

Subgrid Stabilized Defect Correction Methods for the Navier-Stokes Equations

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Abstract

We consider the synthesis of a recent subgrid stabilization method with defect correction methods. The combination is particularly efficient and combines the best algorithmic features of each. We give a thorough numerical analysis of the combination and present some numerical tests which both verify the theoretical predictions and illustrate the methods promise.

1 Introduction

This report studies the synthesis of defect correction methods and subgrid stabilization. Our proposed method adds an eddy viscosity stabilization on only the last few resolved scales on arbitrary, unstructured meshes. Computational considerations for total algorithmic efficiency suggest combining the stabilization method with defect correction when solving *underresolved, equilibrium flow problems*. In this work, we study precisely this combination in that context. We analyze convergence of the combination for the (nonlinear) Navier-Stokes equations. This analysis gives mathematical guidance on the selection of the methods algorithmic parameters. In our accompanying tests, we observe that the subgrid stabilized defect correction method has greater accuracy than the artificial viscosity method without the oscillations reported in the usual (centered) Galerkin finite element method or the unmodified defect correction finite element method.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) denote a bounded, regular flow domain. We consider the approximate solution of the Navier-Stokes equations for internal flow on Ω : find $u : \Omega \rightarrow \mathbb{R}^d$, $p : \Omega \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} -\nu\Delta u + (u \cdot \nabla)u + \nabla p &= f && \text{in } \Omega, \\ \nabla \cdot u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \int_{\Omega} p \, dx &= 0. \end{aligned} \tag{1.1}$$

In (1.1), the coefficient ν is the kinematic viscosity of the fluid and $f \in L^2(\Omega)^d$ is the body force driving the flow.

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Let (\cdot, \cdot) , $\|\cdot\|$ denote the usual L^2 inner product and norm, respectively. Define $X := H_0^1(\Omega) := \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d \times d} \text{ and } v = 0 \text{ on } \partial\Omega\}$, $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0\}$, and define $b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v)$ for all $u, v, w \in X$. Integrating by parts gives the following standard variational formulation of (1.1): find $u \in X$, $p \in Q$ satisfying:

$$\begin{aligned} \nu(\nabla u, \nabla v) + b^*(u, u, v) - (p, \nabla \cdot v) &= (f, v) \quad \forall v \in X, \\ (\nabla \cdot u, q) &= 0 \quad \forall q \in Q. \end{aligned} \quad (1.2)$$

For the finite element discretization, we choose the conforming velocity-pressure finite element spaces, $X_h \subset X$ and $Q_h \subset Q$, satisfying the discrete inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\| \|\nabla v_h\|} \geq \beta > 0, \quad (1.3)$$

where β is independent of h .

The stabilization we consider requires a coarser finite element velocity space, $X_H \subset X$ corresponding to the large scales of the fluid velocity. Since finite element spaces are constructed based upon triangulations of the domain Ω , typically (although not necessarily for our analysis) $X_H \subset X_h \subset X$. We define the following space

$$L_H = \nabla X_H \subset L^2(\Omega)^{d \times d}. \quad (1.4)$$

The stabilized finite element method we consider herein is: find $u_h \in X_h$, $p_h \in Q_h$ and $g_H \in L_H$ satisfying

$$\begin{aligned} (\nu + \alpha)(\nabla u_h, \nabla v_h) + b^*(u_h, u_h, v_h) \\ - \alpha(g_H, \nabla v_h) - (p_h, \nabla \cdot v_h) &= (f, v_h) \quad \forall v_h \in X_h, \\ (\nabla \cdot u_h, q_h) &= 0 \quad \forall q_h \in Q_h, \\ (g_H - \nabla u_h, l_H) &= 0 \quad \forall l_H \in L_H, \end{aligned} \quad (1.5)$$

where α is the user-selected stabilization parameter and typically, $\alpha = \mathcal{O}(h)$. It is easy to verify that the last equality in (1.5) implies that g_H is the L^2 projection of ∇u_h , denoted by $\overline{\nabla u_h}$.

In a typical implementation of (1.5), the variables g_H, l_H in L_H are defined on macro-elements, i.e., elements of the coarse mesh. Thus, solving (1.5) involves coupling of micro-variables (functions in X_h, Q_h) across macro-elements. Thus, although these terms are cheap to evaluate in a residual calculation, the bandwidth of the linearized system arising from (1.5) increases substantially and the solution of the linear system containing these terms is not attractive. This issue is discussed briefly in Layton [31] and at some length John, Kaya and Layton [27] and Anitescu, Layton and Pahlevani[2] for the evolutionary problem.

