Subgrid-scale Eddy Viscosity Methods are Variational Multiscale Methods

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Abstract

We prove that two recently introduced stabilized methods fit into the framework of variational multiscale methods (introduced by Hughes). The interest in this connection is that the two methods allow fluctuations to be nonzero across meshlines. Thus, the connection may lead to other variational multiscale methods which also circumvent this restriction.

1 Introduction

Consider, as a prototypical problem which exhibits solution structure on many different scales, the convection dominated, convection-diffusion equation:

$$\left\{ \begin{array}{l} u_t - \epsilon \Delta u + b \cdot \nabla u = f(x), \ x \in \Omega \subset \mathbf{R}^d, 0 < t \leq T, \\ u = 0 \text{ on } \partial\Omega \text{ and } u(x,0) = u_0(x), \ x \in \Omega. \end{array} \right. \tag{1.1}$$

When ϵ is significantly smaller than computationally feasible meshwidths, (1.1) can exhibit solution structures, such as boundary and interior layers, which can make the numerical solution of (1.1) challenging.

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Recently Hughes and co-workers (see e.g., [12-15]) have introduced a systematic approach to the discretization of multiscale problems including (1.1) in which coupled equations for the resolved scales (of $\geq O(\text{meshwidth})$) and the fluctuations are simultaneously discretized. When the fluctuations are approximated in spaces of bubble functions, the fluctuations can be explicitly eliminated from the coupled system and the approximate effects of the fluctuations on the resolved scales explicitly calculated. This approach is called the "Variational Multiscale Method" (henceforth VMM) and it has been actively developed and, in particular, shown to encompass several popular methods such as the residual free bubble and the streamline diffusion/SUPG method, [2], [3], [4], [6], [8], [11], [12], [13], [14], [15], [18]. On the other hand, this approach also has an often noted fundamental conceptual restriction connected to the use of bubble functions to model fluctuations that it implicitly assumes/imposes the physical constraint that:

(1.2) fluctuations cannot cross mesh lines.

Since the equations for the large scales, $\bar{u}$, and small scales, u', are necessarily coupled, it is not surprising that the equations for the errors in the discretizations of the large and small scales are also coupled. We show in Proposition 2.1 below that the error in the approximation of the large scales is bounded by the error in the best approximation of those scales and the error in the approximation of the small scales. Thus, a good approximation of the large scales can occur two ways. If the coarse mesh is sufficiently fine that essentially $\bar{u} \cong u$ then $u'(= u - \bar{u})$ will be small and a crude approximation of u' will suffice. On the other hand, if the coarse mesh is coarse relative to interesting solution behavior, then more accurate approximation of u' might be needed to ensure that the error in $\bar{u}$ will be small. Since fluctuations do move, an improved approximation of u' seems to require exploring VMM's which circumvent the constraint (1.2).

In this report, we will give a slightly more general interpretation of Hughes' Variational Multiscale Method which both includes the previous work and removes the constraint (1.2) for some methods which arise from the generalization. In particular, we show that a new consistently stabilized method of [17] (see also Kaya [16]) fits into our framework as a generalized variational multiscale method as does another consistently stabilized method of Guermond [10].

2 Mathematical Formulations

Let $X := H_0^1(\Omega)$ denote the Sobolev space $\{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)$ and $v = 0$ on $\partial\Omega\}$ (e.g., Adams [1]) in which solutions of (2.1) are sought and let $(\cdot,\cdot), ||\cdot||$ denote the usual $L^2(\Omega)$ inner product and norm. If $u : [0,T] \to X$ is a solution of (2.1) then integration by parts reveals that u satisfies:

(2.1) $(u_t, v) + a(u,v) = (f,v), \forall\, v \in X, 0 < t \le T,$ and $u(x,0) = u_0 \in X,$

where,

$$a(u,v) := (\epsilon\nabla u, \nabla v) + (b\cdot\nabla u, v). \quad (2.2)$$

Let $\delta > 0$ denote a length scale (or coarse meshwidth) and let $\bar{X} = X_\delta$ denote a closed subspace of X associated with functions varying over length scales δ and larger. Define $X' = \bar{X}^{\perp}$. Since $\bar{X}, X'$ are closed subspaces, we can define projection operators

