Numerical Analysis of a Subgrid Scale Eddy Viscosity Method for Higher Reynolds Number Flow Problems *

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Abstract

We develop a variationally consistent eddy viscosity model, given by [18] for convection diffusion equation, for the Navier-Stokes equations. We also prove that the new model is equivalent to a variational multiscale method. The new method removes the restriction that the fluctuations are quasi-stationary. We then give a complete error analysis. The error analysis is optimal and uniform in Reynolds number in all terms except for those arising from the reaction term in the linearized error equation. In particular, we show that error in the method for the Oseen problem is uniform in the Reynolds number.

1 Introduction

This report gives a numerical analysis of a special subgrid scale eddy viscosity method/model for the Navier-Stokes equations at higher Reynolds number.

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In this method, variationally consistent eddy viscosity is introduced acting only on the discrete fluctuations. This technique is inspired by earlier work in Guermond [9], Hughes [12], Layton [18]. It can also be thought of as an extension to general domains and boundary conditions of the spectral vanishing viscosity idea of Mayday and Tadmor [22]. Specifically, these new method insert eddy viscosity acting only on the smallest resolved mesh scales. We will also show that the new method is equivalent to a variational multiscale method.

Consider the incompressible, viscous Navier-Stokes equations

$$\begin{aligned} u_t + \nabla \cdot (uu) + \nabla p - \nu \Delta u = f, \text{ in } \Omega, \text{ for } 0 < t \leq T, \\ \nabla \cdot u = 0, \text{ in } \Omega, \text{ for } 0 < t \leq T, \\ u(x,0) = u_0(x), \text{ in } \Omega, u = 0 \text{ on } \Gamma, \text{ for } 0 < t \leq T, \end{aligned} \tag{1.1}$$

where u is the fluid velocity, p is the pressure, f is the external force, ν is the kinematic viscosity, and $\Omega \subset \mathbb{R}^d (d = 2$ or $d = 3)$ is a bounded, simply connected domain with polygonal boundary Γ. We also impose the usual normalization condition that $\int_\Omega p dx = 0$.

In (1.1) the control parameter is the Reynolds number, $\nu \sim \frac{1}{Re}$. As Re increases, the flow becomes more sensitive to perturbations, more complex in structure and eventually turbulent. Since many flows in nature occur at high Reynolds number, this report considers the approximate solution of (1.1) for small ν.

The idea we consider consists of stabilizing the discrete equations by adding eddy viscosity in a variationally consistent way. To motivate this approach, consider a first variational formulation of (1.1) in

$$\begin{aligned} X &:= \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{dxd} \text{ and } v = 0 \text{ on } \Gamma\}, \\ L &:= \{S_{ij} \in L^2(\Omega)^{dxd} : S_{ij} = S_{ji}\}, \\ Q &:= L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}. \end{aligned}$$

One nonstandard variational formulation of (1.1) is: find $u : [0,T] \to X$, $p : (0,T] \to Q$ and $g : [0,T] \to L$ satisfying

$$\begin{aligned} &(u_t, v) + b(u,u,v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) + ((2\nu + \nu_T)\nabla^s u, \nabla^s v) \\ &-(\nu_T g, \nabla^s v) = (f,v), \text{ for all } v \in X, q \in Q \end{aligned} \tag{1.2}$$

$$(g - \nabla^s u, l) = 0, \text{ for all } l \in L \tag{1.3}$$

where $(.,.)$ denotes the $L^2(\Omega)$ inner product, $b(u,u,v): X \times X \times X :\to \mathbb{R}$

$$b(u,v,w) = \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v) \tag{1.4}$$

denotes the skew symmetrized trilinear form, ν_T denotes the eddy viscosity parameter. We will prove that (1.2), (1.3) and the usual formulation of (1.1) are equivalent to each other by verification of an inf-sup condition associated with (1.3), (Lemma 2.1).

Next, this report considers finite element discretization of the Navier-Stokes equation based on (1.2), (1.3) as follows. Construct a coarse finite element mesh $\Pi^H(\Omega)$ and a fine mesh $\Pi^h(\Omega)$, where $h << H$ typically. Conforming velocity-pressure finite element spaces are constructed based upon $\Pi^h(\Omega)$ and $\Pi^H(\Omega)$. Let h, H be two mesh widths with $H > h$ and let $X^h \subset X$, $X^H \subset X$, $Q^h \subset Q$ and $L^H \subset L$ be finite element spaces. Consider the approximation (u^h, p^h, g^H) based on the variational formulation (1.2), (1.3) : find $u^h \in X^h, p^h \in Q^h, g^H \in L^H$ satisfying

$$\begin{aligned}(u_t^h, v^h) + b(u^h, u^h, v^h) - (p^h, \nabla \cdot v^h) + (q^h, \nabla \cdot u^h) \\ + ((2\nu + \nu_T)\nabla^s u^h, \nabla^s v^h) - (\boldsymbol{\nu_T g^H}, \boldsymbol{\nabla^s v^h}) = (f, v^h)\end{aligned} \tag{1.5}$$

for all $v^h \in X^h$ and $q^h \in Q^h$ where $\boldsymbol{g^H} \in L^H \subset L^2(\Omega)$ is defined by

$$(\boldsymbol{g^H} - \boldsymbol{\nabla^s u^h}, l^H) = 0, \text{for all } l^H \in L^H. \tag{1.6}$$

Different choices of L^H give rise to different methods. The choice of $L^H = \{0\}$ gives the usual artificial viscosity methods. On the other hand, when $\nabla X^H \subset L^H$, (1.5),(1.6) gives the usual (centered) finite element method.

The effect of the extra (bold term) terms in (1.5) can be seen by using (1.6), to eliminate $\boldsymbol{g^H}$ from (1.5) as follows. (1.6) implies that

$$\boldsymbol{g^H} = P_{L^H} \boldsymbol{\nabla^s u^h}$$

where $P_{L^H} : L \to L^H$ denotes the usual L^2 orthogonal projection. Insertion of this into (1.5) and simplification using properties of orthogonal projection shows that u^h satisfies:

$$\begin{aligned}(u_t^h, v^h) + b(u^h, u^h, v^h) - (p^h, \nabla \cdot v^h) + (q^h, \nabla \cdot u^h) + (2\nu \nabla^s u^h, \nabla^s v^h) \\ + (\nu_T (I - P_{L^H})(\boldsymbol{\nabla^s u^h}), (I - P_{L^H})(\boldsymbol{\nabla^s v^h})) = (f, v^h),\end{aligned} \tag{1.7}$$

for all $v^h \in X^h$ and $q^h \in Q^h$. The formulation (1.5),(1.6) thus introduces implicitly the extra stabilization

$$(\nu_T(I - P_{L^H})(\nabla^s \boldsymbol{u^h}), (I - P_{L^H})(\nabla^s \boldsymbol{v^h})). \tag{1.8}$$

This is an eddy viscosity term acting on the scales between H and h, i.e.,the small scales. The normal use of eddy viscosity (EV) models is in high Re/turbulent problems (see e.g., Smagorinsky [23], Lewandowski [20], Iliescu and Layton [15]) and thus we are specifically interested in the numerical analysis of (1.5),(1.6) for higher Re /smaller ν. Normal EV models have often been noted to be too diffusive and over damp in the large flow structures (large eddies). Indeed, the physical interpretation of EV models is that they model energy lost around the cut off lengthscale due to inertial effects (eddy breakdown to below the meshwidth size). Normal EV, however, removes energy strongly from *all* resolved scales. Thus, the method (1.5),(1.6) is of special interest because (*i*) the eddy viscosity does not act on the large structures and (*ii*) it is introduced in a variationally consistent manner. Another consistent stabilization technique is the variational multiscale method (VMM hereafter) of Hughes and co-workers [11](see also [3, 12, 14, 13, 16]). In the VMM, the original problem is decomposed into two subproblems for the small scales and the large scales. This coupled system is then discretized. This report will give a complete and systematic analysis of the finite element method (1.2),(1.3) for small ν and establish a connection between new method and variational multiscale methods.