For this reason, we consider a further defect correction discretization of (1.2) herein. This combination greatly increases efficiency by shifting the macro-micro coupling to a residual calculation. The method consists of an initialization step followed by k correction steps, where k is the local polynomial degree of X_h :

Initialization Step. Solve for $(u_h^1, p_h^1) \in (X_h, Q_h)$ such that

$$\begin{aligned} (\nu + \alpha)(\nabla u_h^1, \nabla v_h) + b^*(u_h^1, u_h^1, v_h) - (p_h^1, \nabla \cdot v_h) &= (f, v_h) \quad \forall v_h \in X_h, \\ (\nabla \cdot u_h^1, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned} \quad (1.6)$$

k - Correction Steps. Given $(u_h^j, p_h^j) \in (X_h, Q_h)$ for $j = 1, 2, \dots, k$, solve for $(u_h^{j+1}, p_h^{j+1}) \in (X_h, Q_h)$ satisfying

$$\begin{aligned} & (\nu + \alpha)(\nabla(u_h^{j+1} - u_h^j), \nabla v_h) + b^*(u_h^{j+1}, u_h^{j+1}, v_h) - b^*(u_h^j, u_h^j, v_h) - (p_h^{j+1} - p_h^j, \nabla \cdot v_h) \\ & = (f, v_h) - [(\nu + \alpha)(\nabla u_h^j, \nabla v_h) + b^*(u_h^j, u_h^j, v_h) - (p_h^j, \nabla \cdot v_h) - \alpha(g_H^j, \nabla v_h)] \quad \forall v_h \in X_h, \\ & \quad (\nabla \cdot u_h^{j+1}, q_h) = 0 \quad \forall q_h \in Q_h, \\ & \quad (g_H^j - \nabla u_h^j, l_H) = 0 \quad \forall l_H \in L_H. \end{aligned} \tag{1.7}$$

Remark 1.1. *It is typical that while defect correction methods are algorithmically simple to implement, they are cumbersome to write (as above) and can resist analysis. There are also several equivalent formulations of (1.7) and several algorithmic options within the defect correction idea. We stress that this is not an iteration: only k steps are performed where k is the local polynomial degree of X_h . Thus, an asymptotic analysis as $j \rightarrow \infty$ is irrelevant; we analyze herein the method as $h \rightarrow 0$ for fixed j .*

The algorithmic efficiency of the defect correction method (1.6), (1.7) can be seen by rewriting (1.7) as follows: find $(u_h^{j+1}, p_h^{j+1}) \in (X_h, Q_h)$ satisfying

$$\begin{aligned} & (\nu + \alpha)(\nabla u_h^{j+1}, \nabla v_h) + b^*(u_h^{j+1}, u_h^{j+1}, v_h) \\ & \quad - (p_h^{j+1}, \nabla \cdot v_h) = (f, v_h) + \alpha(g_H^j, \nabla v_h) \quad \forall v_h \in L_H, \\ & \quad (\nabla \cdot u_h^{j+1}, q_h) = 0 \quad \forall q_h \in Q_h, \\ & \quad (g_H^j - \nabla u_h^j, l_H) = 0 \quad \forall l_H \in L_H. \end{aligned} \tag{1.8}$$

Since u_h^j is known in (1.8), g_H^j is explicitly calculable by computing the L^2 projection operator of ∇u_h^j into L_H . Since L_H is (typically) a space of discontinuous piecewise polynomials of degree $k - 1$ on a coarse mesh, this projection calculation uncouples into one, well-conditioned small linear system per coarse mesh element. Given g_H^j , the solution u_h^{j+1} then only involves solving an artificial viscosity discretization of the Navier-Stokes equations. If $\alpha = \mathcal{O}(h)$, this is known to lead to linearized systems which can be solved efficiently.

