightly (by $O(k^2)$) to reduce the curvature.

3.2. The local truncation error. Since the discrete curvature and filter are well defined for variable timesteps, the method is determined. It is, as presented in (1.3),

$$\begin{aligned} \text{Step 1} \quad &: \quad \tfrac{y_{n+1}^* - y_n}{k_n} = f(t_{n+1}, y_{n+1}^*), \\ \text{Step 2} \quad &: \quad y_{n+1} = y_{n+1}^* - \tfrac{\nu}{2}\left\{\tfrac{2}{1+\tau}y_{n+1}^* - 2y_n + \tfrac{2\tau}{1+\tau}y_{n-1}\right\}. \end{aligned} \tag{3.2}$$

Step 2 is used to solve for y_{n+1}^* and eliminate the prefilter value by

$$y_{n+1}^* = \frac{1+\tau}{1+\tau-\nu}y_{n+1} - \nu\frac{1+\tau}{1+\tau-\nu}y_n + \frac{\tau\nu}{1+\tau-\nu}y_{n-1}.$$

Eliminating y_{n+1}^* in Step 1 then gives the equivalent 2−step method

$$\begin{aligned} &\frac{1+\tau}{1+\tau-\nu}y_{n+1} - \nu\frac{1+\tau}{1+\tau-\nu}y_n + \frac{\tau\nu}{1+\tau-\nu}y_{n-1} - y_n = \\ &= k_n f(t_{n+1}, \frac{1+\tau}{1+\tau-\nu}y_{n+1} - \nu\frac{1+\tau}{1+\tau-\nu}y_n + \frac{\tau\nu}{1+\tau-\nu}y_{n-1}). \end{aligned} \tag{3.3}$$

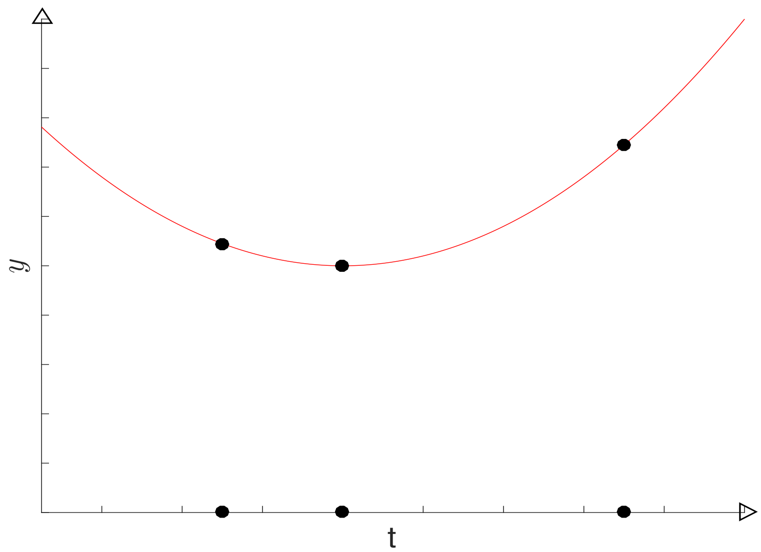

FIG. 3.1. $\kappa = k_{n-1} k_n \phi''(t)$

This yields the following coefficients

$$\alpha_2 = \frac{1+\tau}{1+\tau-\nu} \quad , \quad \beta_2 = \frac{1+\tau}{1+\tau-\nu}$$

$$\alpha_1 = -\frac{1+\tau+\nu\tau}{1+\tau-\nu} \quad , \quad \beta_1 = -\nu\frac{1+\tau}{1+\tau-\nu}$$

$$\alpha_0 = \frac{\tau\nu}{1+\tau-\nu} \quad , \quad \beta_0 = \frac{\tau\nu}{1+\tau-\nu}$$

The $\beta-$coefficients as given above satisfy a standard normalization condition

$$\beta_2 + \beta_1 + \beta_0 = 1.$$

There is a considerable amount known about $2-$step methods, even with varying timesteps. Many of the properties of the method follow from applying the theory in, e.g., Dahlquist [D79], Dahlquist, Liniger and Nevanlinna [DLN83], to the above and its variable timestep analog.

We prove the following.

PROPOSITION 3.3. *The variable timestep method (3.2) is always consistent. It is second order accurate provided*

$$\nu = \frac{\tau(1+\tau)}{1+2\tau}.$$

Moreover, the LTE for $\nu = \frac{\tau(\tau+1)}{1+2\tau}$ *is*

$$LTE = \frac{-(1+4\tau)}{6\tau} k_n^3 y'''(t_n) + \mathcal{O}(k_n^4).$$

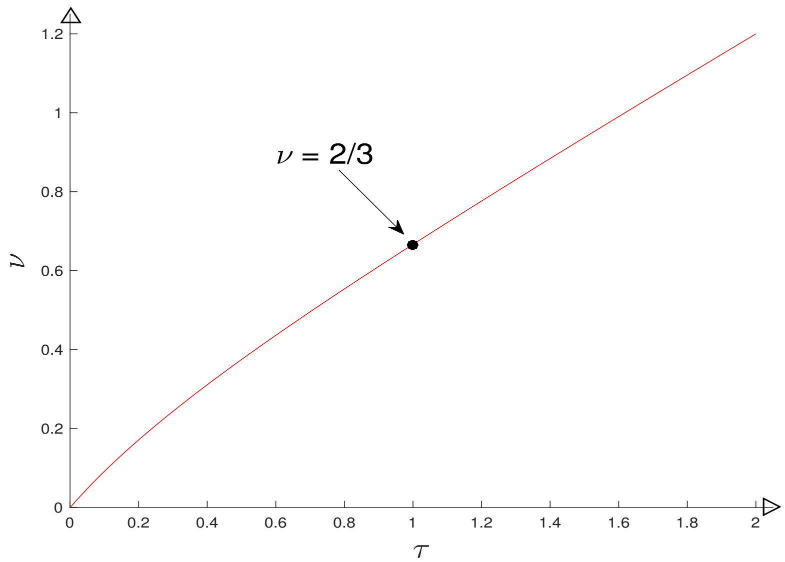

FIG. 3.2. *Curvature reduction and second order choice of* ν

The relation $\nu = \tau(1+\tau)/(1+2\tau)$ for second order accuracy is plotted below.