$$P : X \to \bar{X},\ (I - P) =: Q : X \to X'.$$

determining the precise definition of large scales and small scales via

$$\bar{u} = P\ u \in \bar{X},\ u' = Q\ u \in X'. \quad (2.3)$$

The VMM approach of Hughes and co-workers , [12-15] chooses P, Q to be L^2 orthogonal projections and writes $u = \bar{u} + u'$ in (2.1). Next, set alternately $v = \bar{v} \in X$ and $v = v' \in X'$. This gives the coupled system:

$$(\bar{u}_t, \bar{v}) + a(\bar{u}, \bar{v}) - (\bar{f}, \bar{v}) = (r', \bar{v}),\ \forall\ \bar{v} \in \bar{X}, \quad (2.4)$$

where $(r', \bar{v}) := (f', \bar{v}) - [(u'_t, \bar{v}) + a(u', \bar{v})]$, and

$$(u'_t, v') + a(u', v') - (f', v') = (\bar{r}, v'),\ \forall\ v' \in X', \quad (2.5)$$

where $(\bar{r}, v') := (\bar{f}, v') - [(\bar{u}_t, v') + a(\bar{u}, v')]$.

Remark: With P, Q chosen to be L^2 orthogonal projections, several terms vanish in both residuals.

Thus, as noted, e.g., in [13], the large scales are driven by the projection of the small scale residual onto the large scales and the small scales are driven by the projection of the large scale residual onto the small scales.

The Variational Multiscale Method is a finite element procedure which simultaneously discretizes (2.4) and (2.5). To be specific, let $\Pi_H(\Omega)$ denote an edge-to-edge triangulation of Ω with maximum triangle diameter "H" and with minimum angle bounded away from zero. Choose $X_H := \bar{X}$ and X' to be, respectively, the finite element spaces of C^0 conforming linears and cubic bubble functions

$$(2.6)\begin{cases} \bar{X}_H := \{v \in C^0(\bar{\Omega}) : v_{|_\Delta} \in \mathcal{P}_1(\Delta)\ \forall \text{ triangles } \Delta \in \Pi_H(\Omega)\} \\ X'_b := \{v \in C^0(\bar{\Omega}) : v_{|_\Delta} \in \mathcal{P}_3(\Delta) \cap H^1_0(\Delta), \forall \text{ triangles } \Delta \in \Pi_H(\Omega)\} \end{cases}$$

This choice (2.6) is the prototypical one studied in many reports.

Definition 2.1 *The continuous-in-time Variational Multiscale Method applied to (1.1) is the approximation*

$$u_H : [0,T] \to X_H,\ \ u'_b : [0,T] \to X'_b,$$

satisfying: $u_H(0) = Pu_0, u'_b(0) = Qu_0$ *and*

$$(2.7)\qquad (u_{H,t}, v_H) + a(u_H, v_H) - (\bar{f}, v_H) = -a(u'_b, v_H),\ \ \forall\ v_H \in X_H,$$

$$(2.8)\qquad (u'_{b,t}, v'_b) + a(u'_b, v'_b) - (f', v'_b) + (\mathcal{S}(u'_b), \nabla v'_b) = -a(u_H, v'_b),\ \ \forall\ v'_b \in X'_b,$$

where $(\mathcal{S}(u'_b), \nabla v'_b)$ *denotes an added small-scale stabilization term (for which various choices are possible).* $\square$

The discretization (2.7), (2.8) is conceptually and computationally ingenious: since $u'_b \in X'$, it follows that $u'_b \in H^1_0(\Delta)$ for every triangle Δ in the mesh. Thus, the small scale equation (2.8) reduces to a 1×1 subproblem on each triangle. This can be solved explicitly and inserted into the RHS of (2.7) giving a correction for the effects of the unresolved scales on the resolved scales.

So far it has not been easy to produce a computationally feasible generalization of the VMM framework. The computational attraction of (2.7), (2.8) is linked with the precise choice of X'_b to be functions supported on only one element. This choice of X'_b also imposes the implicit restriction (1.2) that discrete fluctuations are quasi-stationary.

To understand the constraint (2.9), it is worthwhile to consider errors in discretizations of the VMM formulation (2.4), (2.5), such as (2.7), (2.8). To this end, let

$$\bar{e} := \bar{u} - \bar{u}_H,\ \ e' = u' - u'_b$$

denote, respectively the errors in the approximations to the large and small scales.