An alternative formulation of defect correction method is to begin with nonlinear, stabilized artificial viscosity approximation (1.6) for (u_h^1, p_h^1) and then correct by solving the linearized problem instead of the nonlinear one. This has the advantage that only one linear solution is needed per correction step. This variation reads: given (u_h^j, p_h^j) find (u_h^{j+1}, p_h^{j+1}) satisfying:

$$\begin{aligned} & (\nu + \alpha)(\nabla(u_h^{j+1} - u_h^j), \nabla v_h) + b^*(u_h^j, u_h^{j+1} - u_h^j, v_h) + b^*(u_h^{j+1} - u_h^j, u_h^j, v_h) \\ & \quad - (p_h^{j+1} - p_h^j, \nabla \cdot v_h) = (f, v_h) - [(\nu + \alpha)(\nabla u_h^j, \nabla v_h) + b^*(u_h^j, u_h^j, v_h) \\ & \quad \quad - (p_h^j, \nabla \cdot v_h) - \alpha(g_H^j, \nabla v_h)] \quad \forall v_h \in X_h, \\ & \quad (\nabla \cdot u_h^{j+1}, q_h) = 0 \quad \forall q_h \in Q_h, \\ & \quad (g_H^j - \nabla u_h^j, l_H) = 0 \quad \forall l_H \in L_H. \end{aligned} \tag{1.9}$$

The correction (1.9) is in the familiar residual-update form. It can be simplified to read: find (u_h^{j+1}, p_h^{j+1}) satisfying

$$\begin{aligned} & (\nu + \alpha)(\nabla u_h^{j+1}, \nabla v_h) + b^*(u_h^j, u_h^{j+1}, v_h) + b^*(u_h^{j+1}, u_h^j, v_h) \\ & \quad - (p_h^{j+1}, \nabla \cdot v_h) = (f, v_h) + b^*(u_h^j, u_h^j, v_h) + \alpha(g_H^j, \nabla v_h) \quad \forall v_h \in X_h, \\ & \quad (\nabla \cdot u_h^{j+1}, q_h) = 0 \quad \forall q_h \in Q_h, \\ & \quad (g_H^j - \nabla u_h^j, l_H) = 0 \quad \forall l_H \in L_H. \end{aligned} \tag{1.10}$$

1.1 Literature Review for Defect Correction Method

The idea of defect correction is simple and universal. In its initial form, it was considered an algorithmically efficient way to perform Richardson's extrapolation, e.g. Stetter [38]. Since most practical problems do not have enough regularity, the practical importance was not recognized until the work of Hemker [21, 20] and Hemker and Koren [23, 22]. One current view of defect correction method is that it allows for a solution that is nearly nonsingular, ill-conditioned problems through stabilization and correction, for a sample of recent works, see e. g. Altase and Burrage [1], Axelsson and Nikolova [4], Juncu [28], Graziadei, Mattheij and Boonkamp [16], Heinrichs [19, 18], Desideri and Hemker [6], Nemedov and Mattheij [34], Shaw and Crumpton [37]. For example, when applied to viscoelastic fluid flow (Lee [32]), the defect correction method proved to be the key algorithmic idea for computing with a Weissenberg number beyond which other algorithms failed.

Much analytical insight into defect correction method was obtained early for periodic, constant coefficient problems by local model analysis. The first complete convergence theory for defect correction method for convection dominated problem in 1D was performed in Ervin and Layton [8] in which uniform epsilon convergence was proven away from layers. This result was extended to higher dimensions, higher order methods and unstructured meshes Axelsson and Layton [3]. Recently, global uniform in epsilon convergence on Shishkin meshes has been proven in 1D (Frohner, Linss and Roos [11], Frohner and Roos [12], Hemker, Shishkin and Shishkina [24]).

It was noticed early by Hemker [21], that defect correction method overcorrects near layers and should be modified. Various proposals have been advanced, e.g. Hemker [21, 20], Hemker and Koren [22, 23], Ervin and Layton [7]. The one considered herein is to correct the large scales only and leave a small amount of stabilization in the small scales. This is a discretization idea of Layton [31] which is related to ideas of Maday and Tadmor [33], Guermond [17], and Hughes, Mazzei and Jansen [25]. For current work on this discretization for flow problems, see, e.g. Kaya and Riviere [29], John and Kaya [26], John, Kaya and Layton [27].