The general idea of using two-grid discretizations to increase the *efficiency* of methods was pioneered by J. Xu (see, e.g., Marion and Xu [21]) and developed by Girault and Lions [7], [6]. This plus the physical ideas underlying eddy viscosity models and previous work [4] on stabilizations in viscoelasticity lead very naturally to the present method.

In Section 2, we present the notation and recall some basic results used throughout this paper. In Section 3, we describe a natural extension of Hughes' variational multiscale method and prove equivalence between generalized variational multiscale methods and the new method (1.5),(1.6). This connection is known for linear convection diffusion problems, [16]; we show in Section 3, it holds in the more interesting Navier-Stokes case as well. Section 4 gives a fully mathematically rigorous analysis of the method, proving convergence to solutions of (1.5),(1.6). In Section 5, we show the error in the new method for Oseen problem is uniformly in Reynolds number.

2 Notation and Preliminaries

We use standard notation for Sobolev spaces Adams [1]. The $L^2(\Omega)$ norm and inner product of scalar, vector and tensor valued functions will be denoted by $\|.\|$ and $(.,.)$. $H^k(\Omega), k > 0$, denotes the Sobolev space of real-valued functions with square integrable derivatives up to order k equipped with usual norm which we denote by $\|.\|_k$. Since the case of scalar, vector or tensor functions will be clear from the context, we will not distinguish between these cases in the notation of $H^k(\Omega)$. The $H^1(\Omega)$, norm is defined by $\|v\|_1 = \sqrt{\|v\|^2 + \|\nabla v\|^2}$ and is equivalent to $\|\nabla v\|$ and $\|\nabla^s v\|$. The norm of dual space $H^{-k}(\Omega) = (H^k(\Omega) \cap H_0^1(\Omega))^*$ is defined by

$$\|f\|_{-k} = \sup_{v \in (H^k(\Omega)\cap H_0^1(\Omega))^d} \frac{|(f,v)_{L^2}|}{\|v\|_{k,\Omega}}.$$

As usual, V denotes the space of divergence free functions

$$V := \{v \in X : (\nabla \cdot v, q) = 0, \text{for all } q \in Q\}.$$

We assume that the velocity-pressure finite element spaces $X^h \subset X, Q^h \subset Q$ satisfy the discrete inf-sup or Babuska-Brezzi condition:

$$\inf_{q^h \in Q^h} \sup_{v^h \in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|q\|^h \|\nabla v^h\|} \geq \beta > 0. \tag{2.1}$$

This condition is now well understood and numerous example of attractive finite element spaces satisfying (2.1) exist, e.g.,Gunzburger [10], Girault and Raviart [8]. Recall that under this condition, the space of discretely divergence -free functions V^h,

$$V^h := \left\{v^h \in X^h : (\nabla \cdot v^h, q^h) = 0, \text{for all } q^h \in Q^h\right\} \tag{2.2}$$

is well-defined and the natural formulations of the discrete Stokes problem in (X^h, Q^h) and V^h are equivalent. Further, if (2.1) holds and $u \in V$ then (see Girault and Raviart [8]),

$$\inf_{v^h \in V^h} \left\|\nabla(u - v^h)\right\| \leq C(\beta) \inf_{v^h \in X^h} \left\|\nabla(u - v^h)\right\|.$$

Since (1.3) naturally occurs as a constraint associated with (1.2), well-posedness of the continuous reformulation depends upon another inf-sup condition, which we verify next.

Lemma 2.1. *(Inf-Sup Condition) The formulation (1.2),(1.3) satisfies following inf-sup condition:*

$$\inf_{l\in L}\sup_{(g,v)\in(L,X)}\frac{|\ (\nabla v-g,l)\ |}{(\|g\|^2+\|\nabla v\|^2)^{\frac{1}{2}}\,\|l\|}\geq 1 \tag{2.3}$$

Proof. Picking v=0 and g=l establishes the required inequality. □

Corollary 2.1. *The continuous problem (1.2), (1.3) is equivalent to the usual variational formulation of the Navier-Stokes equations in (X, Q).*

We assume that the following approximation assumption, typical of piecewise polynomial velocity-pressure finite element spaces of degree (k, k-1) holds : there is $k \geq 1$ such that for any $u \in (H^{k+1}(\Omega))^d \cap X$ and $p \in (H^k(\Omega) \cap Q)$:

$$\inf_{v^h\in X^h}\left\{\left\|u-v^h\right\|+h\left\|\nabla(u-v^h)\right\|\right\}\leq Ch^{k+1}\left\|u\right\|_{k+1},$$

$$\inf_{q^h\in Q^h}\left\|p-q^h\right\|\leq Ch^k\left\|p\right\|_k.$$

Note that skew symmetrized trilinear form $b(u, u, v)$ introduced in (1.4) has the following properties:

$$b(u,v,w)=-b(u,w,v)\text{ and }b(u,v,v)=0\text{ for all }\ u,v,w\in X. \tag{2.4}$$

We will use the following lemma in Section 4.

Lemma 2.2. *Let $\Omega \subset R^3$,i.e.,$d = 3$.*

$$b(u,v,w)\leq C(\Omega)\sqrt{\|u\|\,\|\nabla^s u\|}\,\|\nabla^s v\|\,\|\nabla^s w\|.$$

Proof. By Lemma 2.1 p.12 of Temam [24]

$$b(u,v,w)\leq C(\Omega)\left\|u\right\|_{1/2}\left\|v\right\|_1\left\|w\right\|_1.$$

The Poincaré-Friedrichs and Korn's inequality implies

$$\left\|v\right\|_1\leq C\left\|\nabla^s v\right\|,\ \left\|w\right\|_1\leq C\left\|\nabla^s w\right\|.$$

Lastly, an interpolation inequality implies

$$\left\|u\right\|_{1/2}\leq C(\Omega)\left\|u\right\|^{1/2}\left\|u\right\|_1^{1/2}\leq C(\Omega)\left\|u\right\|^{1/2}\left\|\nabla^s u\right\|.$$

□

For the error analysis, we assume regularity in time $\|\nabla u\| \in L^4(0,T)$ i.e. $\int_0^T \|\nabla u\|^4 \, dt < \infty$. With this assumption, it is known that the usual Leray [19] weak solution of the Navier-Stokes equations is unique, Ladyzhenskaya [17].

It will be important to introduce the notion of means/large scales and fluctuations/small scales that we use. We shall use over-bar and prime notation to denote large and small scales, respectively. If $X^H \subset X^h$, we can decompose $u^h \in X^h$ into discrete means and fine mesh fluctuations via

$$u^h = u^H + u'_h, u^H \in X^H \, u' \in X'_h, \tag{2.5}$$

where X^H, X'_h denote the coarse mesh space and small scale space. The space for fluctuations X'_h and the decomposition (2.5) are determined by specifying how $u^H \in X^H$ is determined from u^h.

