Model Reduction by Constraints, Discretization of Flow Problems and an Induced Pressure Stabilization

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Abstract

For flow problems, turbulence models and numerical methods can be thought of as model reductions. Herein we consider reduction imposed as a constraint using Lagrange multipliers. Development of this last method recovers the subgrid eddy viscosity method and a new stabilization of the incompressibilty constraint. Analysis of the new incompressibility stabilization reveals that stabilization is only really needed for the finest pressure scales.

1 Introduction

There is currently intense interest in multiscale problems in which the effects of the (unresolved) fine scales upon the (resolved) large scales are essential and must be estimated and accounted for in discretizations. This is also clearly connected with ideas of model reduction. For example, in flow problems all normal approachs can be viewed as model reductions, including the usual Galerkin finite element method (reduction to a finite

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dimensional problem), models in large eddy simulation (reduction to local spacial averages), conventional turbulence models (reduction to time averages) and the much greater reductions of P.O.D. methods. Interesting new approachs includes the variational multiscale method (e.g., [Col02], [HMHMR98], [HMJ00], [HOM01] and much other work also) in which, in essence, model reduction is only applied to the equations of the unresolved fluctuations in an exact representation of the linkage between the means and fluctuations and two-scale/nonlinear Galerkin methods, e.g., Marion and Xu [MX95].

One purpose of this report is to explore a new description of the Galerkin truncation of the Navier-Stokes equations to a finite dimensional problem. A new regularization of the incompressibility constraint evolves from this point of view. We also give an analysis of this regularization and show it has interesting properties.

To begin, consider the incompressible, viscous Navier-Stokes equations in a domain $\Omega \subset \mathbf{R}^d (d = 2 \text{ or } 3)$

$$u_t + \nabla \cdot (u\ u) + \nabla p - \nu \Delta u = f \text{ and } \nabla \cdot u = 0 \tag{1.1}$$

in $\Omega \times (0, T]$ subject to the no-slip condition on $\partial\Omega$, an initial condition and the usual pressure normalization condition

$$u = 0 \text{ on } \partial\Omega,\ u(x,0) = u_0(x) \text{ and } \int_\Omega p\ dx = 0. \tag{1.2}$$

The usual variational formulation of (1.1), (1.2) uses the function spaces

$$X := H_0^1(\Omega)^d,\ Q := L_0^2(\Omega) \text{ and } V := \{v \in X : (\nabla \cdot v, q) = 0\ \forall q \in Q\}.$$

Define the usual trilinear form

$$b(u, v, w) := \int_\Omega u \cdot \nabla v \cdot w\ dx = -\int_\Omega uv : \nabla w\ dx.$$

With these definitions, one variational formulation of (1.1) is: find $u : [0, T] \to X,\ p : (0, T] \to Q$ satisfying $u(x, 0) = u_0(x)$ and

$$\begin{aligned} &(u_t, v) + b(u, u, v) + (\nu \nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v) \text{ for all } v \in X, \\ &(\nabla \cdot u, q) = 0 \text{ for all } q \in Q. \end{aligned} \tag{1.3}$$

In the analysis of (1.3) and in the formulation of numerical methods for (1.1), (1.2), (1.3), it is useful to recall, [GR86], [Gun89], that

$$b(u, v, w) = b^*(u, v, w) := \frac{1}{2} b(u, v, w) - \frac{1}{2} b(u, w, v)$$

for $u \in V,\ v, w \in X$.

In Section 2, we consider the following method of reducing (1.3) to a finite dimensional problem; picking finite dimensional subspaces $\bar{X} \subset X,\ \bar{Q} \subset Q$:

$$\text{(1.4)} \qquad \text{Solve (1.3) subject to the constraint that } u \in \bar{X},\ p \in \bar{Q}.$$

When (1.4) is formulated using two Lagrange multipliers, it naturally leads to a problem in which the infinite dimensional part is captured by the two multipliers. In Section 2, we give a functional analytic formulation of the idea (1.4) and prove the doubly constrained problem is well-posed. When solved exactly we also prove it to be equivalent to the usual Galerkin approximation to (u, p) in $(\bar{X}, \bar{Q})$. Next a choice must be made: either finite dimensional approximation of the multipliers or elimination of the multipliers by penalty methods considered. Both lead to new discretizations. The first would give important information (through λ_1^h, λ_2^h) about the severity of the particular finite dimensional truncation. The second choice (which we consider) leads to multiscale stabilizations of the nonlinear convection terms (previously studied from a different origin in [L02, H02, K02, KL02, G99a, G99b, MX95]) and related stabilization of the incompressibility constraint. This (apparently new) regularization is *simple* and *optimally accurate*. It's analysis (in Section 3) shows that pressure stabilization is really only needed for the finest resolved scales of the pressure. This result is in complete accord with computational experience that violation of the $(\bar{X}, \bar{Q})$ inf-sup condition can lead to pressure oscillation on the finest mesh scales. It is closely connected to the local stabilization scheme of [BB01], related to those using bubble functions (as an implicit model for pressure fluctuations) [Pie89], [BN83], [BP84], [BBFMR92], [FF95], [HFB86], [KS92] and [TL91] and, naturally, yet another realization of the idea of regularization of ill-posed problems, [TA77],

2 Finite Dimensional Reduction Using Constraints

The elaboration of numerical methods for (1.1), (1.2) is based on that of the two methods: the usual Galerkin method and the Lagrange multiplier method of enforcing constraints.