Because of the attractive form of the defect correction method, it is particular efficient when used in conjunction with adaptivity. The first theoretical study and computational testing of defect correction plus adaptivity was in Ervin, Layton and Maubach [9, 10] and Cawood, Ervin, Layton and Maubach [5]. Interesting recent work in this direction has been done by Nikolova [35] and Axelsson and Nikolova [4]. In particular, [10] considers the problems of stationary turbulence with Smagorinsky model. It was noted there that the estimators decompose into residuals associated with the base discretization's numerical error, the defect correction method's update error and the turbulence model's modelling error - an interesting feature of both defect correction method and adaptive solution of various turbulence model.

2 Mathematical Preliminaries

The error analysis we shall perform for the method (1.6), (1.7) will be for nonsingular solutions of the Navier-Stokes equations (1.1), (1.2). We thus collect few useful facts about nonsingular solutions.

Definition 2.1. *Let V and V_h denote respectively the divergence free subspaces of X and*

X_h :

$$\begin{aligned} V &:= \{v \in X : (q, \nabla \cdot v) = 0, \quad \forall q \in Q\}, \\ V_h &:= \{v_h \in X_h : (q_h, \nabla \cdot v_h) = 0, \quad \forall q \in Q_h\}. \end{aligned}$$

Although typically $V_h \subsetneq V$, it is known that under the discrete inf-sup condition (1.3), functions in V are well approximated by ones in V_h (Girault and Raviart [14]).

Lemma 2.1. *Let the discrete inf-sup condition (1.3) holds. Then for any $v \in V$*

$$\inf_{v_h \in V_h} \|\nabla(v - v_h)\| \leq C(1 + \frac{1}{\beta}) \inf_{v_h \in X_h} \|\nabla(v - v_h)\|. \quad (2.1)$$

Proof. We refer to [14] for the proof of this lemma. □

We shall denote by M as a finite constant with

$$M = \sup_{u, v, w \in X} \frac{|b^*(u, v, w)|}{\|\nabla u\| \|\nabla v\| \|\nabla w\|}.$$

Definition 2.2. *u is a nonsingular solution of (1.1) if there is a $\mu = \mu(u, \nu) > 0$ such that*

$$\inf_{v \in V} \sup_{w \in V} \frac{\nu(\nabla v, \nabla w) + b^*(u, v, w) + b^*(v, u, w)}{\|\nabla v\| \|\nabla w\|} \geq \mu > 0. \quad (2.2)$$

Definition 2.3. *u is isolated solution of (1.1), if there is a $\delta > 0$ such that there exists no other solution u' of (1.1) with $\|\nabla(u - u')\| < \delta$.*

The following basic facts are known concerning the equilibrium Navier-Stokes equations (1.1).

Proposition 2.1. (a) *Given $f \in H^{-1}(\Omega)^d$ there exists at least one $(u, p) \in (X, Q)$ satisfying (1.1).*

(b) *For $\|f\|$ small enough, that solution is unique and nonsingular.*

(c) *There is an open dense subset $\mathcal{D} \subset H^{-1}(\Omega)^d$ such that for all $f \in \mathcal{D}$, all solutions of (1.1) are nonsingular and the number of solutions for each $f \in \mathcal{D}$ is finite and odd.*

(d) *A nonsingular solution is isolated.*

(e) *Let u be a nonsingular solution of (1.1) with data f and \tilde{u} another solution with data \tilde{f} . If $\|\nabla(u - \tilde{u})\| \leq \mu(u)/4M$ then*

$$\|\nabla(u - \tilde{u})\| \leq \frac{2}{\mu(u)} \|f - \tilde{f}\|_{-1}.$$

Proof. (a), (b), (c) are well known in the Navier-Stokes equations literature, see, e.g., [14] for (a) and (b) and Temam [39] for (c). Part (e) was proven in Layton [30] and part (d) is a standard result about nonsingular solutions of nonlinear problems. □

Since the set of invertible operators is open and $b^*(\cdot, \cdot, \cdot)$ is continuous in X , it is known that the point of linearization in various terms of (2.2) can be shifted slightly without changing the essential conclusions.

Lemma 2.2. *Let u be a nonsingular solution of (1.1). Then there is a $\delta > 0$ such that for any $\alpha < \delta$, u' and $u'' \in V$ with $\|\nabla(u - u')\| < \delta$, $\|\nabla(u - u'')\| < \delta$ satisfying*

$$\inf_{v \in V} \sup_{w \in V} \frac{(\nu + \alpha)(\nabla v, \nabla w) + b^*(u', v, w) + b^*(v, u'', w)}{\|\nabla v\| \|\nabla w\|} \geq \frac{1}{2} \mu(u).$$

Proof. This is a standard result in nonlinear analysis (Schwartz [36]). \square

It will be important to note that if (1.3) holds the infimum and supremum in Lemma (2.2) can also be taken over V_h .