Definition 2.1. *(Elliptic Projection) $P_E : X \to X^H$ is the projection operator satisfying*

$$(\nabla^s(u - P_E u), \nabla^s v^H) = 0 \text{ for all } v^H \in X^H. \tag{2.6}$$

Definition 2.2. *(Stokes Projection) P_S discrete Stokes projection operator $P_S(u,p) = (u^H, p^H)$ is defined by $(u^H, p^H) \in (X^H, Q^H)$ satisfying*

$$\begin{aligned} (\nabla^s(u - u^H), \nabla^s v^H) - (p - p^H, \nabla \cdot v^H) = 0, \text{ for all } v^H \in X^H \\ (\nabla \cdot (u - u^H), q^H) = 0, \text{ for all } q^H \in Q^H. \end{aligned} \tag{2.7}$$

If $u \in V$ (i.e.$\nabla \cdot u = 0$), this is equivalent to the following: $u^H \in V^H$ satisfies

$$(\nabla^s(u - u^H), \nabla^s v^H) - (p - q^H, \nabla \cdot v^H) = 0, \text{ for all } v^H \in V^H.$$

In Section 3, to present new method (1.5),(1.6) in a variational multiscale framework, we consider the decomposition of deformation tensor as in . Let the spaces $L^H \subset L$, $X^H \subset X^h \subset X$ as defined in Section 2. For the multiscale decomposition of the deformation tensor define $\nabla^s u^h \mathbf{D}^h$ write

$$\mathbf{D}^h = \nabla^s u^h, \text{ for all } u^h \in X^h,$$

and split $\mathbf{D}^h = \mathbf{D}^H + \mathbf{D}^h$ where $\mathbf{D}^H = P_{L^H}\mathbf{D}^h$ and $\mathbf{D}' = (I - P_{L^H})\mathbf{D}^h$.

Lemma 2.3. *Assume that $X^H \subset X^h$ and $L^H = \nabla X^H$. Then,*

$$\boldsymbol{D}^h = \nabla^s(I - P_E)u^h \text{ and } P_{L^H}\boldsymbol{D}^h = \nabla^s P_E u^h$$

where $P_E = \bar{P}$ is the elliptic projection operator into X^H.

Proof. From $\mathbf{D}^H = P_{L^H}\mathbf{D}^h$, it follows that

$$(\mathbf{D}^h - \mathbf{D}^H, l^H) = 0, \text{ for all } l^H \in L^H.$$

Also, $\mathbf{D}^H \in L^H \subset L$ and $L^H = \nabla^s X^H$. Thus, $\mathbf{D}^H = \nabla^s w^H$ for some $w^H \in X^H$. This yields:

$$(\nabla^s(u^h - w^H), \nabla^s v^h) = 0 \text{ for all } v^h \in X^H. \tag{2.8}$$

Then, consider the elliptic projector operator $P_E : X \to X^H$. From this map, define $P_E u^h = w^H$. Then, (2.8) can be written as:

$$(\nabla^s(u^h - P_E u^h), \nabla^s v^h) = 0 \text{ for all } v^h \in X^H.$$

This implies that $P_{L^H}\mathbf{D}^h = \nabla^s P_E u^h$ and $(I - P_{L^H})\mathbf{D}^h = \nabla^s(I - P_E)u^h$. $\square$

We shall assume that the finite dimensional space $L^H = \nabla X^H$ satisfies an inverse-type inequality of the form. Then, from Lemma 2.3

$$\left\| P_{L^H}\nabla\phi^h \right\| = \left\| \nabla (P_E)_{X^H} \right\| \phi^h \leq CH^{-1} \left\| (P_E)_{X^H}\phi^h \right\| \leq CH^{-1} \left\| \phi^h \right\|. \tag{2.9}$$

where $\phi^h \in V^h$ and $P_{L^H},\ P_E$ is the L^2 orthogonal projection, elliptic projection, respectively.

3 The Connection to Variational Multiscale Methods

The Variational Multiscale Method (VMM) is a finite element method for multiscale problem introduced by Hughes [12], which simultaneously discretizes coupled systems of both the large and small scales. The usual 1-scale semi discrete variational formulation of (1.1) is : find $u : [0,T] \to X$, $p : (0,T] \to Q$ satisfying

$$\begin{aligned} &(u_t, v) + b(u,u,v) + (2\nu\nabla^s u, \nabla^s v) - (p, \nabla\cdot v) = (f,v), \text{for all}\, v \in X, \\ &(q, \nabla\cdot u) = 0, \text{ for all } q \in Q. \end{aligned} \tag{3.1}$$

The usual 1-scale semi-discrete finite element method for (3.1) arises by choosing appropriate finite dimensional subspaces $X^h \subset X$ and $Q^h \subset Q$ and determining u^h and p^h satisfying (3.1) restricted to X^h, Q^h. In turbulent flows the effects of small scales velocity in $X \setminus X^h$ upon the large velocity $u^h \in X^h$ are widely thought to be critical for a realistic simulation. Thus, Hughes' the variational multiscale method seeks to incorporate as follows. Following Hughes [12], the velocity and pressure spaces are decomposed into means and fluctuations

$$X = \bar{X} + X' \text{ and } Q = \bar{Q} + Q' \tag{3.2}$$

where $\bar{X} \subset X,\ \bar{Q} \subset Q$ denote closed subspaces of large velocity and pressure scales. In this decomposition as noted Hughes, $\bar{X}$ is chosen as a standard finite element space i.e. X^h. Associated with $\bar{X},\ \bar{Q}$ is a projection operator: $\bar{P} : (X, Q) \to (\bar{X}, \bar{Q})$. The space of fluctuations (X', Q') is the complement of $\bar{X},\ \bar{Q}$ in $X,\ Q$, respectively. Then $(X', Q') := P'(X, Q)$ where $P' = (I - \bar{P})$. By using (3.4), u and p defined as:

$$u = \bar{u} + u' \text{ and } p = \bar{p} + p', \text{ where } (\bar{u}, \bar{p}) = \bar{P}(u, p),\ (u', p') = P'(u, p). \tag{3.3}$$

Inserting this decomposition into (3.1) and setting first $(v, q) = (\bar{v}, \bar{q})$ then $(v, q) = (v', q')$ gives the following coupled equations for $\bar{u}$ and u' :

$$\begin{aligned} (\bar{u}_t + u'_t, \bar{v}) + b(\bar{u} + u', \bar{u} + u', \bar{v}) - (2\nu\nabla^s(\bar{u} + u'), \nabla^s\bar{v}) \\ - (\bar{p} + p', \nabla \cdot \bar{v}) = (\bar{f} + f', \bar{v}) \text{ for all } \bar{v} \in \bar{X} \\ (\bar{u}_t + u'_t, v') + b(\bar{u} + u', \bar{u} + u', v') - (2\nu\nabla^s(\bar{u} + u'), \nabla^s v') \\ - (\bar{p} + p', \nabla \cdot v') = (\bar{f} + f', v') \text{ for all } v' \in X'. \end{aligned}$$

After rearranging, these coupled system yields

$$\begin{aligned} (\bar{u}_t, \bar{v}) + b(\bar{u}, \bar{u}, \bar{v}) - (2\nu\nabla^s\bar{u}, \nabla^s\bar{v}) - (\bar{p}, \nabla \cdot \bar{v}) - (\bar{f}, \bar{v}) \\ = (f', \bar{v}) - a(u', \bar{v}) - (p', \nabla \cdot \bar{v}) \end{aligned} \tag{3.4}$$

where

$$a(u', \bar{v}) = (u'_t, \bar{v}) + b(u', u', \bar{v}) + b(\bar{u}, u', \bar{v}) + b(u', \bar{u}, \bar{v}) + (2\nu\nabla^s u', \nabla^s\bar{v})$$

and

$$\begin{aligned} (u'_t, v') + b(u', u', v') - (2\nu\nabla^s u', \nabla^s v') - (p', \nabla \cdot v') - (f', v') \\ = (\bar{f}, v') - a(\bar{u}, v') - (\bar{p}, \nabla \cdot v') \end{aligned} \tag{3.5}$$

where

$$a(\bar{u}, v') = (\bar{u}_t, v') + b(\bar{u}, \bar{u}, v') + b(u', \bar{u}, v') + b(\bar{u}, u', v') + (2\nu \nabla^s \bar{u}, \nabla^s v').$$

A variational multiscale discretization from (3.4),(3.5) begins by selecting a finite dimensional subspace $\bar{X} := X^h$ for the approximate mean flow $\bar{u} \in \bar{X}, \bar{p} \in \bar{Q}$ and finite dimensional spaces X'_h, Q'_h for the fluctuation. Note that ideally $X'_h \subset X'$ but usually $X'_h \subsetneq X'$.