The Galerkin Formulation

The Galerkin formulation begins with finite dimensional subspaces

$$\overline{X} \subset X \quad , \overline{Q} \subset Q$$

which satisfy the discrete inf-sup condition:

$$\inf_{\overline{q} \in \overline{Q}} \sup_{\overline{v} \in \overline{X}} \frac{(\overline{q}, \nabla \cdot \overline{v})}{||\overline{q}|| \; ||\nabla \overline{v}||} \geq \beta > 0. \tag{2.1}$$

Under (2.1), the discretely div-free subspace V is well-defined and non-trivial, where

$$V := \{\overline{v} \in \overline{X} : (\overline{q}, \nabla \cdot \overline{v}) = 0 \text{ for all } \overline{q} \in \overline{Q}\}. \tag{2.2}$$

The Galerkin formulation is to find $\overline{u} : [0,T] \to \overline{X}$, $\overline{p} : (0,T] \to \overline{Q}$ satisfying

$$(\overline{u}_t, \overline{v}) + b(\overline{u},\ \overline{u}, \overline{v}) + \nu(\nabla \overline{u},\ \nabla \overline{v}) - (\overline{p} \nabla \cdot \overline{v}) + (\overline{q}, \nabla \cdot \overline{u}) = (f, \overline{v}), \tag{2.3}$$

for all $(\overline{v}, \overline{q}) \in (\overline{X}, \overline{Q})$.

The following is well-known, [GR86], [Gun89].

Theorem 2.1 *Under the inf-sup condition (1.3) in $(\overline{X}, \overline{G})$, the Galerkin formulation (2.3) is equivalent to: find $\overline{u} \in \overline{V}$ satisfying:*

$$(\overline{u}_t, \overline{v}) + b(\overline{u}, \overline{u}, \overline{v}) + \nu(\nabla \overline{u}, \nabla \overline{v}) = (f, \overline{v}), \text{ for all } \overline{v} \in \overline{V}. \quad \square \tag{2.4}$$

The Galerkin formulation produces an *approximate* solution $\overline{u}, \overline{p}$ and reduces the original NSE to (nonlinear) equations in the finite dimensional spaces $\overline{X}, \overline{Q}$.

Finite Dimensional Truncation via Constraints

Let Y, L, Y^* be Hilbert spaces with the indicated embeddings

$$Y \hookrightarrow L \hookrightarrow Y^*$$

compact and dense. Abstractly, the nonlinear problem can be represented: given $f \in Y^*$ and $N(\cdot) : Y \to Y^*$, find $\mathcal{U} \in Y$ with

$$N(\mathcal{U}) = f. \tag{2.5}$$

Let Y be decomposed into a finite dimensional space $\overline{Y}$ (representing the large scales) and an infinite dimensional space of fluctuations Y'. Thus, given $\mathcal{U} \in Y$ we can write

$$\mathcal{U} = \overline{\mathcal{U}} + \mathcal{U}', \overline{\mathcal{U}} \in \overline{Y}, \mathcal{U}' \in Y' \text{ where } Y = \overline{Y} \oplus Y'.$$

The problem (2.5) can be truncated to $\overline{Y}$ through the Galerkin method, the nonlinear Galerkin method, an approximate variational multiscale method or through constraints, which we now develop. Thus, consider the following continuous problem which approximates (2.5): find $\overline{w} \in \overline{Y}$ satisfying:

$$N(\overline{w}) = f \text{ subject to } B(\overline{w}) = 0, \tag{2.6}$$

where $B : Y \to Y$ has nullspace Y'. Formally introducing a Lagrange multiplier λ, (2.10) leads to the problem: find $\overline{w} \in \overline{Y}, \lambda \in M$ (the multiplier space which is yet to be specified) satisfying

$$N(\overline{w}) + (B)^*\lambda = f, \text{ and } B\overline{w} = 0. \tag{2.7}$$

This problem is finite dimensional problem in $\overline{w}$ and is infinite dimensional in λ. Provided $M \subset Y^*$, its well-posedness depends upon the structure of $N(\cdot)$ and on the choice of B' and M via the inf-sup condition

$$\inf_{\mu \in M} \sup_{y \in Y} \frac{\langle By, \mu \rangle}{||y||_Y ||\mu||_M} \geq \beta > 0, \tag{2.8}$$

which must be verified.

The abstract theory of mixed methods, e.g., [GR86], [BF83], gives the following.

