University of Pittsburgh
Department of Mathematics

Linear algebra preliminary exam

2 May 2018

Rule: You can use theorems proved in class or in the textbooks (provide proper reference), but if you use a statement from homework or tests you need to provide a proof.

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be a matrix whose components are either 1 or -1. Prove that $\det A = 2^{n-1}m$ where m is an integer.

Problem 2. Assume that V is a finite dimensional complex vector space. Suppose $T, U \in L(V,V)$ are two operators and that $TU-UT$ is nonnegative. Prove that T and U have a common eigenvector.

Problem 3. Prove that for any two matrices $A, B \in \mathbb{R}^{n\times n}$,

$$\det(I-AB) = \det(I-BA).$$

Problem 4.

(a) Assume $A \in \mathbb{C}^{n\times n}$ has n distinct nonzero eigenvalues. Prove that there are exactly 2^n distinct matrices B such that $B^2 = A$ (i.e., in particular, there are no more than 2^n matrices with this property).

(b) How many such matrices $B \in \mathbb{C}^{3\times 3}$ exist if $A = \mathrm{diag}(2,2,1)$. Why?

Problem 5. Let $f\colon \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ be the function defined for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n+1}$ by

$$f(\mathbf{x}, \mathbf{y}) := \sum_{j=1}^{n} x_j \bar{y}_j - x_{n+1}\bar{y}_{n+1}.$$

(a) For $A, B \in \mathbb{C}^{(n+1)\times(n+1)}$, show that $f(A\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, B\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n+1}$ if and only if $A = JB^*J$, where $J = \operatorname{diag}(1, \dots, 1, -1)$.

(b) We define

$$U(n,1) := \{U \in \mathbb{C}^{(n+1)\times(n+1)} \mid f(U\mathbf{x}, U\mathbf{y}) = f(\mathbf{x}, \mathbf{y}) \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^{n+1}\}.$$

Show that if $U \in U(n,1)$ then U is invertible. How do U^{-1} and U^* relate?

Problem 6. Prove that there exists a constant $C > 0$ such that for all matrices $A \in \mathbb{C}^{3\times 3}$, there exists $P \in U(3)$ and a diagonal $D \in \mathbb{C}^{3\times 3}$ such that

$$\|P^*AP - D\|^2 \le C\|A^*A - AA^*\|,$$

where $\|B\| := \operatorname{Tr}(BB^*)^{1/2}$ denotes the Hilbert-Schmidt norm of B.

Hint: Prove the statement for the class of triangular matrices first.