Definition 3.1. *(Hughes et al., [12]) The VMM approximation to (1.1) is a pair* $(\bar{u}, \bar{p}) : [0, T] \to (\bar{X}, \bar{Q})$ *and* $(u'_h, p'_h) : [0, T] \to (X'_h, Q'_h)$ *satisfying* $\bar{u}(0) = \bar{P}u_0, u'_h(0) = P'u_0$ *and*

$$\begin{aligned}&(\bar{u}_t, \bar{v}) + b(\bar{u}, \bar{u}, \bar{v}) + (2\nu \nabla^s \bar{u}, \nabla^s \bar{v}) - (\bar{p}, \nabla \cdot \bar{v})\\ &+ (\bar{q}, \nabla \cdot \bar{u}) - (\bar{f}, \bar{v}) = (f', \bar{v}) - a(u'_h, \bar{v}) + (p'_h, \nabla \cdot \bar{v'_h}),\end{aligned} \tag{3.6}$$

where

$$a(u'_h, \bar{v}) = (u'_{h,t}, \bar{v}) + b(u'_h, u'_h, \bar{v}) + b(\bar{u}, u'_h, \bar{v}) + b(u'_h, \bar{u}, \bar{v}) + (2\nu \nabla^s u'_h, \nabla^s \bar{v}),$$

for all $\bar{v} \in \bar{X},\ \bar{q} \in \bar{Q}$ *and*

$$\begin{aligned}&(u'_{h,t}, v'_h) + b(u'_h, u'_h, v'_h) - (2\nu \nabla^s u'_h, \nabla^s v'_h) - (p', \nabla \cdot v') + (q'_h, \nabla \cdot u'_h)\\ &+ (\nu_T \nabla^s u'_h, \nabla^s v'_h) - (f', v') = (\bar{f}, v'_h) - a(\bar{u}, v'_h) + (\bar{p}, \nabla \cdot v'_h)\end{aligned} \tag{3.7}$$

where

$$a(\bar{u}, v'_h) = (\bar{u}_t, v'_h) + b(\bar{u}, \bar{u}, v'_h) + b(u'_h, \bar{u}, v'_h) + b(\bar{u}, u'_h, v'_h) + (2\nu \nabla^s \bar{u}, \nabla^s v'_h)$$

for all $v'_h \in X'_h$, $q'_h \in Q'_h$ *and where* ν_T *is the turbulent viscosity coefficient in a small scale artificial viscosity type stabilization term.*

Remark 3.1. *The work of Hughes (and co-workers) [12] has explored the choices*

- *the projection operator* $\bar{P}$ *as* L^2 *projection,*
- $(\bar{X}, \bar{Q})$ = *standard velocity-pressure finite element spaces,* (X'_h, Q'_h) = *spaces of bubble functions vanishing on element edges,*
- $\nu_T = (\mu H)^2 \mid \nabla^s u'_h \mid$, *a Smagorinsky eddy viscosity term.*

Remark 3.2. *Different choices of X'_h have been explored. In the case of periodic boundary conditions, X'_h is often constructed as the span of the next few exponentials [13]. For bounded domains,(motivated by Residual Free Bubble(RFB) [3] theory) the choices $(\bar{X}, \bar{Q})$ = standard velocity-pressure finite element spaces, (X'_h, Q'_h) = spaces of bubble functions vanishing on element edges have been explored. One important question which is actively being investigated is how rich the space X'_h must be.*

The VMM has been developed and connected with the other popular methods. For the practical approximation the choice of bubble functions to model fluctuations imposes the following assumption: *the small scales exist only in the interior of element boundaries of the element domains.* This assumption leads to localizing calculations for the small scales in the sense that the problems are elementwise uncoupled. This choice of (X'_h, Q'_h), is made for the same reason as in RFB methods (Hughes [12]). Thus with VMMs, the small scales' equation can be solved element by element and inserted into large scales' equation. The introduced terms then closely represent the effect of the modelled small scales on the large scales.

Definition 3.2. *(Generalized VMM) A generalized VMM is determined by alternate choices of*

- *projection operator $\bar{P}$ defining large scales,*
- *the discrete spaces $\bar{X}$, $\bar{Q}$, X'_h, Q'_h where X'_h, Q'_h are the complements of $\bar{X}$, $\bar{Q}$, respectively,*
- *the small scale stabilization term*

  $$(\nu_T \nabla^s u'_h, \nabla^s v'_h)$$

  is added on the right hand side of (3.7)

Subgrid Eddy Viscosity and VMM

One result of (3.1) is that (1.5),(1.6) is a suitably generalized VMM in the sense of Definition 3.2. To present this we use the Lemma 2.3.

Remark 3.3. *Lemma 2.3 shows that the natural definition of means of $\nabla^s u$ is by L^2 projection and means of u is by elliptic projection.*

Remark 3.4. *The VMM discretization of the incompressibility constraint is:*

$$\begin{aligned}(\nabla \cdot (u^H + u_h'), v^H) &= 0, \text{ for all } v^H \in X^H \\ (\nabla \cdot (u^H + u_h'), v_h') &= 0, \text{ for all } v_h' \in X_h'.\end{aligned} \tag{3.8}$$

At this point an algorithmic choice must be made. If $u^h = u^H + u_h'$ is to be discretely div-free with respect to Q^h then (3.8) is imposed (as stated) as a coupled system. In this case u^H may not be discretely div-free with respect to Q^H. This is the choice we make herein; it leads to the use of the elliptic projection operator P_E. If u^H is to be discretely div-free with respect to Q^H, then (3.8) uncouples and u_h' will not (in general) be discretely div-free with respect to Q_h' nor will u^h be discretely div-free with respect to Q^h. With this choice P_E be should be replaced in our error analysis by the discrete Stokes projection P_S. This issue does not arise in spectral discretization of periodic problem since spectral basis functions are chosen to be exactly div-free.

Theorem 3.1. *Assume that $X^H \subset X^h$ and $L^H = \nabla X^H$. Then (1.5),(1.6) is a generalized VMM wherein:*

- *the projector operator $P_E : X^h \to X^H$ a elliptic projection operator function. Thus, $u^H = P_E u^h$.*
- *X_h', is the complement of X^H, in X^h ,respectively, given by $X_h' = (I - P_E)X^h$. Thus, $u_h' = (I - P_E)u^h$.*
- *the stabilization term is given by $(\nu_T(\nabla^s u^h)', (\nabla^s v^h)')$.*

Proof. From Lemma 2.3, the extra term in equation (1.8) can be written as

$$(\nu_T(I - P_{L^H})(\boldsymbol{\nabla^s u^h}), (I - P_{L^H})(\boldsymbol{\nabla^s v^h})) = (\nu_T(\nabla^s u^h)', (\nabla^s v^h)').$$