Lemma 2.3. *Let u be a nonsingular solution of (1.1). Then there is a $\delta > 0$ such that for any $\alpha < \delta$, u' and $u'' \in V$ or V_h with $\|\nabla(u - u')\| < \delta$, $\|\nabla(u - u'')\| < \delta$ satisfying*

$$\inf_{v_h \in V_h} \sup_{w_h \in V_h} \frac{(\nu + \alpha)(\nabla v_h, \nabla w_h) + b^*(u', v_h, w_h) + b^*(v_h, u'', w_h)}{\|\nabla v_h\| \|\nabla w_h\|} \geq \frac{1}{2} \mu(u).$$

Proof. For the proof see e.g. Girault and Raviart [15]. \square

3 Error Analysis

The basic principle of defect correction method in this context is that each step attempts to increase the rate of convergence by one power of h up to the order of the basic method. To begin, note that (u_h^1, p_h^1) is just the usual artificial viscosity approximation to (u, p) . Since the error analysis for this step is standard (and a special case of the general step in which $(u_h^0, p_h^0) = (0, 0)$), we present the result only. The error analysis uses basic tools from [14], [15] and requires a few basic assumptions on (X_h, Q_h) that assume that (1.3) holds and that (X_h, Q_h) become dense in (X, Q) as $h \rightarrow 0$. Specifically, we assume:

Proposition 3.1. *Let (1.3) holds. For any $\alpha \geq 0$ and $f \in H^{-1}(\Omega)^d$, the algorithm (1.6), (1.7) is well-defined: there exist approximate solutions (u_h^j, p_h^j) for $j = 1, 2, \dots$.*

Proof. Existence of (u_h^1, p_h^1) follows from the fact that (X_h, Q_h) is finite dimensional, the fixed point theory and the following a priori bounds

$$(\alpha + \nu) \|\nabla u_h^1\| \leq \|f\|_{-1}, \tag{3.1}$$

$$\|p_h^1\| \leq \beta^{-1} (2 + M(\alpha + \nu)^{-2}) \|f\|_{-1}. \tag{3.2}$$

The first result (3.1) is obtained by setting $v_h = u_h^1$ in (1.1). The second (3.2) follows from (1.3), exactly as for the usual Galerkin approximation.

Given $(u_h^j, p_h^j) \in (X_h, Q_h)$ the same argument can be only used in the formulation (1.8) to prove existence of (u_h^{j+1}, p_h^{j+1}) provided only $\|g_H^j\|$ is bounded. To see this, note that in the second equation of (1.8), g_H^j is the L^2 projection into L_H of ∇u_h^j . Thus, $\|g_H^j\| \leq \|\nabla u_h^j\| < \infty$, which is the required bound. \square

A similar result is true for the defect correction using the linearization (1.6), (1.9) provided h is small enough.

Proposition 3.2. *Let (1.3) holds. Consider the algorithm (1.6), (1.9) (or equivalently, (1.6), (1.10)). Assume that u is a nonsingular solution of (1.1). For any $f \in H^{-1}(\Omega)$ let $\alpha \geq 0$ tend to zero as h tends to zero. Then, there is $h_0 > 0$ such that for $h \leq h_0$, (u_h^2, p_h^2) exists and is unique. More generally, if u_h^j is close enough to u in X , then (u_h^{j+1}, p_h^{j+1}) exists and is unique.*

Proof. This is a linearization argument. First we note that since (u_h^1, p_h^1) is the artificial viscosity approximation to a nonsingular solution, standard error analysis of branches of nonsingular solution, e.g., [14], [15], shows that $\lim u_h^1 = u$ as h tends to 0, (see Proposition 3.3, which follows for more detail). Thus, by Lemma 2.3, the linearization (1.10) is invertible for h small enough. Thus, (u_h^2, p_h^2) exists uniquely.

The remainder of the proof is an induction argument which follows similarly: once u_h^2 exists uniquely and the linearization (1.10) at u_h^1 is invertible, it will follow that u_h^2 converges to u as h tends to 0 (with appropriate error estimates). This implies that u_h^3 exists uniquely and the argument is repeated.

