Unless directed otherwise, you may use facts proved in the reference textbooks listed in the graduate handbook. Justify all other assertions, including from prior homework or tests.

**Problem 1.** Let $A$ be an $n \times n$ matrix such that $A_{ij} = 0$ whenever $|i - j| \neq 1$. If $n$ is even, describe $\det(A)$ as a product of $\pm 1$ and certain entries of $A$. If $n$ is odd, what is $\det(A)$?

**Problem 2.** For linear operators $S$ and $T$ on a finite-dimensional vector space $V$, prove: $\max\{\text{nullity}(S), \text{nullity}(T)\} \leq \text{nullity}(ST) \leq \text{nullity}(S) + \text{nullity}(T)$.

**Problem 3.** Let $B \in \mathbb{C}^{n \times n}$ be positive self-adjoint, with $\langle Bx, x \rangle > 0$ for all $x \neq 0$, where $\langle x, y \rangle := \sum_{i=1}^{n} x_i \bar{y}_i$ is the standard inner product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

(a) Define a new inner product $(\cdot, \cdot)$ on $\mathbb{C}^n$ by $\langle x, y \rangle := \langle Bx, y \rangle$.

For $M \in \mathbb{C}^{n \times n}$, prove that the adjoint of $M$ with respect to the inner product $(\cdot, \cdot)$ is $B^{-1}M^*B$, where $M^* = M^t$ is the adjoint of $M$ with respect to the standard inner product $(\cdot, \cdot)$. Use this to show that if $A$ is self-adjoint with respect to $(\cdot, \cdot)$ then $B^{-1}A$ is self-adjoint with respect to $(\cdot, \cdot)$.

(b) If $A \in \mathbb{C}^{n \times n}$ is self-adjoint, show that the generalized Rayleigh quotient $\frac{\langle Ax, x \rangle}{\langle Bx, x \rangle}$ attains a maximum $m \in \mathbb{R}$ on $\mathbb{C}^n - \{0\}$ at a vector $v$ satisfying $Av = mBv$.

**Problem 4.** For a one-variable polynomial $p(x)$ let $p'$ be the derivative of $p$, computed in the usual way, and let $E(p)(x) = \frac{1}{2}[p(x) + p(-x)]$ be the even part of $p$. For any fixed $n \in \mathbb{N}$, find the characteristic and minimal polynomials of the operator $T$ defined by $T(p)(x) = (1 + x)E(p')(x)$ on the space $V_n$ of polynomials of degree at most $n$.

**Problem 5.** Let $V$ be a finite-dimensional inner product space. Suppose $A$ and $B$ are orthogonal projections of $V$ with the property that for all $x \in V$, $\|Ax\|^2 + \|Bx\|^2 \leq \|x\|^2$.

Prove that $A + B$ is also an orthogonal projection.

**Problem 6.** Suppose that a matrix $A \in \mathbb{R}^{6 \times 6}$ satisfies the following:

- $\text{rank}(A) > 3$;
- $A^3$ is a projection, but $Ax \neq x$ for all non-zero vectors $x$; and
- There is an $A$-invariant direct sum decomposition $\mathbb{R}^6 = U \oplus V$, such that $\dim U = 2$ and the restriction of $A$ to $U$ is orthogonal, and $\dim V = 4$ and the restriction of $A$ to $V$ is nilpotent.

Describe all possible Jordan forms of $A$ (there is more than one), now regarded as a complex matrix, and prove that these are the only possibilities.