Thus, the new method is equivalent to: find $(u^h, p^h, g^H) \in (X^h, Q^h, L^H)$ satisfying

$$(u_t^h, v^h) + b(u^h, u^h, v^h) - (p^h, \nabla \cdot v^h) + (q^h, \nabla \cdot u^h) + (2\nu \nabla^s u^h, \nabla^s v^h) + (\nu_T (\nabla^s u^h)', (\nabla^s v^h)') = (f, v^h) \tag{3.9}$$

for all $v^h \in X^h$ and $q^h \in Q^h$. In equation (3.9) write $u^h = u^H + u_h'$, and set alternately, $v^h = v^H$ and $v^h = v_h'$. Similarly, as in (3.6), (3.7) this gives the following coupled equations:

$$(u_t^H, v^H) + b(u^H, u^H, v^H) + (2\nu \nabla^s u^H, \nabla^s v^H) - (p^H, \nabla \cdot v^H) + (q^H, \nabla \cdot u^H) - (f^H, v^H) = (f', v^H) - a(u_h', v^H) \text{ for all } v^H \in X^H,\ q^H \in Q^H, \tag{3.10}$$

where

$$a(u_h', v^H) = (u_{h,t}', v^H) + b(u_h', u_h', v^H) + b(u^H, u_h', v^H) + b(u_h', u^H, v^H) - (p_h', \nabla \cdot v^H) + (2\nu \nabla^s u_h', \nabla^s v^H),$$

and

$$(u_{h,t}', v_h') + b(u_h', u_h', v_h') - (2\nu \nabla^s u_h', \nabla^s v_h') - (p', \nabla \cdot v') + (q_h', \nabla \cdot u_h') + (\nu_T \nabla^s u_h', \nabla^s v_h') - (f', v') = (f^H, v_h') - a(u^H, v_h') \text{ for all } v_h' \in X_h',\ q_h' \in Q_h', \tag{3.11}$$

where

$$a(u^H, v_h') = (u_t^H, v_h') + b(u^H, u^H, v_h') + b(u_h', u^H, v_h') + b(u^H, u_h', v_h') - (p^H, \nabla \cdot v_h') + (2\nu \nabla^s u^H, \nabla^s v_h')$$

□

As noted before, the Hughes' VMM uses the assumption that fluctuations vanish identically on the boundaries of the element. This is clearly not viable. Fortunately, this new method (1.5),(1.6), fluctuations allows to be nonzero across boundaries. Indeed, let choose span of linear basis function with vertex in Π^H, for coarse mesh Π^H and span of linear basis function with vertex in Π^h not in Π^H for the complement of X^H i.e. for the fluctuations. It is clear that with this new VMM's approach, fluctuations can be **nonzero across mesh edges**.

4 Convergence Analysis

The first important question in the method (1.5),(1.6) is how the two scales and the eddy viscosity parameter h, H, ν_T should be coupled. The second important question in the method is the dependence of the error upon ν (i.e., Reynolds number). To answer the first question, this section considers the questions of stability, consistency and convergence of the method. We show convergence of the usual semi-discrete finite element approximation of the model as the mesh widths $h, H \to 0$ and give an estimate of the error. This error estimate in Theorem 4.1 then reveals the correct coupling between h, H, ν_T. The semi-discrete approximation is a map $u^h : [0,T] \to V^h$ satisfying

$$\begin{aligned}&(u_t^h, v^h) + b(u^h, u^h, v^h) + (2\nu \nabla^s u^h, \nabla^s v^h) \\ &\quad + \nu_T((I - P_{L^H})\nabla^s u^h, (I - P_{L^H})\nabla^s v^h) = (f, v^h), \text{ for all } v^h \in V^h \end{aligned} \tag{4.1}$$

where V^h is the space of discretely divergence-free functions given by (2.2).

Proposition 4.1. *[Stability of method (1.5),(1.6)] The approximate solution of u^h of (1.5- 1.6) is stable. For any $t > 0$,*

$$\begin{aligned}&\frac{1}{2}\left\|u^h(t)\right\|^2 + \int_0^t (2\nu \left\|\nabla^s u^h\right\|^2 + \nu_T \left\|(I - P_{L^H})\nabla^s u^h\right\|^2) dt' \\ &\qquad \le \frac{1}{2}\left\|u^h(0)\right\|^2 + \int_0^t (f, u^h) dt'.\end{aligned} \tag{4.2}$$

In particular, $\sup_{0 \le t \le T} \left\|u^h\right\| \le C(f, u_0)$.

Proof. Set $u^h = v^h$ in (4.1), use triangle inequality gives:

$$\frac{1}{2}\frac{d}{dt}\left\|u^h\right\|^2 + 2\nu \left\|\nabla^s u^h\right\|^2 + \nu_T \left\|(I - P_{L^H})\nabla^s u^h\right\|^2 \le (f, u^h).$$

The result follows from integrating over$[0, t]$. Stability then follows by applying Cauchy-Schwarz inequality on the right hand side. □

Corollary 4.1. *u^h exist and is unique. If the discrete inf-sup condition (2.1) holds then p^h exists and is unique.*

We note that by adding and subtracting terms, it is easy to see that the true solution (u, p) satisfies

$$
\begin{aligned}
&(u_t, v^h) + b(u, u, v^h) - (p, \nabla \cdot v^h) \\
&+ (2\nu \nabla^s u, \nabla^s v^h) + (\nu_T (I - P_{L^H})\nabla^s u, (I - P_{L^H})\nabla^s v^h) \\
&= (f, v^h) + (\nu_T (I - P_{L^H})\nabla^s u, (I - P_{L^H})\nabla^s v^h) \text{ for all } v^h \in X^h
\end{aligned}
\tag{4.3}
$$

Theorem 4.1. *(Convergence) Suppose* $\nabla u \in L^4(0, T; L^2(\Omega))$. *Then with* $C = C(\Omega)$, *there is a constant* $C^*(T) = \exp(\int_0^T (7 + C\nu^{-3}) \|\nabla u\|^4) dt')$ *such that*

$$
\max_{0 \le t \le T} \|u - u^h\|^2 + C^{**}C(T) \int_0^T [\nu \|\nabla^s (u - u^h)^2\| + \nu_T \|(I - P_{L^H})\nabla^s (u - u^h)^2\|] dt'
$$

$$
\le CC^*(T) \|u(x, 0) - v^h(x, 0)\|^2 + C_1 C^*(T) \inf_{v^h \in X^h} \Big(\int_0^T [(H^{-2} + \nu_T^{-1}) \|(u - v^h)_t\|_{-1}^2
$$

$$
+ \nu \|\nabla^s (u - v^h)\|^2 + \nu_T \|(I - P_{L^H})\nabla^s (u - v^h)\|^2 + \nu_T \|(I - P_{L^H})\nabla^s u\|^2] dt'
$$

$$
+ ((H^{-2} + \nu_T^{-1}) \|\nabla^s (u - v^h)\|_{L^4(0,T;L^2\Omega))} \|u - v^h\|_{L^4(0,T)} (\|\nabla u\|^2_{L^4(0,T;L^2(\Omega))} + 1)\Big)
$$

$$
+ CC^*(T) \inf_{q^h \in Q^h} \int_0^T (H^{-2} + \nu_T^{-1}) \|p - q^h\|^2 dt'. \tag{4.4}
$$

Proof. Let $e = u - u^h$ and $v^h \in V^h$. Then, decompose the error into two parts: $e = \eta - \phi^h$, where $\eta = u - v^h$ and $\phi^h = u^h - v^h$. Let $v^h \in V^h$ denote an approximation of u. Subtracting (4.1) from (4.3) yields,

$$
\begin{aligned}
&(e_t, v^h) + [b(u, u, v^h) - b(u^h, u^h, v^h)] + (2\nu \nabla^s e, \nabla^s v^h) \\
&+ (\nu_T (I - P_{L^H})\nabla^s e, (I - P_{L^H})\nabla^s v^h) = -(p, \nabla \cdot v^h) \\
&+ (\nu_T (I - P_{L^H})\nabla^s u, (I - P_{L^H})\nabla^s v^h).
\end{aligned}
\tag{4.5}
$$