Proposition 2.1 *Suppose N is linear and coercive. Let $M \subset Y^*$, suppose that B is continuous,*

$$\langle By, \mu \rangle \leq C \; ||y||_X \; ||\mu||_M \textit{ for all } y \in Y, \; \mu \in M,$$

and suppose the inf-sup condition (2.8) holds. Then, (2.7) is equivalent to the usual Galerkin approximation to $\mathcal{U}$ in $\bar{Y}$ given by: find $\overline{w} \in \overline{Y}$ satisfying

$$\langle N(\overline{w}), \overline{v} \rangle = \langle f, \overline{v} \rangle, \textit{ for all } \overline{v} \in \overline{Y}. \quad \square$$

Consider this abstract approach applied to the Navier Stokes equations. We choose $Y = (X, Q)$ and $\mathbf{L} := L^2(\Omega)^{d\times d}$.

The most natural interpretation for fluids is (following [Lay02a]) to split the gradient (or deformation tensor) rather than the veocity itself into means and fluctuations as follows. Given $\overline{X} \subset X$ define

$$\overline{\mathbf{L}} := \{\ell \in L : \ \ell = \nabla \overline{v} \text{ for some } \overline{v} \in \overline{X}\}.$$

Associated with $\overline{\mathbf{L}}$ we define $\mathbf{L}' = \overline{\mathbf{L}}^{\perp}$ and let $\overline{P}, P' = I - \overline{P}$ denote the L-orthogonal projectors

$$\overline{P} : \mathbf{L} \to \overline{\mathbf{L}}, \text{ and } P' = (I - \overline{P}) : \mathbf{L} \to \mathbf{L}' := \overline{\mathbf{L}}^{\perp}.$$

Choose

$$M_1 = \mathbf{L}' \text{ and } B_1 : X \to M_1 \text{ via } B_1 w := P'\nabla w, \ w \in X.$$

The adjoint of B_1, B_1^* is then formally

$$B_1^* : M_1 \to X^* \text{ by } B_1^*\mu := \nabla \cdot (P'\mu) \text{ for } \mu \in M_1.$$

The associated bilinear form $b_1(\cdot,\cdot) : X \times M_1 \to \mathbf{R}$ by

$$b_1(v, \mu) := (B_1 v, \mu), \text{ for } v \in X \text{ and } \mu \in M_1.$$

Proposition 2.2 *$b_1(\cdot,\cdot)$: $X \times M_1 \to \mathbf{R}$ satisfies*

(i) continuity:

$$b_1(v, \mu_1) \le C||v||_X \ ||\mu_1||_{M_1} \textit{ for all } v \in X \textit{ and } \mu \in M_1,$$

(ii) the inf-sup condition:

$$\inf_{\mu_1 \in M_1} \sup_{v \in X} \frac{b_1(v, \mu_1)}{||v||_X ||\mu_1||_M} \ \ge C > 0,$$

(iii) $\overline{X} = \ker(B_1)$.

Proof: For (i) we note that $P'\mu_1 = \mu_1$ for $\mu_1 \in M_1$. Thus,

$$b_1(v, \mu_1) = (B_1 v, \mu_1) = (P'\nabla v, \mu_1) = (\nabla v, P'\mu_1) =$$
$$(\nabla v, \mu_1) \le ||\nabla v|| \ ||\mu_1|| \le C||v||_X \ ||\mu_1||_{M_1},$$

by the Poincaré-Friedrich's inequality.

For (ii), let $\mu_1 \in M_1$ be fixed but arbitrary. Consider

$$\frac{b_1(v,\mu_1)}{||v||_X||\mu_1||_{M_1}} = \text{(as in (i))} = \frac{(\nabla v,\mu_1)}{||v||_X||\mu||_{M_1}}.$$

Since $\nabla X = \mathbf{L}$, there is a $v \in X$ with $\nabla v = \mu$. Choosing this $v \in X$ gives:

$$\frac{b(v,\mu_1)}{||v||_X||\mu_1||_{M_1}} = \frac{||\mu_1||^2}{||v||_X||\mu_1||_M} = \frac{||\mu_1||}{||v||_X} = \frac{||\nabla v||}{||v||_X} \geq C > 0,$$

by the Poincaré-Friedrich's inequality.

Part (iii) is immediate from the definitions of P' and that $\overline{L} = \nabla \overline{X}$. □

From (iii) an equivalent formulation to $u \in \overline{X}$ is $B(w) = 0$. We next develop an equality constraint for $p \in \overline{Q}$. The most natural choice is also the simplest one. Choose a finite dimensional subspace

$$\overline{Q} \subset Q \text{ and define } Q' := \overline{Q}^{\perp} \subset Q.$$

Associated with Q' is the orthogonal projector $P'_Q : Q \to Q'$. Define $B_2 q = P'_Q q$, for all $q \in Q$ so that

$$q \in \overline{Q} \text{ if and only if } B_2 q = 0.$$

Accordingly, define M_2 and $b_2 : Q \times M_2 \to \mathbf{R}$ by

$$M_2 := Q' \text{ and } b_2(q,\mu_2) := (B_2 q, \mu_2).$$

Proposition 2.3 *Let $M_2 = Q'$. Then,*

(i) $b_2(\cdot,\cdot)$ is continuous on $Q \times M_2$.