This argument fails if (1.10) is an *iteration* but it is correct since it is a *correction* only performed a fixed number of times. \square

Concerning the error in u_h^1 , we have the following proposition.

Proposition 3.3. *Assume the spaces (X_h, Q_h) satisfy (1.3). Suppose u is a nonsingular solution of the (1.1). Suppose α tends to 0 as h tends to 0. Then, there is $h_0 > 0$ such that for $h \leq h_0$, the error in (u_h^1, p_h^1) satisfies:*

$$\begin{aligned} \|\nabla(u - u_h^1)\| &\leq C(\beta) \left[\frac{2}{\mu(u)} (\alpha + \nu + \frac{2M}{\nu} \|f\|_{-1}) + 1 \right] \inf_{v_h \in X_h} \|\nabla(u - v_h)\| \\ &\quad + \frac{2}{\mu(u)} \left[\inf_{\lambda_h \in Q_h} \|p - \lambda_h\| + \alpha \|\nabla u\| \right], \\ \|p - p_h^1\| &\leq \left(1 + \frac{1}{\beta}\right) \inf_{\lambda_h \in Q_h} \|p - \lambda_h\| + \frac{1}{\beta} (\nu + \alpha + \frac{2M}{\nu} \|f\|_{-1}) \|\nabla(u - u_h^1)\| + \frac{\alpha}{\beta} \|\nabla u\|. \end{aligned}$$

Proof. The proof that $u_h^1 \rightarrow u$ is standard, following, e.g., [14], [15]. We shall only thus give the proof of the error bound since it gives the ideas of the proof of the general case in a simpler context. The true solution (u, p) satisfies for any $v_h \in V_h$, $\lambda_h \in Q_h$:

$$(\nu + \alpha)(\nabla u, \nabla v_h) + b^*(u, u, v_h) - (p - \lambda_h, \nabla \cdot v_h) = (f, v_h) + \alpha(\nabla u, \nabla v_h). \quad (3.3)$$

Let $\tilde{u} \in V_h$ be an approximation to u and write $e^1 = u - u_h^1 = \eta - \phi_h$ where $\phi_h = u_h^1 - \tilde{u}$ and $\eta = u - \tilde{u}$. Subtracting from (3.3), the equation (1.6) for (u_h^1, p_h^1) gives:

$$(\nu + \alpha)(\nabla e^1, \nabla v_h) + b^*(u, u, v_h) - b^*(u_h, u_h, v_h) = (p - \lambda_h, \nabla \cdot v_h) + \alpha(\nabla u, \nabla v_h). \quad (3.4)$$

The nonlinear term can be rewritten as

$$\begin{aligned} b^*(u, u, v_h) - b^*(u_h, u_h, v_h) &= b^*(e^1, u, v_h) + b^*(u_h, e^1, v_h) \\ &= b^*(\eta, u, v_h) - b^*(\phi_h, u, v_h) + b^*(u_h, \eta, v_h) - b^*(u_h, \phi_h, v_h). \end{aligned}$$

Using this decomposition of $b^*(\cdot, \cdot, \cdot)$ and splitting $e^1 = \eta - \phi_h$ gives

$$\begin{aligned} (\nu + \alpha)(\nabla \phi_h, \nabla v_h) + b^*(\phi_h, u, v_h) + b^*(u_h, \phi_h, v_h) &= (\nu + \alpha)(\nabla \eta, \nabla v_h) + b^*(\eta, u, v_h) \\ + b^*(u_h, \eta, v_h) - (p - \lambda_h, \nabla \cdot v_h) - \alpha(\nabla u, \nabla v_h), \quad \forall (v_h, \lambda_h) \in (V_h, Q_h). \end{aligned} \quad (3.5)$$

Applying standard bounds to the right hand side of (3.5) gives

$$\begin{aligned} & \frac{1}{\|\nabla v_h\|} [(\nu + \alpha)(\nabla \phi_h, \nabla v_h) + b^*(\phi_h, u, v_h) + b^*(u_h, \phi_h, v_h)] \\ & \leq (\nu + \alpha)\|\nabla \eta\| + M(\|\nabla u\| + \|\nabla u_h\|)\|\nabla \eta\| + \|p - \lambda_h\| + \alpha\|\nabla u\|. \end{aligned}$$

Taking supremum over the $v_h \in V_h$, using Lemma 2.3 and á priori bounds on $\|\nabla u\|$ and $\|\nabla u_h\|$ yield

$$\frac{1}{2}\mu(u)\|\nabla \phi_h\| \leq [\alpha + \nu + M\|f\|_{-1}(\frac{1}{\nu} + \frac{1}{\nu + \alpha})]\|\nabla \eta\| + \|p - \lambda_h\| + \alpha\|\nabla u\|.$$

By using triangle inequality, taking the infimum over $v^h \in V^h$, $\lambda^h \in Q^h$ and using Lemma 2.1, one obtains the required result.