Proof. The equivalent 2−step method corresponds to the coefficients

$$\alpha_2 = \tfrac{1+\tau}{1+\tau-\nu} \quad , \quad \beta_2 = \tfrac{1+\tau}{1+\tau-\nu}$$

$$\alpha_1 = -\tfrac{1+\tau+\nu\tau}{1+\tau-\nu} \quad , \quad \beta_1 = -\nu\tfrac{1+\tau}{1+\tau-\nu}$$

$$\alpha_0 = \tfrac{\tau\nu}{1+\tau-\nu} \quad , \quad \beta_0 = \tfrac{\tau\nu}{1+\tau-\nu}$$

By a Taylor expansion (the Appendix), the method is consistent if and only if the following two condition are satisfied,

$$Condition\ 1 : \alpha_2 + \alpha_1 + \alpha_0 = 0$$

$$Condition\ 2 : \alpha_2 \ - \tfrac{1}{\tau}\alpha_0 \ - (\beta_2 \ + \beta_1 \ + \beta_0 \ ) = 0$$

The first two consistency conditions identically holds. Indeed,

$$Condition\ 1 :$$
$$\frac{1+\tau}{1+\tau-\nu} + (-\nu\frac{1+\tau}{1+\tau-\nu} - 1) + \frac{\tau\nu}{1+\tau-\nu} = 0$$

$$\Updownarrow$$

$$1+\tau-\nu[1+\tau] - [1+\tau-\nu] + \tau\nu = 0 \Leftrightarrow 0 = 0,$$

and similarly for Condition 2

$$\mathit{Condition\ 2}:$$
$$\frac{1+\tau}{1+\tau-\nu}-\frac{1}{\tau}\frac{\tau\nu}{1+\tau-\nu}-(\frac{1+\tau}{1+\tau-\nu}-\nu\frac{1+\tau}{1+\tau-\nu}+\frac{\tau\nu}{1+\tau-\nu})=0$$
$$\Updownarrow$$
$$\tau[1+\tau]-\tau\nu-(\tau[1+\tau]-\tau\nu[1+\tau]+\tau\tau\nu)=0\Leftrightarrow 0=0.$$

Therefore the method is always consistent. The method is second order accurate if and only if

$$\frac{1}{2}\alpha_2\ +\frac{1}{2\tau^2}\alpha_0\ -\beta_2\ +\frac{1}{\tau}\beta_0=0$$
$$\Updownarrow$$
$$\tau^2\alpha_2\ +\alpha_0\ -2\tau^2\beta_2\ +2\tau\beta_0=0$$
$$\Updownarrow$$
$$\tau^2\frac{1+\tau}{1+\tau-\nu}+\frac{\tau\nu}{1+\tau-\nu}-2\tau^2\frac{1+\tau}{1+\tau-\nu}+2\tau\frac{\tau\nu}{1+\tau-\nu}=0$$
$$\Updownarrow$$
$$\tau^2[1+\tau]+\tau\nu-2\tau^2[1+\tau]+2\tau^2\nu=0$$
$$\Updownarrow$$
$$\tau[1+\tau]+\nu-2\tau[1+\tau]+2\tau\nu=0$$
$$\Updownarrow$$
$$\tau[1+\tau]+\nu[1+2\tau]-2\tau[1+\tau]=0$$
$$\Updownarrow$$
$$\nu[1+2\tau]=-\tau[1+\tau]+2\tau[1+\tau]=\tau+\tau^2$$
$$\Updownarrow$$
$$\nu=\frac{\tau+\tau^2}{1+2\tau},$$

as claimed. That the LTE for $\nu=\frac{\tau(\tau+1)}{1+2\tau}$ is

$$LTE=\frac{-(1+4\tau)}{6\tau}k_n^3y'''(t_n)+\mathcal{O}(k_n^4)$$

is a calculation of the first non-zero term of the LTE expansion. ▯

Remark 3.4. *BDF2 is related to the method herein. The normal, fully variable BDF2 method is given by*

$$\frac{2\tau+1}{\tau+1}y_{n+1}-(\tau+1)y_n+\frac{\tau^2}{\tau+1}y_{n-1}=k_nf(t_{n+1},y_{n+1}). \tag{3.4}$$

By comparison, the equivalent, variable step linear multistep method herein is

$$\frac{1+\tau}{1+\tau-\nu}y_{n+1}-\nu\frac{1+\tau}{1+\tau-\nu}y_n+\frac{\tau\nu}{1+\tau-\nu}y_{n-1}-y_n=$$
$$=k_nf(t_{n+1},\frac{1+\tau}{1+\tau-\nu}y_{n+1}-\nu\frac{1+\tau}{1+\tau-\nu}y_n+\frac{\tau\nu}{1+\tau-\nu}y_{n-1}),$$

For $\nu = \tau(1+\tau)/(1+2\tau)$ the LHS is again the same as (variable step) BDF2 while the RHS differs.

3.3. Stability for variable step sizes. As defined by Dahlquist, Liniger and Nevanlinna [DLN83] equation (1.12) p.1072, a variable step size method is $A-$stable if, when applied as a one-leg scheme to

$$y' = \lambda(t)y, \mathrm{Re}(\lambda(t)) \leq 0,$$

solutions are always bounded for any sequence of step sizes and any such $\lambda(t)$. We analyze $A-$stability for variable step sizes applying the same conditions as for constant step sizes since they were derived in Dahlquist [D79], Dahlquist, Liniger and Nevanlinna [DLN83] for variable step, $2-$step methods. Specifically, we apply the characterization in Dahlquist [D79], Lemma 4.1 page 3, 4 (specifically rearranging the equation on page 4 following (4.1)), which states that the method is $A-$stable if

$$\begin{cases} -\alpha_1 \geq 0, \\ 1 - 2\beta_1 \geq 0 \text{ and} \\ 2(\beta_2 - \beta_0) + \alpha_1 \geq 0 \end{cases} \tag{3.5}$$

The coefficients for (3.3) are

$$\begin{aligned} \alpha_2 &= \tfrac{1+\tau}{1+\tau-\nu} &, \quad \beta_2 &= \tfrac{1+\tau}{1+\tau-\nu} \\ \alpha_1 &= -\tfrac{1+\tau+\nu\tau}{1+\tau-\nu} &, \quad \beta_1 &= -\nu\tfrac{1+\tau}{1+\tau-\nu} \\ \alpha_0 &= \tfrac{\tau\nu}{1+\tau-\nu} &, \quad \beta_0 &= \tfrac{\tau\nu}{1+\tau-\nu}. \end{aligned}$$