(ii)

$$\inf_{\mu_2 \in M_2} \sup_{q \in Q} \frac{(B_2 q,\mu_2)}{||q||_Q||\mu_2||_{M_2}} \geq 1 > 0.$$

Proof: Part (i) is simply the Cauchy-Schwarz inequality. For part (ii), given $\mu_2 \in M_2$, pick $q = \mu_2$. Then $B_2 q_2 = \mu_2 (\in Q')$. □

Corollary 2.1 *With the choices $M_1 := \mathbf{L}'$ and $M_2 := Q'$ we have*

(i) the following inf-sup condition in $M = (M_1, M_2)$ holds:

$$\inf_{\mu=\mu_1,\mu_2 \in M} \sup_{(v,q)\in(X,Q)} \frac{b_1(v,\mu_1) + b_2(q,\mu_2)}{(||\mu_1||^2_{M_1} + ||\mu_2||^2_{M_2})^{1/2}(||v||^2_X + ||q||^2_Q)^{1/2}} \geq C > 0,$$

(ii) the constrained problem: find $(w, q) \in (X, Q), \lambda = (\lambda_1, \lambda_2) \in M := (M_1, M_2)$ satisfying

$$(2.9) \qquad \begin{cases} (w_t, v) + b(w, w, v) + (\nu\nabla w, \nabla v) - (q, \nabla \cdot v) + (r, \nabla \cdot w) \\ +b_1(v, \lambda_1) + b_2(r, \lambda_2) = (f, v), \text{ for all } v \in X,\ r \in Q, \\ b_1(w, \mu_1) = 0, \text{ for all } \mu_1 \in M_1, \\ b_2(q, \mu_2) = 0, \text{ for all } \mu_2 \in M_2. \end{cases}$$

is equivalent to the Galerkin approximation (2.3) for $(\overline{u}, \overline{p}) \in (\overline{X}, \overline{Q})$.

Proof: Part (i) follows exactly like the proofs of Propositions 2.2 and 2.3. Part (ii) is a special case of Proposition 2.1. □

3 Penalty Induces Subgrid Eddy Viscosity and Fine Scale Pressure Stabilization

Consider the constrained problem (2.9), rewritten at the level of the continuous problem: find $w, q, \lambda_1\lambda_2$ satisfying:

$$(3.1) \qquad \begin{cases} w_t + \nabla \cdot (w\ w) - \nu\Delta w + \nabla q + (B_1)^*\lambda_1 + (B_2)^*\lambda_2 = f, \\ \nabla \cdot w = 0,\ B_1 w = 0, \text{ and } B_2 q = 0. \end{cases}$$

One standard method of approximating such constrained problems is to replace the constraints by penalty terms. We explore that idea in this section.

Pick small penalty parameters $\epsilon_1, \epsilon_2 > 0$. The standard penalty approach is to replace $B_1 w = 0$ and $B_2 q = 0$ by

$$(3.2) \qquad B_1 w_\epsilon = \epsilon_1 \lambda_1 \text{ and } B_2 q_\epsilon = \epsilon_2 \lambda_2.$$

Consider $b_1(v, \lambda_1)$ and $b_2(q, \lambda_2)$. With (2.2)

$$b_1(v, \lambda_1) = (B_1 v, \lambda_1) = (\text{since } \lambda_1 = \epsilon_1^{-1} B_1 w_\epsilon) = \epsilon_1^{-1}(B_1 v, B_1 w_\epsilon), \text{ and}$$
$$b_2(r, \lambda_2) = (B_2 r, \lambda_2) = \text{ using (2.2)} = \epsilon_2^{-1}, (B_2 r, B_2 q).$$

Thus, λ_1, λ_2 are eliminated from the penalty approximation to (3.1). We obtain the problem: find: $w_\epsilon : [0,T] \to X,\ q_\epsilon : (0,T] \to Q$ satisfying:

$$(3.3)\quad (w_{\epsilon t}, v) + b(w_\epsilon, w_\epsilon, v) + (\nu \nabla w_\epsilon, \nabla v) - (q_\epsilon, \nabla \cdot v) + (\nu \nabla w_\epsilon, \nabla v)$$
$$+\epsilon_1^{-1}(P'(\nabla w_\epsilon), P'(\nabla v)) + \epsilon_2^{-1}(P'_Q q_\epsilon, P'_Q r) = (f, v), \text{ for all } v \in X,\ r \in Q.$$

The terms multiplied by ϵ_1^{-1} in method (3.3) are almost identical to those in the regularization by subgrid eddy viscosity, [Lay02a], [KL02], with the identification of $\epsilon_2^{-1} = \nu_T$.