Proposition 2.1 *Suppose* $b \in L^\infty(\Omega), \nabla \cdot b = 0,\ 0 < T < \infty$, *and* $\epsilon < ||b||_{L^\infty}$. *Then the errors* $\bar{e} = \bar{u} - u_H$ *and* $e' = u' - u'_b$ *satisfy*

$$
\begin{aligned}
&||\bar{e}||^2_{L^\infty(0,T;L^2(\Omega))} + \epsilon||\nabla \bar{e}||^2_{L^2(0,T;L^2(\Omega))} \leq \\
&\leq C\Big\{||\bar{u}(0) - u_H(0)||^2 + \epsilon^{-1} \inf_{v^H \in X_H} \Big[||\bar{u}_t - v_{H,t}||^2_{L^2(0,T;H^{-1}(\Omega))} + \\
&||\nabla(\bar{u} - v_H)||^2_{L^2(0,T;L^2(\Omega))}\Big]\Big\} + \epsilon^{-1} \int_0^T \left(\epsilon||\nabla e'(s)||^2 + ||e'(s)||^2\right) ds.
\end{aligned}
$$

Proof. Subtracting the equation (2.7) for u_H from the equation (2.4) for $\bar{u}$ gives the following equation for $\bar{e}$:

$$(\bar{e}_t\ v_H) + a(\bar{e}, v_H) = -a(e', v_H), \text{ for all } v_H \in X_H.$$

In the usual manner, decompose $\bar{e}$ as $\bar{e} = \eta - \Phi_H$ where $\eta = \bar{u} - v_H$ and $\Phi_H = u_H - v_H$ where v_H approximates $\bar{u}$ well in X_H. Setting v_H in the above error equation to be Φ_H gives:

$$
\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}||\Phi_H||^2 + \epsilon||\nabla\Phi_H||^2 \leq ||\eta_t||_{-1}\ ||\nabla\Phi_H|| \\
&+C(||\nabla\eta||\ ||\nabla\Phi_H|| + \epsilon||\nabla e'||\ ||\nabla\Phi_H||) + C||e'||\ ||\nabla\Phi_H||.
\end{aligned}
$$

Young's inequality then implies:

$$
\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}\ ||\Phi_H||^2 + \frac{\epsilon}{2}||\nabla\Phi_H||^2 \\
&\leq C\epsilon^{-1}[||\eta_t||^2_{-1} + ||\nabla\eta||^2 + ||\nabla e'||^2], \text{ or} \\
&||\Phi_H(t)||^2 + \int_0^t \epsilon||\nabla\Phi_H(s)||^2\ ds \leq ||\Phi_H(0)||^2 + \\
&+ \int_0^t \epsilon^{-1}[||\eta_t(s)||^2_{-1} + ||\nabla\eta(s)||^2 + \epsilon||\nabla e'(s)||^2 + ||e'||^2]ds.
\end{aligned}
$$

The triangle inequality completes the proof. □

As expected, the errors in the approximations to $\bar{u}$ and u' are coupled. The error in $\bar{u}$ is driven by the error in approximating u'. This suggests two natural ways to obtain a good approximation to the large scales $\bar{u}$:

1. Approximate $\bar{u}$ very well on a very fine mesh. In this case u' will be small and a crude approximation (to a small quantity) is acceptable.

2. Approximate $\bar{u}$ on a less fine mesh but use a better approximation of u'.

Clearly, the larger u' is the better the computational model for u' must be to ensure small errors in $\bar{u}$. In case 1, the restriction (2.9) is perhaps no problem. In case 2, however, the computational model of u' should be improved.

Definition 2.2 *A generalized VMM is determined by:*

(a) the projection operator P defining the large scales.

(b) the finite dimensional space $\bar{X}_H \subset \bar{X}$ and its complement X'_b in which (2.4), (2.5) are discretized.

(c) the small scale stabilization term $(\mathcal{S}(u'_b), \nabla v'_b)$ added to the discretization of the small scale equation (2.5). $\square$

It is not immediately obvious if such generalization is useful, i.e., if interesting and realizable methods, can arise from it. To answer this question in the affirmative, we prove our two main results that the methods of [17] and of Guermond [10] are generalized VMM's, which allow approximations to fluctuations which can move across edges of the mesh.