PROPOSITION 3.5. *The method (2.4) is $A-$stable for*

$$-\frac{1+\tau}{1+2\tau} \leq \nu \leq \min\{\frac{1+\tau}{3\tau}, 1+\tau\}.$$

Proof. We check the 3 conditions. The first is

$$-\alpha_1 \geq 0$$
$$\Updownarrow$$
$$\frac{1+\tau+\nu\tau}{1+\tau-\nu} \geq 0.$$

Considering cases this holds if and only if

$$-\frac{1+\tau}{\tau} \leq \nu \leq 1+\tau.$$

The second is

$$1 - 2\beta_1 \geq 0$$
$$\Updownarrow$$
$$1 + 2\nu\frac{1+\tau}{1+\tau-\nu} \geq 0$$

Since Condition 1 requires $\nu \leq 1+\tau$ a case is eliminated and this holds provided

$$-\frac{1+\tau}{1+2\tau} \leq \nu.$$

Condition 3 is

$$2(\beta_2 \ -\beta_0)+\alpha_1 \geq 0$$
$$\Updownarrow$$
$$2\left(\frac{1+\tau}{1+\tau-\nu} - \frac{\tau\nu}{1+\tau-\nu}\right) - \frac{1+\tau+\nu\tau}{1+\tau-\nu} \geq 0.$$

Since Condition 1 requires $\nu \leq 1+\tau$ this holds if and only if

$$1+\tau \geq +3\nu\tau$$
$$\Updownarrow$$
$$\nu \leq \frac{1+\tau}{3\tau}.$$

Since

$$\min\{\frac{1+\tau}{\tau}, \frac{1+\tau}{1+2\tau}\} = \frac{1+\tau}{1+2\tau}$$

the result follows. □

Since the filter is curvature reducing only for $0<\nu<1+\tau$ it is sensible to restrict the values to

$$0<\nu \leq \min\{\frac{1+\tau}{3\tau}, 1+\tau\}.$$

We plot next the region in Figure 3.3, below the dark curve, in the (τ,ν) plane of variable step $A-$stability. Also plotted, the dashed curve, is the choice of $\nu=\nu(\tau)$ that yields second order accuracy. We see that constant or reducing the timestep ensures $A-$stability while increasing the timestep one must either accept first order accuracy with $A-$stability or second order with some reduced (and yet undetermined) $A(\theta)-$stability, $\theta<\pi/2$.

Remark 3.6. *For variable step BDF2 the same conditions can be applied. The result after some algebra is that the third condition for $A-$stability holds for*

$$\tau \leq 1.$$

This is the same constraint that occurs for the method herein when ν is restricted to the curve of second order accuracy in Figure 3.3.

3.4. Modified equation analysis.

Consider oscillation equation

$$y'(t) = i\omega y(t) \quad \text{and} \quad y(0)=1. \tag{3.6}$$

The linear multistep method (1.3) is generally a first order approximation to oscillation equation (3.6) and second order for the choice $\nu=\tau(1+\tau)/(1+2\tau)$. To delineate the distribution of error between phase error and amplitude error we construct the

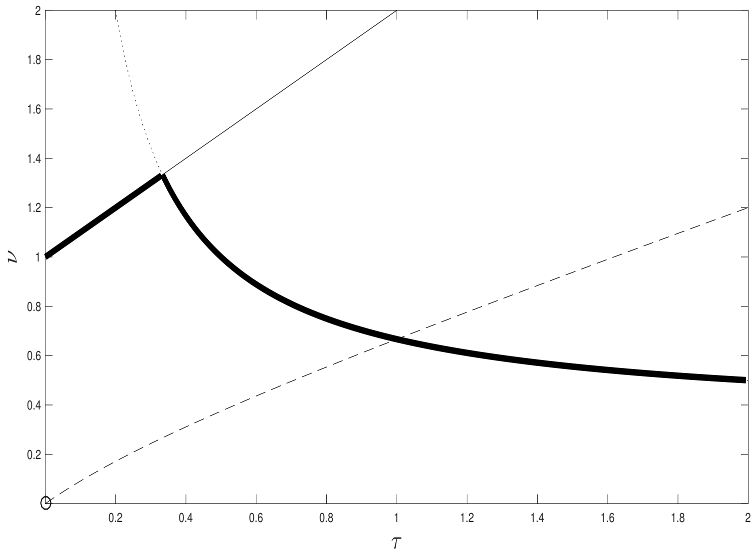

FIG. 3.3. *A-stable for $\nu \leq$ dark curve, Dashed Curve $= O(k^2)$*

modified equation of the method for the oscillation equation. We note that the modified equation is based on an expansion that assumes implicitly condition $|\omega k_n| < 1$.

PROPOSITION 3.7. *The three term modified equation of oscillation equation(3.6) for (1.3) is*

$$\begin{aligned} &u'(t) = i\omega u(t) + k_n C_1 (i\omega)^2 u(t) + k_n^2 C_2 (i\omega)^3 u(t) + k_n^3 C_3 (i\omega)^4 u(t), \\ &u(0) = 1. \end{aligned} \tag{3.7}$$

where C_1, C_2, C_3, are

$$\begin{aligned} C_1 &= \frac{\tau + \tau^2 - \nu - 2\nu\tau}{2\tau(1+\tau-\nu)} \\ C_2 &= \frac{2\tau^4 + \tau^3(4-5\nu) + \nu(1+2\nu) + \tau\nu(1+6\nu) + \tau^2(2-5\nu+6\nu^2)}{6\tau^2(1+\tau-\nu)^2} \\ C_3 &= \frac{6\tau^6 + \tau^5(18-20\nu) - \tau\nu^2(31+24\nu) + \tau^2\nu(4-33\nu-36\nu^2)}{24\tau^3(1+\tau-\nu)^3} \\ &+ \frac{\tau^4(18-39\nu+23\nu^2) + \tau^3(6-16\nu+13\nu^2-24\nu^3) - \nu(1+8\nu+6\nu^2)}{24\tau^3(1+\tau-\nu)^3}. \end{aligned}$$

Proof. The general three term modified equation of oscillation equation(3.6) takes the form

$$u' = i\omega u + k_n g_1(u) + k_n^2 g_2(u) + k_n^3 g_3(u) \quad \text{and} \quad u(0) = 1.$$

Thus,

$$
\begin{aligned}
u'' &= -\omega^2 u + i\omega k_n g_1(u) + i\omega k_n^2 g_2(u) \\
&\quad + i\omega k_n g_1'(u)u + k_n^2 g_1'(u) g_1(u) + i\omega k_n^2 g_2'(u)u + \mathcal{O}(k_n^3) \\
u''' &= -i\omega^3 u - \omega^2 k_n g_1(u) - 2\omega^2 k_n g_1'(u)u + \mathcal{O}(k_n^2) \\
u^{(4)} &= w^4 u + \mathcal{O}(k_n).
\end{aligned}
$$