Another interesting feature of (3.3) is the treatment of the incompressibility constraint. Setting $v = 0$ in (3.3) shows that w_ϵ satisfies:

$$(3.4)\qquad (\nabla \cdot w_\epsilon, r) + \epsilon_2^{-1}(P'_Q q_\epsilon, P'_Q r) = 0, \text{ for all } r \in Q.$$

In (2.4) write $r = \overline{r} + r'$ where $\overline{r} \in \overline{Q},\ r' \in Q'$. Setting alternately $r = \overline{r}$ and $r = r'$ reduces (3.4) to two constitutive subproblems

$$(\nabla \cdot w_\epsilon, \overline{r}) = 0, \text{ for all } \overline{r}, \text{ and } (\nabla \cdot w_\epsilon, r') + \epsilon_2^{-1}(p'_Q q_\epsilon, r') = 0, \text{ for all } r' \in Q'.$$

These can be written as:

$$(3.5)\qquad \overline{P}_Q(\nabla \cdot w_\epsilon) = 0 \text{ and } P'_Q(\nabla \cdot w_\epsilon + \epsilon_2^{-1} q_\epsilon) = 0.$$

Pressure Regularization is Only Needed on Finest Scales

The method (3.3) introduces a new regularization of the pressure/ incompressibility coupling by (3.4), (3.5). This regularization is a possible way to circumvent the discrete inf-sup condition in a more accurate way in discretizations of (3.3). We consider now exactly this question for the Stokes problem:

$$(3.6)\qquad -\Delta u + \nabla p = f,\ \nabla \cdot u = 0, \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

The most common examples of a "bad" choice of velocity-pressure spaces are the linear-constant pair and the linear-linear pair. Accordingly, choose meshes $T^H(\Omega),\ T^h(\Omega), (H < h)$ and for $\mu = h, H$.

$$X^\mu := \{v \in C^0(\Omega) : v|_\Delta \in \mathcal{P}_1(\Delta) \text{ for all } \Delta \text{ in } T^\mu(\Omega)\} \cap H_0^1(\Omega)$$
$$Q_j^\mu := \{q \in H^j(\Omega) : q|_\Delta \in \mathcal{P}_j(\Delta) \text{ for all } \Delta \text{ in } T^\mu(\Omega)\} \cap L_0^2(\Omega), j = 0, 1.$$

Consider the problem: find $(u_h, p_h) \in (X_h, Q_h)$ satisfying

$$(3.7)\qquad \begin{cases} (\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = (f, v_h), \text{ for all } v_h \in X_h \\ (\nabla \cdot u_h, q_h) + \epsilon^{-1}((I - P_{Q_H})p_h, (I - P_{Q_H})q_h) = 0, \\ \text{for all } q_h \in Q_h. \end{cases}$$

It will be convenient to denote $(I - P_{Q^H})q_h$ by $(q_h)'$.

Lemma 3.1 *Any solution of (3.7) satisfies the á priori bound*

$$\frac{1}{2}||\nabla u_h||^2 + \epsilon^{-1}||(p_h)'||^2 \leq \frac{1}{2}||f||_{-1}^2.$$

Proof: Set $v_h = u_h$ and $q_h = p_h$ in (2.7). Adding the equations gives

$$||\nabla u_h||^2 + \epsilon^{-1}||(p_h)'||^2 = (f, u_h) \leq ||f||_{-1}\ ||\nabla u_h||,$$

from which the result follows. □

If we write

$$Q_h = Q_H + Q'$$

then the basic formulation gives control of the portion of the discrete pressure in Q' (Lemma 3.1). Thus, the component of the discrete pressure in Q^H needs careful consideration. To that end, write $p_h = p_H + p', p_H = P_{Q_H} p_h \in Q_H$ and $p' = (I_h - P_{Q_H})p_h \in Q'$. Similarly, we write $q^h = q^H + q'$ and alternately set $q^h = q^H$ and $q^h = q'$ in (2.7). This uncouples the discrete incompressibility constraint into its large scale and fine scale component parts:

$$(\nabla \cdot u^h, q^H) = 0,\ \text{for all}\ q^H \in Q^H,\ \text{and}\ (\nabla \cdot u^h, q') + \epsilon^{-1}(p', q') = 0,\ \text{for all}\ q' \in Q'.$$

The next lemma contains the key ingredients to the error analysis that follows.

