University of Pittsburgh
Department of Mathematics

Linear Algebra Preliminary Exam

16 August 2018

Rule: You can use theorems proved in class or in the textbooks (provide proper reference), but if you use a statement from homework or tests you need to provide a proof.

Problem 1. Let A be a Hermitian matrix and $B = \text{Re } A$, the real part of A. Show that

$$\max_{\mu\in\sigma(B)} \mu \leq \max_{\lambda\in\sigma(A)} \lambda$$

Here $\sigma(A)$, $\sigma(B)$ are the spectrums of A, B respectively.

Problem 2. Suppose $T_1, ..., T_{n+1}$ are pairwise commuting linear operators on an n-dimensional vector space V. Suppose

$$T_1T_2...T_{n+1} = 0$$

Prove that in the above equation at least one of the factors can be removed without changing its validity.

Problem 3. Let A, B be two square complex matrices.

(a) Prove that if A and B are diagonalizable and $AB = BA$, then

$$e^{A+B} = e^A e^B.$$

(b) Prove that the conclusion of part (a) is false if A and B are both diagonalizable but do not commute.

Problem 4. Let A and B be two $n \times n$ real orthogonal matrices. Show that if

$$\det A + \det B = 0,$$

then

$$\det(A + B) = 0.$$

Problem 5. Prove that a complex 3×3 matrix A is nilpotent if and only if $\operatorname{Tr} A^k = 0$ for $k = 1, 2, 3$.

Problem 6. For an invertible matrix $P \in \mathbb{R}^{n\times n}$, let $T_P \colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the linear map defined by $T_P(M) := PMP^{-1}$ for any $M \in \mathbb{R}^{n\times n}$.

(a) For an orthogonal matrix O, show that the space $S(n) \subset \mathbb{R}^{n\times n}$ of symmetric $n \times n$ matrices is invariant under T_O.

(b) Now let O be the 3×3 rotation matrix:

$$O = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute the minimal and characteristic polynomials of T_O on $S(3)$.

Hint: Note that $O^4 = I$.