Consider (1.3) applied to oscillation equation,

$$
y_{n+1} - \frac{\nu\tau + 1 + \tau}{\tau + 1} y_n + \frac{\nu\tau}{1+\tau} y_{n-1} = i\omega k_n y_{n+1} - i\omega k_n \nu y_n + \frac{\nu\tau}{1+\tau} i\omega k_n y_{n-1}.
$$

Rearrange term and eliminate y_{n+1}, we obtain

$$
y_{n+1} = \frac{1}{1 - i\omega k_n} \left(\frac{\nu\tau + 1 + \tau}{\tau + 1} y_n - \frac{\nu\tau}{1+\tau} y_{n-1} - i\omega k_n \nu y_n + \frac{\nu\tau}{1+\tau} i\omega k_n y_{n-1} \right).
$$

Since $|\omega k_n| < 1$, we can use approximation of $\frac{1}{1-i\omega k_n} = 1 + i\omega k_n - \omega^2 k_n^2 - i\omega^3 k_n^3 + \omega^4 k_n^4 + \mathcal{O}(k_n^5)$. Therefore,

$$
\begin{aligned}
y_{n+1} &= \frac{\nu\tau + 1 + \tau}{\tau + 1} y_n - \frac{\nu\tau}{1+\tau} y_{n-1} + \left(\frac{1 + \tau - \nu}{\tau + 1} \right) i\omega k_n y_n \\
&\quad + \left(\frac{\nu - 1 - \tau}{\tau + 1} \right) \omega^2 k_n^2 y_n \\
&\quad + \left(\frac{\nu - 1 - \tau}{\tau + 1} \right) i\omega^3 k_n^3 y_n + \left(\frac{1 + \tau - \nu}{\tau + 1} \right) \omega^4 k_n^4 y_n + \mathcal{O}(k_n^5)
\end{aligned}
$$

The local truncation error of variable stepsize method (1.3) with modified equations is

$$
\begin{aligned}
LTE &= u(t_{n+1}) - y_{n+1} \\
&= u(t_{n+1}) - \frac{\nu\tau + 1 + \tau}{\tau + 1} y_n + \frac{\nu\tau}{1+\tau} y_{n-1} - \left(\frac{1 + \tau - \nu}{\tau + 1} \right) i\omega k_n y_n \\
&\quad - \left(\frac{\nu - 1 - \tau}{\tau + 1} \right) \omega^2 k_n^2 y_n - \left(\frac{\nu - 1 - \tau}{\tau + 1} \right) i\omega^3 k_n^3 y_n \\
&\quad - \left(\frac{1 + \tau - \nu}{\tau + 1} \right) \omega^4 k_n^4 y_n + \mathcal{O}(k_n^5)
\end{aligned}
$$

Assume that numerical solution of all previous time steps are exact i.e. $y_i = u(t_i)$ for all $i = 1 \cdots n$,

$$
\begin{aligned}
LTE &= u(t_{n+1}) - \frac{\nu\tau + 1 + \tau}{\tau + 1} u(t_n) + \frac{\nu\tau}{1+\tau} u(t_{n-1}) - \left(\frac{1 + \tau - \nu}{\tau + 1} \right) i\omega k_n u(t_n) \\
&\quad - \left(\frac{\nu - 1 - \tau}{\tau + 1} \right) \omega^2 k_n^2 u(t_n) - \left(\frac{\nu - 1 - \tau}{\tau + 1} \right) i\omega^3 k_n^3 u(t_n) \\
&\quad - \left(\frac{1 + \tau - \nu}{\tau + 1} \right) \omega^4 k_n^4 u(t_n) + \mathcal{O}(k_n^5).
\end{aligned}
$$

Apply the Taylor expansion of $u(t_{n-1})$, $u(t_{n+1})$ at time t_n and substitute $u(t_{n+1})$,$u(t_{n-1})$,$u'(t_n)$, $u''(t_n)$, $u'''(t_n)$ and $u^{(4)}(t_n)$ in LTE, we get

$$LTE =$$
$$\left[\left(\frac{1+\tau-\nu}{1+\tau}\right) g_1(u(t_n)) - \frac{1}{2}\omega^2 u(t_n) - \frac{\nu}{2(\tau+\tau^2)}\omega^2 u(t_n) - \left(\frac{\nu-1-\tau}{\tau+1}\right)\omega^2 u(t_n)\right] k_n^2$$
$$+\left[\left(\frac{1}{2}+\frac{\nu}{2(\tau+\tau^2)}\right) i\omega g_1(u(t_n)) + \left(\frac{1}{2}+\frac{\nu}{2(\tau+\tau^2)}\right) i\omega g_1'(u(t_n))u(t_n)\right.$$
$$\left.-\left(\frac{1}{6}-\frac{\nu}{6(\tau^2+\tau^3)}\right) i\omega^3 u(t_n) - \left(\frac{\nu-1-\tau}{\tau+1}\right) i\omega^3 u(t_n) + \frac{1+\tau-\nu}{1+\tau} g_2(u(t_n))\right] k_n^3$$
$$+\left[\frac{1+\tau-\nu}{1+\tau} g_3(u(t_n)) + \left(\frac{1}{2}+\frac{\nu}{2(\tau+\tau^2)}\right) i\omega g_2(u(t_n))\right.$$
$$+\left(\frac{1}{2}+\frac{\nu}{2(\tau+\tau^2)}\right) g_1(u(t_n)) g_1'(u(t_n)) + \left(\frac{1}{2}+\frac{\nu}{2(\tau+\tau^2)}\right) i\omega g_2'(u(t_n))u(t_n)$$
$$-\left(\frac{1}{6}-\frac{\nu}{6(\tau^2+\tau^3)}\right)\omega^2 g_1(u(t_n)) - \left(\frac{1}{6}-\frac{\nu}{6(\tau^2+\tau^3)}\right) 2\omega^2 g_1'(u(t_n))u(t_n)$$
$$\left.+\left(\frac{1}{24}+\frac{\nu}{24(\tau^3+\tau^4)}\right) w^4 u(t_n) - \left(\frac{1+\tau-\nu}{\tau+1}\right)\omega^4 u(t_n)\right] k_n^4$$