Lemma 3.2 *Suppose (X^h, Q^H) satisfies the discrete inf-sup condition*

$$\inf_{q^H \in Q^H} \sup_{v^h \in X^h} \frac{(q^H, \nabla \cdot v^h)}{||q^H||\ ||\nabla v^h||} \geq \beta > 0. \tag{3.8}$$

Then, any solution of (2.7) satisfies the á priori bounds:

$$||\nabla u^h|| \leq ||f||_{-1},\ ||p'|| \leq \frac{\epsilon}{2}||f||_{-1},\ ||p^H|| \leq \beta^{-1}(2 + \frac{\epsilon^{1/2}}{2})||f||_{-1},$$

and thus

$$||\nabla u^h||^2 + ||p^h||^2 \leq C(\beta)(1+\epsilon)||f||_{-1}^2. \tag{3.9}$$

Proof: The first two á priori bounds follow immediately from Lemma 3.1. The third follows from the first two and the (X^h, Q^H) inf-sup condition as follows. From (3.7),

$$(p^H + (p^h)', \nabla \cdot v^h) = (\nabla u^h, \nabla v^h) - (f, v^h),\ \text{so}$$
$$(p^H, \nabla \cdot v^h) \leq ||\nabla u^h||\ ||\nabla v^h|| + ||f||_{-1}||\nabla v^h|| + ||(p^h)'||\ ||\nabla v^h||.$$

Dividing by $||\nabla v^h||$, taking the supremum over $v^h \in X^h$ gives

$$\beta||p^H|| \leq ||\nabla u^h|| + ||f||_{-1} + ||p^h)'||.$$

Inserting the first two á priori bounds into the RHS yields the claimed estimate. $\square$

Using the stability proof in Lemma 3.2 we can immediately conclude that the following projection operator associated with the formulation (3.7) is well defined and optimally accurate.

Definition 3.1 *Let (X^h, Q^H) satisfy (3.8). Given $(u,p) \in (X,Q)$ and $\epsilon > 0$, the stabilized projection $P_\epsilon(u,p) = (\tilde{u},\tilde{p}) \in (X^h, Q^h)$ is defined to be the solution of the linear system*

$$\begin{aligned} &(\nabla(u-\tilde{u}), \nabla v^h) - (p-\tilde{p}, \nabla \cdot v^h) = 0, \ \forall\ v^h \in X^h, \\ &(\nabla \cdot (u-\tilde{u}), q^h) - \epsilon^{-1}((P_{Q^h} - P_{Q^H})(p-\tilde{p}), q^h) = 0 \ \forall\ q^h \in Q^h. \end{aligned}$$

Proposition 3.1 *Let $0 < \epsilon \leq 1$. If (3.8) holds then the projection operator P_ϵ is well defined and has uniformly bounded norm:*

$$||\nabla \tilde{u}|| + ||\tilde{p}|| \leq C(\beta)(||\nabla u|| + ||p||). \tag{3.10}$$

Thus, the error in the projection $(\tilde{u},\tilde{p}) = P_\epsilon(u,p)$ is quas-ioptimal

$$||\nabla(u-\tilde{u})|| + ||p-\tilde{p}|| \leq C(\beta)\{\inf_{v^h \in X^h} ||\nabla(u-u^h)|| + \inf_{q^h \in Q^h} ||p-q^h||\}. \tag{3.11}$$

Proof: The first claimed inequality follows exactly the proof of Lemma 3.2. This proves uniqueness which implies existence. The á priori estimate (3.7) implies that P_ϵ is stable, uniformly in ϵ, and hence quasioptimally convergent, uniformly in ϵ. $\square$

Next, we consider the error in the method. The error equations are used to bound the velocity error and the error in the pressure fluctuations. The (X_h, Q_H) inf-sup condition (3.8) is used to bound the error in the mean pressures.

Theorem 3.1 *Suppose (3.8) holds* $0 < \epsilon \leq 1$. *Then, the errors in the approximation (3.7) satisfy*

$$||\nabla(u-u_h)|| \leq C(\beta)\{\inf_{v_h\in X^h} ||\nabla(u-v_h)||$$
$$+\epsilon^{-1/2} \inf_{q_h\in Q_h} ||(p-q_h)'||\} + 2\epsilon^{-1/2}||p'||,$$
$$||p-p_h|| \leq C(\beta)\{\inf_{v_h\in X_h} ||\nabla(u-v_h)|| \inf_{q_h\in Q_h} ||p-q_h|| + (1+\epsilon^{-1/2})||p'||\}.$$

Proof: With the results of Proposition 3.1 we can quickly perform the error analysis of the method. Indeed, let (u,p) be the solution of the Stokes problem and (u_h,p_h) the approximation given by (3.7). Write the velocity error as

$$u-u_h = (u-\tilde{u}) - (u_h-\tilde{u}),\ \tilde{u} = P_\epsilon(u,p)$$

and let $\eta = u-\tilde{u}$, $\phi_h := u_h - \tilde{u}$. Then, it is easy to see, subtracting (3.7) from the equation for (u,p) and using the definition of $(\tilde{u},\tilde{p})$, that

$$(3.12)(\nabla\phi_h,\nabla v_h) - (p_h-\tilde{p}_h,\nabla\cdot v_h) = (\nabla\eta,\nabla v_h) - (p-\tilde{p},\nabla\cdot v_h) = 0,$$