3 Fluctuations Can Move.

Accompanying $\Pi_H(\Omega)$, let $\Pi_h(\Omega)$ denote a fine mesh (so typically $h << H$). Define the space of vector L^2 piecewise constants L_H to be:

$$L_H := \{\ell_H \in L^2(\Omega)^d : \ell_H|_\Delta \in \mathcal{P}_0(\Delta),\ \forall \text{ elements } \Delta \in \Pi_H(\Omega)\},$$

and let X^h denote the fine mesh, piecewise linears:

$$X_h := \{v^h \in H_0^1(\Omega) : v_h|_\Delta \in \mathcal{P}_1(\Delta),\ \forall \text{ elements } \Delta \in \Pi_h(\Omega)\}.$$

We let $P_H : X \to X_H,\ P_h : X \to X_h$ denote the L^2 orthogonal projection and $I : X_h \to X_h$ the identity on X_h. The method of [17] computes simultaneous approximations of u and ∇u by: $u_h : [0,T] \to X_h, \mathbf{g}_H : [0,T] \to L_H$ satisfy $u_h(0) = P_h u_0$ and

$$(u_{h,t}, v_h) + ((\epsilon + \alpha(h))\nabla u_h, \nabla v_h) + (b \cdot \nabla u_h, v_h) \tag{3.1}$$
$$-(\alpha(h)\mathbf{g}_H, \nabla v_h) = (f, v_h), \ \forall\ v_h \in X_h,$$
$$(\mathbf{g}_H - \nabla u_h, \ell_H) = 0, \ \forall\ \ell_H \in L_H. \tag{3.2}$$

Theorem 3.1 *Let $\Pi_h(\Omega)$ be a refinement of $\Pi_H(\Omega)$. Suppose $X_H \subset X_h$ and $L_H = \nabla X_H$. Then, (2.9) (2.10) is a generalized VMM wherein:*

(a) The projection operator $P = P_E := X \to X_H$ is the elliptic projection operator into X_H, satisfying:

$$(\nabla(w - P_E w), \nabla v_H) = 0, \ \forall\ v_H \in X_H.$$

Thus, $u_H = P_E u_h$, $u'_h = (I - P_E)u_h$.

(b) X_H is as defined above and X'_h is the complement of X_H in X_h given by

$$X_h := (I - P_E)X_H$$

(c) the stabilization is given by:

$$(\mathcal{S}(u'_h), \nabla v'_h) := (\alpha(h)\nabla u'_h, \nabla v'_h).$$

Proof: As (3.2) means $\mathbf{g}_H = P_H(\nabla u_h), \mathbf{g}_H$ can be eliminated from (3.1). Doing so, and using orthogonality reduces (3.1) to:

$$(u_{h,t}, v_h) + a(u_h, v_h) + (\alpha(h)(I - P_H)\nabla u_h, (I - P_H)\nabla v_h) = (f, v^h).$$

Using Lemma 3 of [17], $(I - P_H)\nabla u_h = \nabla(I - P_E)u_h$. By the definition of u'_h this implies $(\alpha(I - P_H)\nabla u_h, (I - P_H)\nabla v_h) = (\alpha\nabla u'_h, \nabla v'_h)$ and (3.1), (3.2) is equivalent to:

$$(u_{h,t}, v_h) + a(u_h, v_h) + (\alpha\nabla u'_h, \nabla v'_h) = (f, v_h), \ \forall\ v_h \in X_h \tag{3.3}$$

In (3.3) write $u_h = u_H + u'_h$ and set, alternately, $v_h = v_H$ and $v_h = v'_h$. The definition of $u_H = P_E u_h$ implies $(\nabla u'_h, \nabla v_H) = 0$. Thus, this gives:

$$(u_{H,t}, v_H) + a(u_H, v_H) - (P_H f, v_H) = (u'_{h,t}, v_H) - a(u'_h, v_H), \tag{3.4}$$
$$\forall\ v_H \in X_H,$$
$$(u'_{h,t}, v'_h) + a(u'_h, v'_h) + (\alpha\nabla u'_h, \nabla v'_h)$$
$$-((I - P_H)f, v'_h) = -(u_{H,t}, v'_h) - a(u_H, v'_h), \ \forall\ v'_h \in X'_h. \tag{3.5}$$

Comparing (3.4), (3.5) to Definition 2.2 completes the proof. □

It was already noted in [17] that the interesting method of Guermond [10] can arise from the formulation (3.1), (3.2) by a different choice of finite element spaces. Thus, by essentially the same argument we obtain the analogous result for the method of Guermond [10].