Setting coefficient of k_n^2 term equal to zero to find $g_1(u)$

$$g_1(u) = \frac{2\nu\tau - \tau - \tau^2 + \nu}{2\tau(1+\tau-\nu)}\omega^2 u = C_1(i\omega)^2 u.$$

We use $g_1(u)$ and set coefficient of k_n^3 equal to zero, we obtain $g_2(u)$ as following,

$$\begin{aligned} g_2(u) &= -\frac{2\tau^4 + \tau^3(4-5\nu) + \nu(1+2\nu) + \tau\nu(1+6\nu) + \tau^2(2-5\nu+6\nu^2)}{6\tau^2(1+\tau-\nu)^2} i\omega^3 u \\ &= C_2(i\omega)^3 u. \end{aligned}$$

Finally, we use $g_1(u)$ and $g_2(u)$ and set coefficient of k_n^4 to zero, we get

$$\begin{aligned} g_3(u) = &\left[\frac{6\tau^6 + \tau^5(18-20\nu) - \tau\nu^2(31+24\nu) + \tau^2\nu(4-33\nu-36\nu^2)}{24\tau^3(1+\tau-\nu)^3}\right. \\ &\left. + \frac{\tau^4(18-39\nu+23\nu^2) + \tau^3(6-16\nu+13\nu^2-24\nu^3) - \nu(1+8\nu+6\nu^2)}{24\tau^3(1+\tau-\nu)^3}\right]\omega^4 u \\ &= C_4(i\omega)^4 u. \end{aligned}$$

□

REMARK 3.8. *The variable stepsize method (1.3) is generally a first order approximation oscillation equation (3.6) and fourth order approximation to modified equation (3.7).*

3.5. The phase and amplitude error. We use modified equation to analyze phase and amplitude error. Let denote e_n as error, then

$$\begin{aligned} e_n &= y(t_n) - y_n \\ &= y(t_n) - u(t_n) + u(t_n) - y_n. \end{aligned}$$

Since $u(t_n) - y_n$ has fourth order approximation, then $y(t_n) - u(t_n)$ gives the leading order error generated by variable stepsize method (1.3) (see Durran [D13]).

THEOREM 3.9. *The phase and amplitude error of variable stepsize method (1.3) is*

$$\begin{aligned} R - 1 &= -C_2(\omega k_n)^2 + \mathcal{O}((\omega k_n)^4) \\ |A| - 1 &= -C_1(\omega k_n)^2 + C_3(\omega k_n)^4 + \mathcal{O}((\omega k_n)^6). \end{aligned}$$

where C_1, C_2, C_3 are defined in (3.7).

Proof. Consider exact solution of oscillation equation (3.6) and modified equation (3.7),

$$\begin{aligned} y(t) &= e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \\ u(t) &= e^{i\omega t + k_n C_1 (i\omega)^2 t + k_n^2 C_2 (i\omega)^3 t + k_n^3 C_3 (i\omega)^4 t} \\ &= e^{-k_n C_1 \omega^2 t + k_n^3 C_3 \omega^4 t} \left[\cos(\omega t - k_n^2 C_2 \omega^3 t) + i\sin(\omega t - k_n^2 C_2 \omega^3 t)\right] \end{aligned}$$

Thus, phase error is

$$R - 1 = \frac{arg(u(t))}{arg(y(t))} - 1 = \frac{\omega t - k_n^2 C_2 \omega^3 t}{\omega t} - 1 = -C_2(\omega k_n)^2.$$

and amplitude error is

$$|A| - 1 = e^{-k_n C_1 \omega^2 t + k_n^3 C_3 \omega^4 t} - 1.$$

take $t = k_n$ since we are looking for local error and use

$$e^{-k_n^2 C_1 \omega^2 + k_n^4 C_3 \omega^4} \approx 1 - C_1(\omega k_n)^2 + C_3(\omega k_n)^4$$

Thus,

$$|A| - 1 = -C_1(\omega k_n)^2 + C_3(\omega k_n)^4.$$

□

REMARK 3.10. *The phase and amplitude error of backward Euler method is recovered and consistent with Durran's book result when $\nu = 0$ and $\tau = 1$.*

REMARK 3.11. *The variable stepsize method (1.3) has second order accuracy and fourth order amplitude error when $C_1 = 0$ i.e. $\nu = \frac{\tau(\tau+1)}{2\tau+1}$.*

4. Some numerical tests.

We give a few numerical illustrations next. The first tests is for the Lorenz system and the results are compared with backward Euler (step 1 without step 2) and BDF2. The self-adaptive RKF4-5 solution is taken as the benchmark solution. The second tests are for a linear and nonlinear exactly conservative systems. The third test is from Sussman [S10]. His example is one for which fully, nonlinearly implicit backward Euler preserves Lyapunov stability of the steady state while the (commonly used in CFD) linearly implicit method does not. This property is one reason for the fully implicit method being used in complex applications. We test if adding Step 2 preserves this property.

4.1. The Lorenz system. Consider the Lorenz system:

$$\frac{dX}{dt} = \sigma(Y - X),$$
$$\frac{dY}{dt} = -XZ + rX - Y,$$
$$\frac{dY}{dt} = XY - bZ.$$

The test chooses parameter values from Durran [D91] ,

$$\sigma = 12, r = 12, b = 6 \text{ and}$$
$$(X_0, Y_0, Z_0) = (-10, -10, 25).$$

The system is solved over the time interval $[0, 5]$ with Backward Euler, Backward Euler plus filter and BDF2 with constant timestep. A reference solution is obtained by adaptive RK4-5. We present solutions of the Lorenz system for several time steps in Figure 4.1.