by the choice of $(\tilde{u},\tilde{p})$, and that

$$(\nabla\cdot\phi_h,q_h)+\epsilon^{-1}((p_h-\tilde{p})',q_h') = (\nabla\cdot\eta,q_h)+\epsilon^{-1}(\eta',q_h')-\epsilon^{-1}(p',q_h') = -\epsilon^{-1}(p',q_h'),$$
(3.13)

again by the choice of $(\tilde{u},\tilde{p})$. Setting $v_h = \phi_h$ and $q_h = p_h - \tilde{p}$ and adding the two equations gives

$$||\nabla\phi_h||^2 + \frac{\epsilon^{-1}}{2}||(p_h-\tilde{p})'||^2 = -\epsilon^{-1}(p',(p_h-\tilde{p})') \leq \frac{\epsilon^{-1}}{2}||(p_h-\tilde{p})'||^2 + \frac{\epsilon^{-1}}{2}||p'||^2.$$

Thus,

$$(3.14) \qquad ||\nabla\phi_h||^2 + \frac{\epsilon^{-1}}{2}||(p_h-\tilde{p})'||^2 \leq \frac{\epsilon^{-1}}{2}||p'||^2$$

and the triangle inequality implies

$$||\nabla(u-u_h)||^2 + \epsilon^{-1}||(p-p_h)'||^2 \leq$$
$$\leq C(\beta)\{\inf_{v^h\in X^h}||\nabla(u-v^h)||^2 + \epsilon^{-1}\inf_{q^h\in Q_h}||(p-q_h)'||^2\} + \epsilon^{-1}||p'||^2,$$

which is the basic error bound for the velocity and the pressure fluctuations. Next the error in the mean pressure field must be considered following the ideas in the proof of Lemma 3.2.

The first error equation (3.12) gives

$$(p_h - \tilde{p}, \nabla \cdot v_h) = (\nabla \phi_h, \nabla v_h)$$

Let $\zeta := p_h - \tilde{p}$ and $\overline{\zeta} := P_{Q_H}(p_h - \tilde{p})$. Writing $p_h - \tilde{p} = \overline{\zeta} + (p_h - \tilde{p})'$ gives:

$$\begin{aligned} (\overline{\zeta}, \nabla \cdot v_h) &= (\nabla \phi_h, \nabla v_h) - ((p_h - \tilde{p})', \nabla \cdot v_h) \\ &\leq (||\nabla \phi_h|| + ||(p_h - \tilde{p})'||)||\nabla v_h||. \end{aligned}$$

Since $\overline{\zeta} \in Q_H$, the (X_h, Q_H) inf-sup condition (3.8) immediately implies

$$\begin{aligned} &\beta||\overline{\zeta}|| \leq \sup_{v_h \in X_h} \frac{\overline{(\zeta, \nabla \cdot v_h)}}{||\nabla v_h||} \leq (||\nabla \phi_h|| + ||(p_h - \tilde{p})'||) \\ &\leq \text{from (3.14)} \ \leq (1 + \frac{\epsilon^{-1/2}}{\sqrt{2}})||p'||. \end{aligned}$$

Combining this bound with the estimate (following trivially from (3.14)) $||(\tilde{p}_h - \tilde{p})'|| \leq \frac{1}{2}||p'||$, gives

$$||p - p_h|| \leq C(\beta)(1 + \epsilon^{-1/2})||p'||,$$

and the triangle inequality completes the proof. □

To provide analytic guidance on the choice of ϵ and H with respect to h, it's necessary to consider specific choices of finite element spaces. The most natural choices are low order spaces and two immediately come to mind: linear-linear elements and linear-constant elements.

Example: linear-linear elements.

Let $S^h := \{\phi_h(x) : \phi_h \in C^0(\overline{\Omega}) \text{ and } \phi_h|_K \in \mathcal{P}_1(K) \text{ for all } K \in \Pi^h(\Omega)\}$ denote the usual space of continuous, piecewise linears. Define

$$X_h := \{v : v_j \in S^h\} \cap H_0^1(\Omega), \ Q_h := S^h \cap L_0^2(\Omega). \tag{3.15}$$

This choice is interesting because it can be used for both compressible and incompressible flows. First note that, generically, if $\Pi^h(\Omega)$ is one refinement of $\Pi^H(\Omega)$, then (X_h, Q_H) satisfies the inf-sup condition (3.8). This is well known in the folklore of the field. We record it here to make the association (3.8) concrete.