Theorem 3.2 *The method of [10] is a generalized VMM in the sense of Definition 2.2.* □

4 Remarks

If we consider a complement of X_H in X_h to be the span of the usual, nodal basis functions associated with vertices in $\Pi_h(\Omega)$ <u>not</u> in $\Pi_H(\Omega)$, then it is clear that fluctuations in the method of [17] are only required to be zero on the vertices of $\Pi_H(\Omega)$ and not on all the edges of $\Pi_h(\Omega)$ or $\Pi_H(\Omega)$.

The computational feasibility of the method (3.1), (3.2) stems from the fact that the basis functions in L_H are supported on one (macro) element. It can be seen by rewriting it as:

$$(u_{h,t}, v_h) + a(u_h, v_h) + (\alpha \nabla u_h, \nabla v_h) = (f, v_h) - (\alpha P_H(\nabla u_h), P_H(\nabla v_h)). \tag{4.1}$$

The extra term $(\alpha P_H(\nabla u_h), P_H(\nabla v_h))$ is not convenient to assemble. However, since P_H is the L^2 orthogonal projection into piecewise constants, calculating a residual of $P_H(\nabla u_h)$ is simple, embarassingly parallel and cheap.

Thus, this term can simply be lagged (with low order timestepping). If a fully implicit method is used, a linear system must be solved which includes this term. An iterative method must be used. Each iteration requires a residual calculation with $P_H(\nabla u_h)$ which, as noted above, is cheap.

More generally, the computational feasibility of the method (4.1) arises from the fact that it uses a multiscale decomposition of ∇u_h

$$\nabla u_h = P_H(\nabla u_h) + (I - P_H)(\nabla u_h)$$

so that discontinuous elements may be used for $P_H(\nabla u_h)$ whereas continuous ones are required for u_h.

It is interesting that the definition of the large eddies by elliptic projection is precisely a discrete analog of Germano's idea of differential filtering [9], which has been developed in a discrete sense by Fischer and Mullen [7] and Dunca [5].

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When ϵ is significantly smaller than computationally feasible meshwidths, (1.1) can exhibit solution structures, such as boundary and interior layers, which can make the numerical solution of (1.1) challenging.

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$$(\bar{u}_t, \bar{v}) + a(\bar{u}, \bar{v}) - (\bar{f}, \bar{v}) = (r', \bar{v}),\ \forall\ \bar{v} \in \bar{X}, \tag{2.4}$$

where $(r', \bar{v}) := (f', \bar{v}) - [(u'_t, \bar{v}) + a(u', \bar{v})]$, and

$$(u'_t, v') + a(u', v') - (f', v') = (\bar{r}, v'),\ \forall\ v' \in X', \tag{2.5}$$

where $(\bar{r}, v') := (\bar{f}, v') - [(\bar{u}_t, v') + a(\bar{u}, v')]$.

Remark: With P, Q chosen to be L^2 orthogonal projections, several terms vanish in both residuals.

Thus, as noted, e.g., in [13], the large scales are driven by the projection of the small scale residual onto the large scales and the small scales are driven by the projection of the large scale residual onto the small scales.

The Variational Multiscale Method is a finite element procedure which simultaneously discretizes (2.4) and (2.5). To be specific, let $\Pi_H(\Omega)$ denote an edge-to-edge triangulation of Ω with maximum triangle diameter "H" and with minimum angle bounded away from zero. Choose $X_H := \bar{X}$ and X' to be, respectively, the finite element spaces of C^0 conforming linears and cubic bubble functions

$$(2.6)\begin{cases} \bar{X}_H := \{v \in C^0(\bar{\Omega}) : v_{|_\Delta} \in \mathcal{P}_1(\Delta)\ \forall \text{ triangles } \Delta \in \Pi_H(\Omega)\} \\ X'_b := \{v \in C^0(\bar{\Omega}) : v_{|_\Delta} \in \mathcal{P}_3(\Delta) \cap H^1_0(\Delta), \forall \text{ triangles } \Delta \in \Pi_H(\Omega)\} \end{cases}$$

This choice (2.6) is the prototypical one studied in many reports.