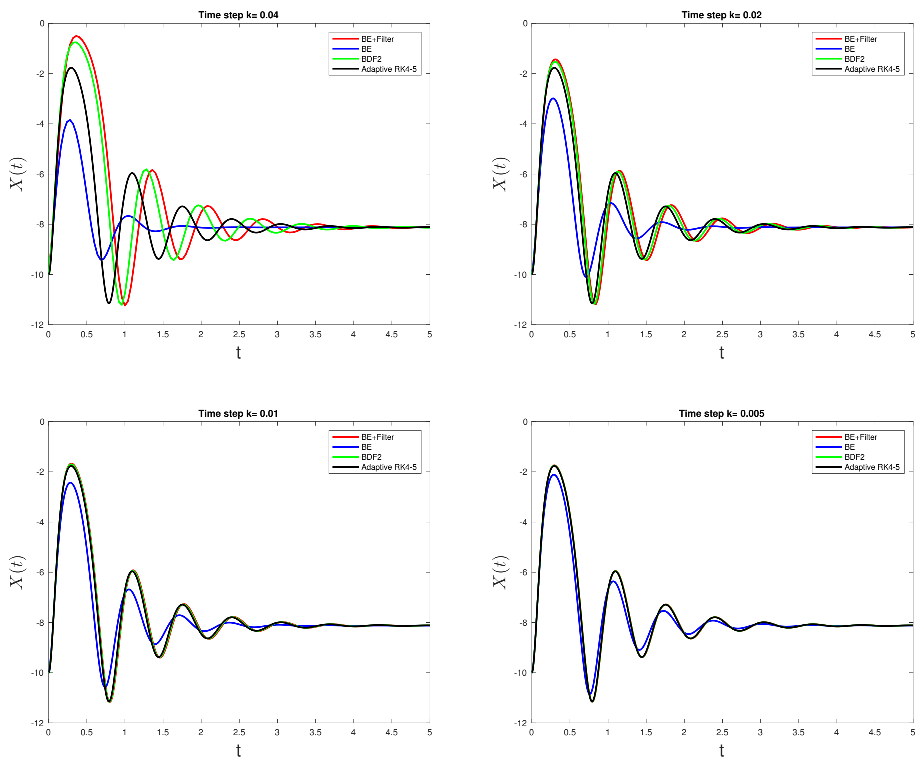

FIG. 4.1. *Lorenz system*

For moderately small time steps BE over damps severely while both BDF2 and BE+filter are accurate, even for constant timesteps. Both have a small phase error that accelerates waves slightly. For large enough time steps, all are inaccurate in different ways.

4.2. Periodic and quasi-periodic oscillations.

4.2.1. Periodic oscillations. Consider a simple pendulum problem test problem from Li and Trenchea [LT15], Williams [W13] given by

$$\frac{d\theta}{dt} = \frac{v}{L} \text{ and}$$
$$\frac{dv}{dt} = -g\sin\theta,$$

where θ, v, L and g denote, respectively, angular displacement, velocity along the arc, length of the pendulum, and the acceleration due to gravity. Set

$$\theta(0) = 0.9\pi,\ v(0) = 0,\ g = 9.8 \text{ and } L = 49$$

to observe the long-time behavior of the numerical solutions in Figure 4.2.

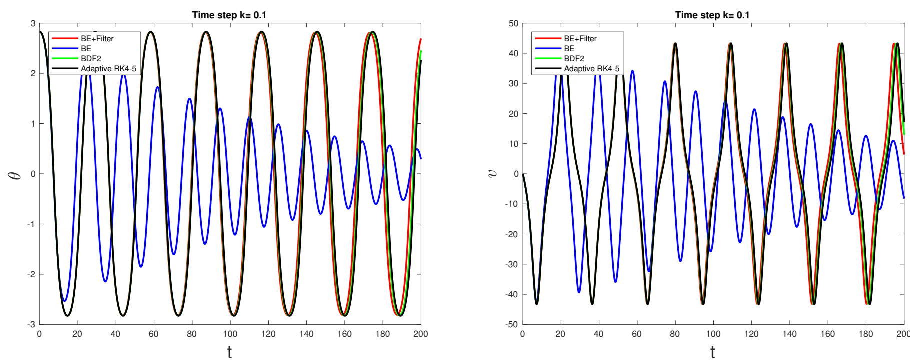

FIG. 4.2. *Simple pendulum*

Consistently with test 1, the phase and amplitude errors in both BE+filter and BDF2 are small while both are large for BE. Adding the filter step to BE has greatly increased accuracy.

4.2.2. Quasi-periodic oscillations. We solve the IVP written as a first order system

$$x'''' + (\pi^2 + 1)x'' + \pi^2 x = 0, 0 < t < 20,$$
$$x(0) = 2, x'(0) = 0, x''(0) = -(1 + \pi^2), x'''(0) = 0.$$

This has exact solution $x(t) = cos(t) + cos(\pi t)$, the sum of two periodic functions with incommensurable periods, hence quasi-periodic, Corduneanu [C89]. We solve using BE+filter with fixed timestep $k = 0.1$ and with a rudimentary adaptive BE+Filter method. In the latter we use initial timestep $k = 0.1$, the heuristic estimator (1.2), tolerance $TOL = 0.1$, 0.4 and adapt by timestep halving and doubling. The plots of both with the exact solution are next in Figure 4.3.

This test suggests that quasi-periodic oscillations are a more challenging test than periodic. Adaptivity is required but even simple adaptivity suffices to obtain an accurate solution.

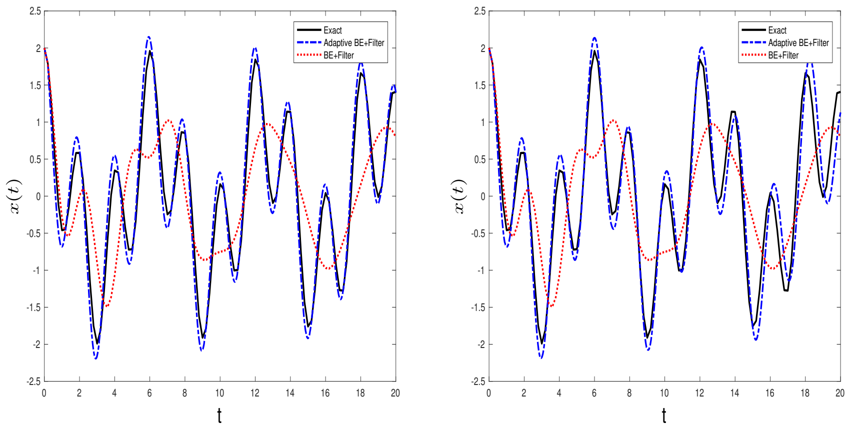

FIG. 4.3. *Quasi-periodic oscillations with TOL* $= 0.1$*(left) and* 0.4*(right).*

4.3. The example of Sussman. Next we present solutions to the test problem of Sussman [S10]. He pointed out that the fully, nonlinearly implicit backward Euler method approaches steady state while the linearly implicit only does so for sufficiently small timestep. In all cases the approximate solution approaches steady state as does the behavior of the true solution. The nonlinear system is

$$\begin{aligned} \tfrac{du_1}{dt} + u_2 u_2 + u_1 &= 1, \\ \tfrac{du_2}{dt} - u_2 u_1 + u_2 &= 1, \end{aligned}$$

with initial value $(u_1, u_2) = (0, 0)$.