Lemma 3.3 *Given the mesh $\Pi^H(\Omega)$. Suppose that given any element $K_H \in \Pi^H(\Omega)$, the mesh $\Pi^h(\Omega)$ includes the centroid of K_H as a node and as edges the line segments connecting it to the vertices of K_H. Then, (X_h, Q_H) satisfies the inf-sup condition (3.8). More generally, if for any $K_H \in \Pi^H(\Omega), X_h$ includes any function ϕ_K which is a generalized bubble function in the sense of H3 p. 254 Pierre [Pie89], then (X_h, Q_H) satisfies the inf-sup condition.*

Proof: In the first case the basis function associated with the centroid node can be used to replace the cubic bubble function in the proof in the original paper of Arnold, Brezzi and Fortin [ABF84]. In the second case the result follows Pierre [Pie89] pages 254-257. □

Thus, we can take $H = h/2$ and $||p'|| = ||p - p_H|| \leq Ch^2||p||_2$. The error estimate in the previous theorem then gives

$$||\nabla(u - u_h)|| \leq C(\beta, u, p)\{h + \epsilon^{-1/2}h^2\},$$
$$||p - p_h|| \leq C(\beta, u, p)\{h + (1 + \epsilon^{-1/2})h^2\}.$$

The velocity error is optimal provided $h^2 \leq \epsilon \leq 1$ and the pressure error is also $O(h)$.

Example: Linear-Constant Elements. With X_h being C^0 piecewise linears as in the previous example, let

$$Q_h := \{q_h : q_h|_K \in \mathcal{P}_0(K) \text{ for all } K \in \Pi^h(\Omega)\} \cap L_0^2(\Omega)$$

This choice fails the (X_h, Q_h) inf-sup condition ([GR86]) but is otherwise very natural and simple. The following is known, [GR86] [BF83].

Lemma 3.4 *Let $\Pi^h(\Omega)$ be generated from $\Pi^H(\Omega)$ by one uniform refinement (connecting mid-edges to subdivide each triangle into 4 congruent ones). Then (X_h, Q_H) satisfies the inf-sup condition (3.8).* □

With this choice $h = H/2$ and $||p'|| \leq C(p)h$ and the error estimates of the theorem become

$$||\nabla(u - u^h)|| \leq C(\beta, u, p)\{h + \epsilon^{-1/2}h\}, \text{ and } ||p - p_h|| \leq C(\beta, u, p)(1 + \epsilon^{-1/2})h$$

Thus, the error is optimal for velocity and pressure provided only $\epsilon = O(1)$.

4 Connections and Conclusions

The overarching goal in the numerical analysis of turbulent flow is to obtain more accurate approximations of the large scales by having a more accurate representation of the mean effects of the small scales upon the large scales. One approach is the variational multiscale method in which a simple approximation of u' is used to obtain a better approximation of $\overline{u}$. A related idea herein would be to use the constrained formulation of Section 2 discretized by a simple approximation of λ_1 and λ_2. We have explored instead simple regularizations of the constraints associated with λ_1 and λ_2. These simple regularizations lead to interesting new algorithms even for the incompressibility constraint in the Stokes problem.

There is a huge volume of work studying related regularization/stabilization methods for the Stokes problem's incompressibility constraint. For high enough order elements, such regularizations are not necessary. High order elements also usually suffer from less grid orientation sensitivity than low order elements. On the other hand, for nonlinear problems with sensitive solutions, so-called, grid-convergence is elusive and there is always a need for "one more refinement". For such problems low order elements will always be useful. For low order elements, stabilization has proven to be an essential tool.

There are (at least) three general types of stabilizations: bubble function augmentation of velocity spaces, (ii) weighted least squares addition of local strong residuals, and (iii) stabilization by perturbation of $\nabla^h \cdot u^h = 0$, and many connections have been explored in the cited literature among these approaches. In Section 3, we consider the induced stabilization which is of the third type.

We show that stabilization of the incompressibility is only needed for the smallest resolved scales in p_h:

$$(\nabla \cdot u_h, q_h) + \epsilon(p_h', q_h') = 0. \tag{4.1}$$

In contrast, artificial compression regularization

$$(\nabla \cdot u_h, q_h) + \epsilon(p_h, q_h) = 0 \tag{4.2}$$

stabilizes all scales and commits a significant consistency error. Brezzi-Pitkaranta type regularization is similar but overweights the smallest of resolved scales by using first derivative information

$$(\nabla \cdot u_h, q_h) + \epsilon(\nabla p_h, \nabla q_h) = 0. \tag{4.3}$$

By this overweighting, it has a smaller consistency error than (4.2) (and hence greater accuracy) but greater than (4.1).

The first time the idea occurs of stabilization of only the finest resolved pressure scales is in the recent and very interesting paper of Becker and Braak [BB01], who study stabilizations of the form

$$(\nabla \cdot u_h, q_h) + \sum_{\Delta_H \in \Pi^H(\Omega)} \text{diam } (\Delta_H)^2 (\nabla p_h - P_H(\nabla p_h), \nabla q_h)_{\Delta_H} = 0.$$

where Q_h is the space of conforming linears and P_H is the projection into piecewise constants on $\Pi^H(\Omega)$. This can be interpreted as a fine scale stabilization of Brezzi-Pitkananta type; Becker and Braak show that it improves the accuracy of the pressure on structured meshes and reduces errors near walls on unstructured meshes.

The general idea of using multiscale stabilizations for discrete models of physical processes is one which is certainly worthy of further study.

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