Definition 2.1 *The continuous-in-time Variational Multiscale Method applied to (1.1) is the approximation*

$$u_H : [0,T] \to X_H,\ \ u'_b : [0,T] \to X'_b,$$

satisfying: $u_H(0) = Pu_0, u'_b(0) = Qu_0$ *and*

$$(u_{H,t}, v_H) + a(u_H, v_H) - (\bar{f}, v_H) = -a(u'_b, v_H),\ \ \forall\ v_H \in X_H, \tag{2.7}$$

$$(u'_{b,t}, v'_b) + a(u'_b, v'_b) - (f', v'_b) + (\mathcal{S}(u'_b), \nabla v'_b) = -a(u_H, v'_b),\ \ \forall\ v'_b \in X'_b, \tag{2.8}$$

where $(\mathcal{S}(u'_b), \nabla v'_b)$ *denotes an added small-scale stabilization term (for which various choices are possible).* $\square$

The discretization (2.7), (2.8) is conceptually and computationally ingenious: since $u'_b \in X'$, it follows that $u'_b \in H^1_0(\Delta)$ for every triangle Δ in the mesh. Thus, the small scale equation (2.8) reduces to a 1×1 subproblem on each triangle. This can be solved explicitly and inserted into the RHS of (2.7) giving a correction for the effects of the unresolved scales on the resolved scales.

So far it has not been easy to produce a computationally feasible generalization of the VMM framework. The computational attraction of (2.7), (2.8) is linked with the precise choice of X'_b to be functions supported on only one element. This choice of X'_b also imposes the implicit restriction (1.2) that discrete fluctuations are quasi-stationary.

To understand the constraint (2.9), it is worthwhile to consider errors in discretizations of the VMM formulation (2.4), (2.5), such as (2.7), (2.8). To this end, let

$$\bar{e} := \bar{u} - \bar{u}_H,\ \ e' = u' - u'_b$$

denote, respectively the errors in the approximations to the large and small scales.

Proposition 2.1 *Suppose* $b \in L^\infty(\Omega), \nabla\cdot b = 0,\ 0 < T < \infty$, *and* $\epsilon < ||b||_{L^\infty}$. *Then the errors* $\bar{e} = \bar{u} - u_H$ *and* $e' = u' - u'_b$ *satisfy*

$$\begin{aligned}
&||\bar{e}||^2_{L^\infty(0,T;L^2(\Omega))} + \epsilon||\nabla\bar{e}||^2_{L^2(0,T;L^2(\Omega))} \leq \\
&\leq C\Big\{||\bar{u}(0) - u_H(0)||^2 + \epsilon^{-1} \inf_{v^H \in X_H} \Big[||\bar{u}_t - v_{H,t}||^2_{L^2(0,T;H^{-1}(\Omega))} + \\
&||\nabla(\bar{u} - v_H)||^2_{L^2(0,T;L^2(\Omega))}\Big]\Big\} + \epsilon^{-1} \int_0^T \left(\epsilon||\nabla e'(s)||^2 + ||e'(s)||^2\right) ds.
\end{aligned}$$

Proof. Subtracting the equation (2.7) for u_H from the equation (2.4) for $\bar{u}$ gives the following equation for $\bar{e}$:

$$(\bar{e}_t\ v_H) + a(\bar{e}, v_H) = -a(e', v_H),\ \text{for all } v_H \in X_H.$$

In the usual manner, decompose $\bar{e}$ as $\bar{e} = \eta - \Phi_H$ where $\eta = \bar{u} - v_H$ and $\Phi_H = u_H - v_H$ where v_H approximates $\bar{u}$ well in X_H. Setting v_H in the above error equation to be Φ_H gives:

$$\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}||\Phi_H||^2 + \epsilon||\nabla\Phi_H||^2 \leq ||\eta_t||_{-1}\ ||\nabla\Phi_H|| \\
&+C(||\nabla\eta||\ ||\nabla\Phi_H|| + \epsilon||\nabla e'||\ ||\nabla\Phi_H||) + C||e'||\ ||\nabla\Phi_H||.
\end{aligned}$$