By using (4.5) and setting $v^h = \phi^h$ gives:

$$
\begin{aligned}
&(\phi_t^h, \phi^h) + (2\nu\nabla^s\phi^h, \nabla^s\phi^h) + (\nu_T(I - P_{L^H})\nabla^s\phi^h, (I - P_{L^H})\nabla^s\phi^h) \\
&= (\eta_t, \phi^h) + [b(u, u, \phi^h) - b(u^h, u^h, \phi^h)] + (2\nu\nabla^s\eta, \nabla^s\phi^h) \\
&+ (\nu_T(I - P_{L^H})\nabla^s\eta, (I - P_{L^H})\nabla^s\phi^h) - (p - q^h, \nabla\cdot\phi^h) \\
&+ (\nu_T(I - P_{L^H})\nabla^s u, (I - P_{L^H})\nabla^s\phi^h).
\end{aligned} \tag{4.6}
$$

Note that, since $\phi^h \in V^h$, $(q^h, \nabla\cdot\phi^h) = 0$, we can write

$$
(p, \nabla\cdot\phi^h) = (p - q^h, \nabla\cdot\phi^h), \;\; q^h \in Q^h.
$$

We want to bound the terms on the right hand side of (4.6). Consider first the convection terms in (4.6). Adding and subtracting terms yields:

$$
b(u, u, \phi^h) - b(u^h, u^h, \phi^h) = b(e, u, \phi^h) + b(u^h, e, \phi^h).
$$

By writing $e = \eta - \phi^h$, and using skew-symmetry, this reduces to

$$
b(u, u, \phi^h) - b(u^h, u^h, \phi^h) = b(\eta, u, \phi^h) - b(\phi^h, u, \phi^h) + b(u^h, \eta, \phi^h).
$$

By using Lemma 2.1., Young's inequality, the projection operator, P_{L^H} and $\left\|\nabla^s\phi^h\right\|^2 = (\left\|P_{L^H}\nabla^s\phi^h\right\|^2 + \left\|(I - P_{L^H})\nabla^s\phi^h\right\|^2)^{\frac{1}{2}}$, the terms are bounded by as follows:

$$
\begin{aligned}
b(\eta, u, \phi^h) &\le C\left\|\nabla u\right\|\left\|\eta\right\|^{\frac{1}{2}}\left\|\nabla^s\eta\right\|^{\frac{1}{2}}\left\|\nabla^s\phi^h\right\|^2 \\
&\le CH^{-2}\left\|\nabla u\right\|^2\left\|\eta\right\|\left\|\nabla\eta\right\| + \left\|\phi^h\right\|^2 \\
&+ \frac{\nu_T}{4}\left\|(I - P_{L^H})\nabla^s\phi^h\right\|^2 + \nu_T^{-1}\left\|\nabla u\right\|^2\left\|\eta\right\|\left\|\nabla^s\eta\right\| \\
b(\phi^h, u, \phi^h) &\le \left\|\nabla^s\phi^h\right\|^{\frac{3}{2}}\left\|\phi^h\right\|^{\frac{1}{2}}\left\|\nabla u\right\| \\
&\le \epsilon\left\|\nabla^s\phi^h\right\|^2 + \frac{C}{\epsilon^3}\left\|\phi^h\right\|^2\left\|\nabla u\right\|^4 \\
b(u^h, \eta, \phi^h) &\le C\left\|\nabla u^h\right\|^{\frac{1}{2}}\left\|u^h\right\|^{\frac{1}{2}}\left\|\nabla^s\eta\right\|\left\|\nabla^s\phi\right\| \\
&\le (CH^{-2} + \nu_T^{-1})\left\|\nabla u^h\right\|\left\|\eta\right\|\left\|\nabla^s\eta\right\| + \frac{1}{2}\left\|\phi^h\right\|^2 \\
&+ \frac{\nu_T}{8}\left\|(I - P_{L^H})\nabla^s\phi^h\right\|^2
\end{aligned}
$$

The remaining terms in (4.6) are estimated by the Cauchy Schwarz, Young's, and Hölder inequalities as follows

$$
\left\|(\eta_t, \phi^h)\right\| \le CH^{-2}\left\|\eta_t\right\|_{-1}^2 + \left\|\phi^h\right\|^2 + \frac{\nu_T}{4}\left\|(I - P_{L^H})\nabla^s\phi^h\right\|^2 + 4\nu_T^{-1}\left\|\eta_t\right\|_{-1}^2
$$

$$2\nu \|\nabla^s \eta\| \left\|\nabla^s \phi^h\right\| \leq 2\nu \|\nabla^s \eta\|^2 + \frac{\nu}{2} \left\|\nabla^s \phi^h\right\|^2$$

$$\begin{aligned}
\nu_T \|(I - P_{L^H})\nabla^s \eta\| \left\|(I - P_{L^H})\nabla^s \phi^h\right\| &\leq 4\nu_T \|(I - P_{L^H})\nabla^s \eta\|^2 \\
&+ \frac{\nu_T}{16} \left\|(I - P_{L^H})\nabla^s \phi^h\right\|^2 \\
\nu_T \|(I - P_{L^H})\nabla^s u\| \left\|(I - P_{L^H})\nabla^s \phi^h\right\| &\leq 4\nu_T \|(I - P_{L^H})\nabla^s u\|^2 \\
&+ \frac{\nu_T}{16} \left\|(I - P_{L^H})\nabla^s \phi^h\right\|^2
\end{aligned}$$

$$\begin{aligned}
\|(p - q^h, \nabla \cdot \phi^h)\| \leq H^{-2} \left\|p - q^h\right\|^2 + \left\|\phi^h\right\|^2 + C\nu_T^{-1} \left\|p - q^h\right\|^2 \\
+ \frac{\nu_T}{8} \left\|(I - P_{L^H})\nabla^s \phi^h\right\|^2.
\end{aligned}$$

Inserting these bounds into (4.6) and setting $\epsilon = \nu/2$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left\|\phi^h\right\|^2 + \nu \left\|\nabla^s \phi^h\right\|^2 + \frac{7\nu_T}{16} \left\|(I - P_{L^H})\nabla^s \phi^h\right\|^2 \\
&\quad \leq (C_1 H^{-2} + 4\nu_T^{-1}) \|\eta_t\|_{-1}^2 + C_2(H^{-2} + C_3\nu_T^{-1}) \|\nabla u\| \|\eta\| \|\nabla^s \eta\| \\
&\quad + C_4(H^{-2} + C_5\nu_T^{-1}) \left\|\nabla^s u^h\right\|^2 \|\eta\| \|\nabla^s \eta\| + 2\nu \|\nabla^s \eta\|^2 \\
&\quad + 4\nu_T \|(I - P_{L^H})\nabla^s \eta\|^2 + 4\nu_T \|(I - P_{L^H})\nabla u\|^2 \\
&\quad + \left\|\phi^h\right\|^2 (\frac{7}{2} + \nu^{-3} \|\nabla u\|^4) + (H^{-2} + \nu_T^{-1}) \left\|p - q^h\right\|^2
\end{aligned}$$