For the pressure estimate (just as for the Stokes problem) we begin with the error equation for $v_h \in X_h$ (rather than V_h):

$$(p - p_h^1, \nabla \cdot v_h) = (\nu + \alpha)(\nabla e^1, \nabla v_h) + b^*(e^1, u, v_h) - b^*(u_h, e^1, v_h) - \alpha(\nabla u, \nabla v_h).$$

Write $p - p_h^1 = p - \lambda_h - (p_h^1 - \lambda_h)$ where $\lambda_h \in Q_h$ approximates p well. Then

$$\begin{aligned} (p_h - \lambda_h, \nabla \cdot v_h) &= (p - \lambda_h, \nabla \cdot v_h) - (\nu + \alpha)(\nabla e^1, \nabla v_h) \\ &\quad - b^*(e^1, u, v_h) + b^*(u_h, e^1, v_h) + \alpha(\nabla u, \nabla v_h) \\ &\leq [\|p - \lambda_h\| + (\nu + \alpha)\|\nabla e^1\| + M(\|\nabla u\| + \|\nabla u_h\|)]\|\nabla e^1\| \\ &\quad + \alpha\|\nabla u\|\|\nabla v_h\|. \end{aligned}$$

Dividing by $\|\nabla v_h\|$, taking the supremum over $v_h \in X_h$, using the inf-sup condition (1.3) and the triangle inequality, we have

$$\|p - p_h\| \leq (1 + \frac{1}{\beta})\|p - \lambda_h\| + \frac{1}{\beta}(\nu + \alpha)\|\nabla e^1\| + \frac{M}{\beta}(\|\nabla u\| + \|\nabla u_h\|)\|\nabla e^1\| + \frac{\alpha}{\beta}\|\nabla u\|.$$

Finally, using á priori bounds (3.1), $\nu\|\nabla u\| \leq \|f\|_{-1}$ gives the required result. \square

Concerning the error in the method we consider the variant (1.6), (1.9) in which one linearized problem is solved per correction step. Intuitively, one would expect that the defect correction method (1.6), (1.7) would be more robust and more accurate. On the other hand, complete error analysis of defect correction method with nonlinear correction (1.6), (1.7) is an open problem in the case of large data and nonsingular solutions.

Proposition 3.4. *Consider (1.6), (1.9). Let u be a nonsingular solution of the (1.1) and suppose (1.3) holds. Then, there is $\delta > 0$ such that if $\|\nabla(u - u_h^j)\| < \delta$, for $j = 1, 2, \dots$:*

$$\begin{aligned} \frac{1}{2}\mu(u)\|\nabla(u - u_h^{j+1})\| &\leq C(\beta)(\nu + \alpha + 2M(\delta + \|\nabla u\|)) \inf_{v_h \in X_h} \|\nabla(u - v_h)\| \\ &\quad + \inf_{\lambda_h \in Q_h} \|p - \lambda_h\| + \alpha\|\nabla u - \overline{\nabla u}\| \\ &\quad + M\|\nabla(u - u_h^j)\|^2 + \alpha\|\overline{\nabla(u - u_h^j)}\|, \\ \beta\|p - p_h^{j+1}\| &\leq C \inf_{\lambda_h \in Q_h} \|p - \lambda_h\| + [\nu + 2\alpha + 2M\|\nabla u\|]\|\nabla(u - u_h^{j+1})\| \\ &\quad + M\|\nabla(u - u_h^j)\|^2 + \alpha\|\overline{\nabla(u - u_h^j)}\| + \alpha\|\nabla u - \overline{\nabla u}\|. \end{aligned}$$