Young's inequality then implies:

$$\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}\ ||\Phi_H||^2 + \frac{\epsilon}{2}||\nabla\Phi_H||^2 \\
&\leq C\epsilon^{-1}[||\eta_t||^2_{-1} + ||\nabla\eta||^2 + ||\nabla e'||^2],\ \text{or} \\
&||\Phi_H(t)||^2 + \int_0^t \epsilon||\nabla\Phi_H(s)||^2\ ds \leq ||\Phi_H(0)||^2 + \\
&+ \int_0^t \epsilon^{-1}[||\eta_t(s)||^2_{-1} + ||\nabla\eta(s)||^2 + \epsilon||\nabla e'(s)||^2 + ||e'||^2]ds.
\end{aligned}$$

The triangle inequality completes the proof. □

As expected, the errors in the approximations to $\bar{u}$ and u' are coupled. The error in $\bar{u}$ is driven by the error in approximating u'. This suggests two natural ways to obtain a good approximation to the large scales $\bar{u}$:

1. Approximate $\bar{u}$ very well on a very fine mesh. In this case u' will be small and a crude approximation (to a small quantity) is acceptable.

2. Approximate $\bar{u}$ on a less fine mesh but use a better approximation of u'.

Clearly, the larger u' is the better the computational model for u' must be to ensure small errors in $\bar{u}$. In case 1, the restriction (2.9) is perhaps no problem. In case 2, however, the computational model of u' should be improved.

Definition 2.2 *A generalized VMM is determined by:*

(a) the projection operator P defining the large scales.

(b) the finite dimensional space $\bar{X}_H \subset \bar{X}$ and its complement X_b' in which (2.4), (2.5) are discretized.

(c) the small scale stabilization term $(\mathcal{S}(u_b'), \nabla v_b')$ added to the discretization of the small scale equation (2.5). □

It is not immediately obvious if such generalization is useful, i.e., if interesting and realizable methods, can arise from it. To answer this question in the affirmative, we prove our two main results that the methods of [17] and of Guermond [10] are generalized VMM's, which allow approximations to fluctuations which can move across edges of the mesh.

3 Fluctuations Can Move.

Accompanying $\Pi_H(\Omega)$, let $\Pi_h(\Omega)$ denote a fine mesh (so typically $h << H$). Define the space of vector L^2 piecewise constants L_H to be:

$$L_H := \{\ell_H \in L^2(\Omega)^d : \ell_H|_\Delta \in \mathcal{P}_0(\Delta),\ \forall \text{ elements } \Delta \in \Pi_H(\Omega)\},$$

and let X^h denote the fine mesh, piecewise linears:

$$X_h := \{v^h \in H_0^1(\Omega) : v_h|_\Delta \in \mathcal{P}_1(\Delta),\ \forall \text{ elements } \Delta \in \Pi_h(\Omega)\}.$$

We let $P_H : X \to X_H,\ P_h : X \to X_h$ denote the L^2 orthogonal projection and $I : X_h \to X_h$ the identity on X_h. The method of [17] computes simultaneous approximations of u and ∇u by: $u_h : [0,T] \to X_h, \mathbf{g}_H : [0,T] \to L_H$ satisfy $u_h(0) = P_h u_0$ and

$$(u_{h,t}, v_h) + ((\epsilon + \alpha(h))\nabla u_h, \nabla v_h) + (b \cdot \nabla u_h, v_h) - (\alpha(h)\mathbf{g}_H, \nabla v_h) = (f, v_h),\ \forall\ v_h \in X_h, \tag{3.1}$$

$$(\mathbf{g}_H - \nabla u_h, \ell_H) = 0,\ \forall\ \ell_H \in L_H. \tag{3.2}$$

Theorem 3.1 *Let $\Pi_h(\Omega)$ be a refinement of $\Pi_H(\Omega)$. Suppose $X_H \subset X_h$ and $L_H = \nabla X_H$. Then, (2.9) (2.10) is a generalized VMM wherein:*

(a) The projection operator $P = P_E := X \to X_H$ is the elliptic projection operator into X_H, satisfying:

$$(\nabla(w - P_E w), \nabla v_H) = 0,\ \forall\ v_H \in X_H.$$

Thus, $u_H = P_E u_h,\ u'_h = (I - P_E)u_h$.