Since by assumption $\nabla u \in L^4(0, T; L^2(\Omega))$, Gronwall's inequality implies

$$\begin{aligned}
&\max_{0 \leq t \leq T} \left\|\phi^h\right\|^2 + C_1(T) \int_0^T [2\nu \left\|\nabla^s \phi^h\right\|^2 + \frac{7\nu_T}{8} \left\|(I - P_{L^H})\nabla^s \phi^h\right\|^2] dt' \\
&\leq C^*(T) \left\|\phi^h(0)\right\|^2 + CC^*(T)[\int_0^T (H^{-2} + \nu_T^{-1}) \|\eta_t\|_{-1}^2 + \nu \|\nabla^s \eta\|^2 \\
&+ \nu_T \|(I - P_{L^H})\nabla^s \eta\|^2 + \nu_T \|(I - P_{L^H})\nabla u\|^2 \\
&+ (H^{-2} + \nu_T^{-1}) \|\eta\| \|\nabla^s \eta\| (\|\nabla u\|^2 + \left\|\nabla u^h\right\|^2)] dt' \\
&+ CC^*(T) \inf_{q^h \in Q^h} \int_0^T (H^{-2} + \nu_T^{-1}) \left\|p - q^h\right\|^2 dt'
\end{aligned}$$

where $C^*(T) = \exp(\int_0^T (7 + C\nu^{-3}) \|\nabla u\|^4) dt'$ We can bound the remaining terms by using Cauchy-Schwarz inequality in $L^2(0,T)$ and Proposition 4.1. Thus,

$$\int_0^T \|\nabla u\|^2 \|\eta\| \ \|\nabla^s \eta\| \, dt' \leq \|\nabla u\|^2_{L^4(0,T;L^2(\Omega))} \|\nabla^s \eta\|_{L^4(0,T;L^2(\Omega))} \|\eta\|_{L^4(0,T;L^2(\Omega))}$$

$$\int_0^T \left\|\nabla u^h\right\|^2 \|\eta\| \ \|\nabla^s \eta\| \, dt' \leq C \|\nabla^s \eta\|_{L^4(0,T;L^2(\Omega))} \|\eta\|_{L^4(0,T;L^2(\Omega))} .$$

Applying the triangle inequality we have the infimum taken over only V^h instead of X^h. Under the discrete inf-sup condition, it is known that if $\nabla . u = 0$ the infimum over V^h can be replaced by infimum taken over X^h, Girault and Raviart [8],Theorem 1.1, p.59. Thus the final result follows. □

The error estimates which are similar to Theorem 4.1. can be used as guide to pick parameter scalings by balancing error terms in the case of smooth solutions. To illustrate this let us consider the Mini-element of Arnold, Brezzi and Fortin [2].

Corollary 4.2. *Suppose u,p are sufficiently smooth. Assume that $\Pi^h(\Omega)$ be refinement of $\Pi^H(\Omega)$ $(h << H)$. Let $X^h, X^H, L^H := \nabla X^H, Q^h$ denote a finite element space of cubic bubble functions, piecewise linear on a coarser mesh width $H > h$, piecewise constants on coarser mesh, piecewise linear on a mesh width of h, respectively,*

$$\begin{aligned}
X^h &:= \left\{v \in C^0(\bar{\Omega}) : v\,|_\Delta \in P_3(\Delta) \cap H^1_0(\Delta), \textit{ for all triangles } \Delta \textit{ in} \, \Pi^h(\Omega)\right\} \\
X^H &:= \left\{v \in C^0(\bar{\Omega}) : v\,|_\Delta \in P_1(\Delta), \textit{ for all triangles} \, \Delta \textit{ in} \, \Pi^H(\Omega)\right\} \\
L^H &:= \left\{l^H \in L^2(\Omega)^d : l^H\,|_\Delta \in P_0(\Delta), \textit{ for all triangles} \, \Delta \textit{in} \, \Pi^H(\Omega)\right\} \\
Q^h &:= \left\{v \in C^0\bar{\Omega} : v\,|_\Delta \in P_1(\Delta), \textit{ for all triangles } \Delta \textit{ in} \, \Pi^h(\Omega)\right\}.
\end{aligned}$$

With the choices

$$\nu_T \sim h, \ \ h \sim H^2,$$

the error in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is bounded by $C(u,\nu)h$.

Proof. By using approximation assumptions given Section 2,

$$\max_{0\le t\le T}\left\|u-u^h\right\|^2+C(T)\int_0^T[\frac{\nu}{2}\left\|\nabla^s(u-u^h)\right\|^2+\frac{\nu_T}{4}\left\|(I-P_{L^H})\nabla^s u\right\|^2]dt'$$
$$\le C^*(u,p)((H^{-2}+\nu_T^{-1})h^6+\nu h^2+\nu_T h^2+\nu_T H^2+H^{-2}h^3+\nu_T^{-1}h^3$$
$$+H^{-2}h^3+(H^{-2}+\nu_T^{-1})h^2).$$

For balancing term neglect higher order terms i.e., $H^{-2}h^6, \nu_T^{-1}h^6, H^{-2}h^3$. It can be seen easily that with the natural choice of $\nu_T \sim h,\ H^2 \sim h$ the error is order h. □

Remark 4.1. *If one consider Taylor-Hood element, with the choices*

$$X^h := \left\{v\in C^0(\bar\Omega): v\mid_\Delta\in P_2(\Delta),\ \textit{for all triangles}\ \Delta\ \textit{in}\,\Pi^h(\Omega)\right\}$$
$$X^H := \left\{v\in C^0(\bar\Omega): v\mid_\Delta\in P_2(\Delta),\ \textit{for all triangles}\,\Delta\ \textit{in}\,\Pi^H(\Omega)\right\}$$
$$L^H := \left\{l^H\in L^2(\Omega)^d: l^H\mid_\Delta\in P_1(\Delta),\ \textit{for all triangles}\,\Delta\textit{in}\,\Pi^H(\Omega)\right\}$$
$$Q^h := \left\{v\in C^0\bar\Omega: v\mid_\Delta\in P_1(\Delta),\ \textit{for all triangles}\ \Delta\ \textit{in}\,\Pi^h(\Omega)\right\}.$$

$$\nu_T\sim h,\ \ h\sim H^2,$$

the error in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ *is bounded by* $C(u,\nu)h^2$.

5 Reynolds Number Dependence for the Oseen Problem

Consider the solution of the time dependent linearized Navier-Stokes equations i.e. the Oseen problem

$$\begin{aligned} u_t-\nu\Delta u+\mathbf{b}\cdot\nabla u+\nabla p=f,&\ \text{ in }\ \Omega,\ \text{ for } 0<t\le T,\\ \nabla\cdot u=0,&\ \text{ in }\ \Omega,\ \text{ for } 0<t\le T,\\ u(x,0)=u_0(x),&\ \text{ in }\Omega, u=0 \text{ on }\Gamma,\ \text{ for } 0<t\le T,\end{aligned} \tag{5.1}$$

where $\mathbf{b}(x)$ is a smooth vector field with $\nabla\cdot\mathbf{b}=0$ and $\mathbf{b}=0$ on Γ. Simplifying the proof from the Navier-Stokes case, we show semi-discrete approximation of (5.1) for new method is convergent. Indeed, the variational formulation of (5.1) is

$$(u_t,v)+(\nu\nabla u,\nabla v)+(\mathbf{b}\cdot\nabla u,v)-(p,\nabla\cdot v)+(q,\nabla\cdot u)=(f,v). \tag{5.2}$$

for all $(v, q) \in (X, Q)$. We consider the new method and the same finite element discretization as Navier-Stokes case for Oseen problem. Thus, the semi discrete finite element approximation is a map $u^h : [0, T] \to V^h$ satisfying

$$(u_t^h, v^h) + (\nu \nabla u^h, \nabla v^h) + (\mathbf{b} \cdot \nabla u^h, v^h) \\ + (\nu_T (I - P_{L^H}) \nabla u^h, (I - P_{L^H}) \nabla v^h) = (f, v^h), \text{ for all } v^h \in V^h \quad (5.3)$$

where V^h is the space of discretely divergence-free functions. For error analysis, we need a second equation including u. Multiply (5.1) by $v^h \in V^h$ and integrate over Ω. Rearrangements gives for any $v^h \in V^h$

$$(u_t, v^h) + (\nu \nabla u, \nabla v^h) + (\mathbf{b} \cdot \nabla u, v) - (p - q^h, \nabla \cdot v^h) \quad (5.4)$$

$$+(\nu_T (I - P_{L^H}) \nabla u, (I - P_{L^H}) \nabla v^h) = (\nu_T (I - P_{L^H}) \nabla u, (I - P_{L^H}) \nabla v^h) + (f, v^h).$$