In all cases, adding the filter step did not alter Lyapunov stability of the equilibrium state.

5. Conclusions. While a satisfactory, variable timestep BDF2 method exists, the combination of backward Euler plus a curvature reducing time filter gives another option that is conceptually clear and easily added by one additional line to a legacy code based on the implicit method. Both the theory and the tests both show that adding the filter step to backward Euler greatly increases accuracy.

REFERENCES

[A72] R.A. ASSELIN, *Frequency filter for time integration*, Mon. Weather Review 100(1972) 487-490.

[D78] G. DAHLQUIST, *Positive functions and some applications to stability questions for numerical methods*, 1-29 in: Recent advances in numerical analysis, (editors: C. de Boor and G. Golub) Academic Press, 1978.

[D79] G. DAHLQUIST, *Some properties of linear multistep and one-leg methods for ordinary differential equations,* Conference Proceeding, 1979 SIGNUM Meeting on Numerical ODE's, Champaign, Ill., available at: *http://cds.cern.ch/record/1069163/files/CM-P00069449.pdf* .

[DLN83] G. DAHLQUIST, W. LINIGER AND O. NEVANLINNA, *Stability of two step methods for variable integration steps,* SIAM J. Numer. Anal. 20(1983) 1071-1085.

[D13] D.R. DURRAN, *Numerical methods for wave equations in geophysical fluid dynamics*, Vol. 32. Springer Science & Business Media, 2013.

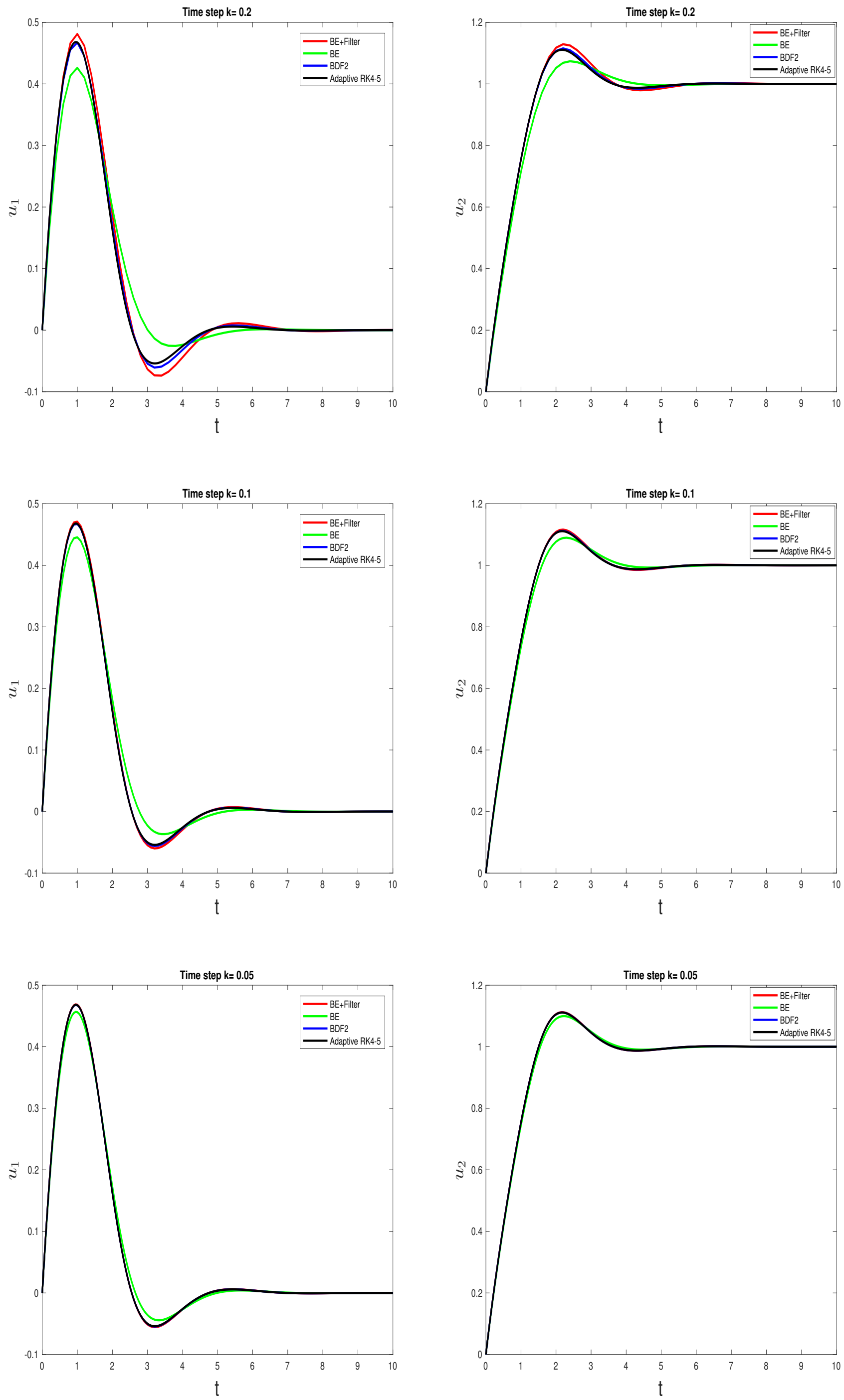

FIG. 4.4. *The example of Sussman*

[G83] R.D. GRIGORIEFF, *Stability of multi-step methods on variable grids*, Numer. Math.

42(1983) 359-377.
[K03] E. Kalnay, *Atmospheric Modeling, data assimilation and predictability*, Cambridge Univ. Press, Cambridge, 2003.
[LT15] Y. Li and C. Trenchea, *Analysis of time filters used with the leapfrog scheme*, technical report, 2015, available at: *http://www.mathematics.pitt.edu/research/technical-reports.*
[N17] L. Najman, *Modern approaches to discrete curvature*, Lecture Notes in Mathematics, Springer, Berlin, 2017.
[N84] O. Nevanlinna, *Some remarks on variable step integration*, Z. Angew. Math. Mech. 64(1984)315-316.
[R69] A. Robert, *The integration of a spectral model of the atmosphere by the implicit method*, Proc. WMO/IUGG Symposium on NWP, Japan Meteorological Soc. , Tokyo, Japan, pp. 19-24, 1969.
[S10] M. Sussman, *A stability example*, technical report, 2010, available at: *http://www.mathematics.pitt.edu/sites/default/files/research-pdfs/stability.pdf.*
[W11] P.D. Williams, *The RAW Filter: An Improvement to the Robert–Asselin Filter in Semi-Implicit Integrations*, Mon. Weather Rev., 139 (2011) 1996–2007.
[W13] P.D. Williams, *Achieving seventh-order amplitude accuracy in leapfrog integration,* Monthly Weather Review 141.9 (2013), 3037-3051.
[D91] D.R. Durran, *The third order Adams-Bashforth method: An attractive alternative to leapfrog time differencing*, Monthly weather review 119.3 (1991), 702-720.
[C89] C. Corduneanu, *Almost periodic functions*, Chelsea, 1989.

