Ph.D. Preliminary Examination (Analysis)

April, 2011

Instructions: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let $\{f_n\}$ be a sequence of real-valued functions on $[0,1]$ such that

(a) $|f_n(x) - f_n(y)| \le \dfrac{|\sqrt{x} - \sqrt{y}|}{1 + |x-y|}$ for all $n \in \mathbf{N}$, $x, y \in [0,1]$;

(b) $\left| \int_0^1 f_n(x)dx \right| \le 1$ for all $n \in \mathbf{N}$.

Prove that $\{f_n\}$ has a subsequence that converges uniformly on [0,1].

2. Let (M,d) be a metric space such that

$$d(x,z) \le \max\{d(x,y), d(y,z)\}$$

for all $x, y, z \in M$. For any $x \in M, r > 0$, the set $B(x,r) = \{y \in M : d(y,x) < r\}$ is called an open ball in M.

(i) Prove that every open ball in M is a closed set.

(ii) Prove that if two open balls in M have a common point, then one of them is contained in the other.

3. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \dfrac{(x^2-y)y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Determine whether f is differentiable at $(0,0)$ and prove your conclusion.

4. Let $f \in [0,\infty) \to \mathbf{R}$ be a differentiable function on $[0,\infty)$ such that $f(0) = \alpha > 0$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2}$$

for every $x \in [0,\infty)$. Prove that $\lim_{x\to\infty} f(x)$ exists (as a real number).

5. Let $n, m \in \mathbf{N}$ and $f : \mathbf{R}^n \to \mathbf{R}^m$ be C^1 on $\mathbf{R}^n$. Suppose that for every $a \in \mathbf{R}^n$, $Df(a)$ has rank m. Prove that $f(\mathbf{R}^n)$ is open.

6. For $x = (x_1, x_2) \in \mathbf{R}^2$ let $|x| = \sqrt{x_1^2 + x_2^2}$. Let $D = \{x \in \mathbf{R}^2 : |x| \le 1\}$ and $f : D \to \mathbf{R}$ be continuous on D. Prove that

$$\lim_{n\to\infty} \int\int_D (n+2)|x|^n f(x)dA = \int_0^{2\pi} f(\cos t, \sin t)dt.$$