Theorem 5.1. *There is a constant $C = e^{\alpha(t-T)}$ independent of ν such that*

$$\max_{0 \le t \le T} \left\| u - u^h \right\|^2 + C \int_0^T [\nu \left\| \nabla (u - u^h)^2 \right\| + \nu_T \left\| (I - P_{L^H}) \nabla (u - u^h)^2 \right\|] dt'$$

$$\le C \left\| u(x, 0) - v^h(x, 0) \right\|^2 + C \inf_{v^h \in X^h} \Big(\int_0^T [(H^{-2} + \nu_T^{-1}) \left\| (u - v^h)_t \right\|_{-1}^2$$

$$+ \nu \left\| \nabla (u - v^h) \right\|^2 + \nu_T \left\| (I - P_{L^H}) \nabla (u - v^h) \right\|^2 + \nu_T \left\| (I - P_{L^H}) \nabla u \right\|^2$$

$$+ H^{-2} \left\| P_{L^H} \mathbf{b} (u - v^h) \right\|^2 + \nu_T^{-1} \left\| (I - P_{L^H}) \mathbf{b} (u - v^h) \right\|^2] dt' \Big)$$

$$+ C \inf_{q^h \in Q^h} \int_0^T (H^{-2} + \nu_T^{-1}) \left\| p - q^h \right\|^2 dt'.$$

Proof. For the error analysis subtract (5.3) from (5.4) and set $e = u - u^h = \eta - \phi^h$ where $\eta = u - v^h$ and $\phi^h = u^h - v^h$. Following the same procedure in the proof of Theorem 4.1 gives the result. □

Remark 5.1. *One of the important property of this result is that new method does improve its robustness with respect to Reynolds number. The Oseen problem captures the convection terms contribution to the linearized problem but omits the reaction term. Thus, this error estimates shows the new method ensures the uniformly in ν in the error contribution from the convection terms.*

To show the advantages of this new method we consider also the error in usual Galerkin formulation of Oseen problem ($\nu_T = 0$). The usual Galerkin variational formulation of Oseen problem is (5.2) and semi discrete approximation is

$$(u_t^h, v^h) - \nu(\nabla u^h, \nabla v^h) - (\mathbf{b} \cdot \nabla u^h, v^h) = (f, v^h), \text{ for all } v^h \in V^h. \tag{5.5}$$

Then, if we subtract (5.5) from (5.2), we get the following theorem.

Theorem 5.2. *Let u^h be the usual Galerkin finite element approximation of the Oseen problem. Then, there is a constant C such that*

$$\max_{0\leq t\leq T} \left\|u - u^h\right\|^2 + \int_0^T \nu \left\|\nabla(u - u^h)^2\right\| \leq C \inf_{v^h \in X^h} \Big(\int_0^T \nu^{-1} \left\|(u - v^h)_t\right\|_{-1}^2$$

$$+ \nu \left\|\nabla(u - v^h)\right\|^2 + \nu^{-1} \left\|\mathbf{b}(u - v^h)\right\|^2 dt' \Big) + C \inf_{q^h \in Q^h} \int_0^T \nu^{-1} \left\|p - q^h\right\|^2 dt'.$$

Remark 5.2. *If we compare Theorem (5.1) and Theorem (5.2), it is clear that the basic error estimate for the usual Galerkin formulation of Oseen problem depends badly on ν. In contrast, this new approach leads to an error estimate with error constants which are uniform in ν for the Oseen problem.*

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] D. N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo **21** (1984), 337–344.

[3] F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, $b = \int g$, Computer Methods in Applied Mechanics and Engineering **145** (1997), 329–339.

[4] M. Fortin, R. Guenette, and R. Pierre, *Numerical analysis of the modified EVSS method*, Comp. Meth. Appl. Mech. Eng. **143** (1997), 79–95.

[5] G. P. Galdi, *An introduction to the Mathematical theory of the Navier-Stokes Equation Vol 1: Linearized Theory*, vol. 38, 1994.

[6] V. Girault and J.-L. Lions, *Two-grid finite element schemes for the transient Navier-Stokes*, Math. Modelling and Num. Anal. **35**.

[7] ———, *Two-grid finite element schemes for the steady Navier-Stokes problem in polyhedra*, Portugal. Math. **58** (2001), 25–57.

[8] V. Girault and P. A. Raviart, *Finite element approximation of the Navier-Stokes equations*, Springer-Verlag, Berlin, 1979.

[9] J. L. Guermond, *Stabilization of Galerkin approximations of transport equations by subgrid modelling*, M2AN **33** (1999), 1293–1316.

[10] M. Gunzburger, *Finite Element Methods for Viscous Incompressible Flow: A Guide to Theory, Practice, and Algorithms*, Academic Press, Boston, 1989.

[11] T. J. R. Hughes, *The multiscale phenomena: Green's functions, the Dirichlet- to-Neumann formulation, subgrid-scale models, bubbles and the origin of stabilized methods*, vol. 127, 1995.

[12] T. J. R. Hughes, L. Mazzei, and K. E. Jansen, *Large eddy simulation and the variational multiscale method*, Comput. Visual Sci. **3** (2000), 47–59.

[13] T. J. R. Hughes, L. Mazzei, A. A. Oberai, and A. Wray, *The multiscale formulation of large eddy simulation: decay of homogenous isotropic turbulence*, Physics of Fluids **13** (2001), 505–512.

[14] T. J. R. Hughes, A. A. Oberai, and L. Mazzei, *Large eddy simulation of turbulent channel flows by the variational multiscale method*, Physics of Fluids **13** (2001), 1784–1799.

[15] T. Iliescu and W. J. Layton, *Approximating the larger eddies in fluid motion III: Boussinesq model for turbulent fluctuations*, Analele Stiintifice ale Universităţii "Al. I. Cuza" Iassi, Tomul XLIV, s.i.a.,Matematica **44** (1998), 245–261.

[16] S. Kaya and W. Layton, *Subgrid-scale eddy viscosity methods are variational multiscale methods*, University of Pittsburgh,Technical report (2002).

[17] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.

[18] W. J. Layton, *A connection between subgrid scale eddy viscosity and mixed methods*, Appl. Math. and Comput. **133** (2002), 147–157.

[19] J. Leray, *Sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math **63** (1934), 193–248.

[20] R. Lewandowski, *Analyse Mathematique et oceanographie*, Masson, Paris, 1997.

[21] M. Marion and J. Xu, *Error estimates for a new nonlinear Galerkin method based on two-grid finite elements*, SIAM J. Numer. Anal. **32** (1995), 1170–1184.

[22] Y. Mayday and E. Tadmor, *Analysis of spectral vanishing viscosity method for periodic conservation laws*, SINUM **6** (1923), 389–440.

[23] J. Smagorinsky, *General circulation experiments with the primitive equation,I: The basic experiment*, Month. Weath. Rev. **91** (1963), 99–164.

[24] R. Temam, *Navier-Stokes Equations and Nonlinear Functional analysis*, SIAM, Philadelphia, 1995.