6. Appendix: 2 step methods.

The results were often developed by applying theory of 2−step methods. We collect here in this appendix some of the results applied. For constant timestep, in the standard form of a 2−step method is

$$\alpha_2 y_{n+1} + \alpha_1 y_n + \alpha_0 y_{n-1} = kf(t_{n+1}, \beta_2 y_{n+1} + \beta_1 y_n + \beta_0 y_{n-1}). \tag{6.1}$$

This can be normalized in various ways; one normalization is to rescale so the $\beta-$coefficients satisfy the standard normalization condition

$$\beta_2 \ + \beta_1 \ + \beta_0 = 1.$$

The local truncation error is developed by expanding in a standard way in Taylor series, giving

$$LTE = [\alpha_2 \ + \alpha_1 \ + \alpha_0]\, y(t_n) + k[\alpha_2 \ - \alpha_0 \ - (\beta_2 \ + \beta_1 \ + \beta_0 \ )]y'(t_n)$$
$$+k^2[\frac{\alpha_2}{2} + \frac{\alpha_0}{2} - \beta_2 \ + \beta_0]y''(t_n) + k^3[\frac{\alpha_2}{6} - \frac{\alpha_0}{6} - \frac{\beta_2}{2} \ - \frac{\beta_0}{2}]y'''(t_n) + \mathcal{O}(k^4).$$

A method is consistent if and only if the first two terms in the LTE expansion are zero and second order accurate if and only if the third term vanishes.

Next consider variable timesteps. The 2-step method for variable timestep is

$$\alpha_2 y_{n+1} + \alpha_1 y_n + \alpha_0 y_{n-1} = k_n f(t_{n+1}, \beta_2 y_{n+1} + \beta_1 y_n + \beta_0 y_{n-1}) \tag{6.2}$$

where the coefficients will depend on τ where

$$\tau = \frac{k_n}{k_{n-1}} \text{ and thus } k_n = \tau k_{n-1}.$$

The LTE expansion for $t_{n+1} - t_n = k_n$, $t_n - t_{n-1} = k_{n-1}$ is now

$$LTE = [\alpha_2 \ + \alpha_1 \ + \alpha_0]\, y(t_n) +$$
$$k_n[\alpha_2 - \alpha_0 \frac{1}{\tau} \ - (\beta_2 \ + \beta_1 \ + \beta_0 \ )]y'(t_n) + {k_n}^2[\frac{1}{2}\alpha_2 \ + \frac{1}{2\tau^2}\alpha_0 \ - \beta_2 \ + \frac{1}{\tau}\beta_0]y''(t_n) +$$
$$+{k_n}^3[\frac{1}{6}\alpha_2 \ - \frac{1}{6\tau^3}\alpha_0 \ - \frac{1}{2}\beta_2 \ - \frac{1}{2\tau^2}\beta_0]y'''(t_n) + \mathcal{O}(k_n^4)$$

The method is consistent if and only if the first two terms are zero and second order accurate if and only if the third term vanishes:

$$Consistent \Leftrightarrow \begin{cases} \alpha_2 + \alpha_1 + \alpha_0 = 0 \ and \\ \alpha_2 \ - \frac{1}{\tau}\alpha_0 \ - (\beta_2 \ + \beta_1 \ + \beta_0 \ ) = 0 \end{cases}$$

$$Second\ order \Leftrightarrow \frac{1}{2}\alpha_2 \ + \frac{1}{2\tau^2}\alpha_0 \ - \beta_2 \ + \frac{1}{\tau}\beta_0 = 0.$$

As defined by, e.g., Dahlquist, Liniger and Nevanlinna [DLN83] equation (1.12) p.1072, a variable step size method is $A-$stable if, when applied as a one-leg scheme to

$$y' = \lambda(t)y, \ \ \mathrm{Re}(\lambda(t)) \leq 0,$$

solutions are always bounded for any sequence of step sizes. Conditions for variable stepsize, $A-$stability were derived for variable step, 2-step methods in Dahlquist [D79]. The characterization in Dahlquist [D79], Lemma 4.1 page 3, 4 (specifically rearranging the equation on page 4 following (4.1)), states that the method is $A-$stable if

$$\begin{cases} -\alpha_1 \geq 0, \\ 1 - 2\beta_1 \geq 0 \text{ and} \\ 2(\beta_2 \ - \beta_0) + \alpha_1 \geq 0. \end{cases} \tag{6.3}$$

Adaptivity. The combination of backward Euler plus filter lends itself to adaptive implementation. There are various choices that must be made in such an implementation. We have purposefully made the simplest one of each option. With simple timestep halving and doubling the general adaptive method implemented was as follows.

$$\begin{aligned}
\text{Given:} \quad & y_n, y_{n-1}, k_{n-1}, k_n \text{ and } Tol \\
\text{Choose:} \quad & \nu = \frac{k_n(k_n + k_{n-1})}{k_{n-1}(2k_n + k_{n-1})} \\
\text{Compute:} \quad & y_{n+1}^{prefilter}, y_{n+1}^{postfilter} \text{ by} \\
\frac{y_{n+1} - y_n}{k_n} \ & = f(t_{n+1}, y_{n+1}), \\
y_{n+1} \Leftarrow & \ y_{n+1} - \frac{\nu}{2}\left\{\frac{2k_{n-1}}{k_n + k_{n-1}}y_{n+1} - 2y_n + \frac{2k_n}{k_n + k_{n-1}}y_{n-1}\right\} \\
EST \ & = |y_{n+1}^{prefilter} - y_{n+1}^{postfilter}| \\
\text{If:} \quad & EST \geq Tol, \ k_n \Leftarrow k_n/2 \text{ and repeat step} \\
\text{If:} \quad & EST \leq \frac{Tol}{8}, \ k_{n+1} = 2k_n \text{ and next step} \\
\text{Else:} \quad & k_{n+1} = k_n \text{ and next step}
\end{aligned}$$