(b) X_H is as defined above and X'_h is the complement of X_H in X_h given by

$$X_h := (I - P_E)X_H$$

(c) the stabilization is given by:

$$(\mathcal{S}(u'_h), \nabla v'_h) := (\alpha(h)\nabla u'_h, \nabla v'_h).$$

Proof: As (3.2) means $\mathbf{g}_H = P_H(\nabla u_h), \mathbf{g}_H$ can be eliminated from (3.1). Doing so, and using orthogonality reduces (3.1) to:

$$(u_{h,t}, v_h) + a(u_h, v_h) + (\alpha(h)(I - P_H)\nabla u_h, (I - P_H)\nabla v_h) = (f, v^h).$$

Using Lemma 3 of [17], $(I - P_H)\nabla u_h = \nabla(I - P_E)u_h$. By the definition of u'_h this implies $(\alpha(I - P_H)\nabla u_h, (I - P_H)\nabla v_h) = (\alpha \nabla u'_h, \nabla v'_h)$ and (3.1), (3.2) is equivalent to:

$$(u_{h,t}, v_h) + a(u_h, v_h) + (\alpha\nabla u'_h, \nabla v'_h) = (f, v_h),\ \forall\ v_h \in X_h \tag{3.3}$$

In (3.3) write $u_h = u_H + u'_h$ and set, alternately, $v_h = v_H$ and $v_h = v'_h$. The definition of $u_H = P_E u_h$ implies $(\nabla u'_h, \nabla v_H) = 0$. Thus, this gives:

$$(u_{H,t}, v_H) + a(u_H, v_H) - (P_H f, v_H) = (u'_{h,t}, v_H) - a(u'_h, v_H),\ \forall\ v_H \in X_H, \tag{3.4}$$

$$(u'_{h,t}, v'_h) + a(u'_h, v'_h) + (\alpha\nabla u'_h, \nabla v'_h) - ((I - P_H)f, v'_h) = -(u_{H,t}, v'_h) - a(u_H, v'_h),\ \forall\ v'_h \in X'_h. \tag{3.5}$$

Comparing (3.4), (3.5) to Definition 2.2 completes the proof. □

It was already noted in [17] that the interesting method of Guermond [10] can arise from the formulation (3.1), (3.2) by a different choice of finite element spaces. Thus, by essentially the same argument we obtain the analogous result for the method of Guermond [10].

Theorem 3.2 *The method of [10] is a generalized VMM in the sense of Definition 2.2.* □

4 Remarks

If we consider a complement of X_H in X_h to be the span of the usual, nodal basis functions associated with vertices in $\Pi_h(\Omega)$ <u>not</u> in $\Pi_H(\Omega)$, then it is clear that fluctuations in the method of [17] are only required to be zero on the vertices of $\Pi_H(\Omega)$ and not on all the edges of $\Pi_h(\Omega)$ or $\Pi_H(\Omega)$.

The computational feasibility of the method (3.1), (3.2) stems from the fact that the basis functions in L_H are supported on one (macro) element. It can be seen by rewriting it as:

$$(u_{h,t}, v_h) + a(u_h, v_h) + (\alpha \nabla u_h, \nabla v_h) = (f, v_h) - (\alpha P_H(\nabla u_h), P_H(\nabla v_h)). \tag{4.1}$$

The extra term $(\alpha P_H(\nabla u_h), P_H(\nabla v_h))$ is not convenient to assemble. However, since P_H is the L^2 orthogonal projection into piecewise constants, calculating a residual of $P_H(\nabla u_h)$ is simple, embarassingly parallel and cheap.

Thus, this term can simply be lagged (with low order timestepping). If a fully implicit method is used, a linear system must be solved which includes this term. An iterative method must be used. Each iteration requires a residual calculation with $P_H(\nabla u_h)$ which, as noted above, is cheap.

More generally, the computational feasibility of the method (4.1) arises from the fact that it uses a multiscale decomposition of ∇u_h

$$\nabla u_h = P_H(\nabla u_h) + (I - P_H)(\nabla u_h)$$

so that discontinuous elements may be used for $P_H(\nabla u_h)$ whereas continuous ones are required for u_h.

It is interesting that the definition of the large eddies by elliptic projection is precisely a discrete analog of Germano's idea of differential filtering [9], which has been developed in a discrete sense by Fischer and Mullen [7] and Dunca [5].

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