Proof. The variational formulation of (1.1) can be rewritten as follows: for any $v_h \in V_h$ and $\lambda_h \in Q_h$,

$$\begin{aligned} & (\nu + \alpha)(\nabla u, \nabla v_h) + b^*(u_h^j, u, v_h) + b^*(u, u_h^j, v_h) - (p - \lambda_h, \nabla \cdot v_h) \\ = & (f, v_h) + [b^*(u_h^j, u, v_h) + b^*(u, u_h^j, v_h) - b^*(u, u, v_h)] \\ & + \alpha(\overline{\nabla u}, \nabla v_h) + \alpha(\nabla u - \overline{\nabla u}, \nabla v_h). \end{aligned} \quad (3.6)$$

The square bracketed term on the right hand side of (3.6) becomes:

$$b^*(u_h^j, u, v_h) + b^*(u, u_h^j, v_h) - b^*(u, u, v_h) = -b^*(u - u_h^j, u - u_h^j, v_h) + b^*(u_h^j, u_h^j, v_h). \quad (3.7)$$

Let $e^{j+1} = u - u_h^{j+1}$, $e^j = u - u_h^j$ and note that $g_H = \overline{\nabla u_h}$ (by (1.10)). With this notation subtract (1.10) from the equation (3.6) and use the last equation (3.7) for the nonlinear terms on the right hand side. This gives:

$$\begin{aligned} & (\nu + \alpha)(\nabla e^{j+1}, \nabla v_h) + b^*(u_h^j, e^{j+1}, v_h) + b^*(e^{j+1}, u_h^j, v_h) \\ = & (p - \lambda_h, \nabla \cdot v_h) - b^*(e^j, e^j, v_h) + \alpha(\overline{\nabla e^j}, \nabla v_h) \\ & + \alpha(\nabla u - \overline{\nabla u}, \nabla v_h) \quad \forall (v_h, \lambda_h) \in (V_h, Q_h). \end{aligned}$$

The remainder of the proof follows that of Proposition 3.3: We first split $e^{j+1} = \eta - \phi_h$, $\eta = u - \tilde{u}$ and $\phi_h = u_h^{j+1} - \tilde{u}$ where $\tilde{u} \in V_h$ approximates u well. By using this decomposition, nonlinear terms can be written as

$$b^*(u_h^j, e^{j+1}, v_h) + b^*(e^{j+1}, u_h^j, v_h) = b(u_h^j, \eta, v_h) - b(u_h^j, \phi_h, v_h) + b(\eta, u_h^j, v_h) - b(\phi_h, u_h^j, v_h).$$

Then, the use of splitting error and Lemma 2.3 give the following inequality for ϕ_h :

$$\begin{aligned} \frac{1}{2}\mu(u)\|\nabla\phi_h\| & \leq (\nu + \alpha + \|\nabla u_h^j\|)\|\nabla\eta\| \\ & \quad + \|p - \lambda_h\| + M\|\nabla e^j\|^2 + \alpha\|\overline{\nabla e^j}\| + \alpha\|\nabla u - \overline{\nabla u}\|. \end{aligned}$$

Since u_h^j is close enough to $u \in X$, we have $\|\nabla u_h^j\| \leq \delta + 2\|\nabla u\|$. The triangle inequality then implies

$$\begin{aligned} \frac{1}{2}\mu(u)\|\nabla e^{j+1}\| & \leq (\nu + \alpha + \frac{1}{2}\mu(u) + 4M\|\nabla u\|)\|\nabla\eta\| \\ & \quad + \|p - \lambda_h\| + M\|\nabla e_j\|^2 + \alpha\|\overline{\nabla e^j}\| + \alpha\|\nabla u - \overline{\nabla u}\|. \end{aligned}$$

The pressure bound also follows from the case $j = 1$. □

The error estimate Proposition 3.4 has four terms. The first term $\|\nabla(u - v_h)\|$ is the error in the best approximation to $(u, p) \in (X_h, Q_h)$. The second term $\|\nabla e_j\|^2$ is the linearization error. Since this is quadratic, it is typically a higher order term. The third, $\alpha\|\overline{\nabla e_j}\|$, shows that each step of defect correction method improves the error in the previous step by one power of h (recall that typically $\alpha = \mathcal{O}(h)$). The last term $\alpha\|\nabla u - \overline{\nabla u}\|$ arises from the error of the stabilized discretization.

As a result of the Proposition 3.4, we can give the following corollaries.

Corollary 3.1. *In addition to the assumptions of the Proposition 3.4, suppose $h \leq H \rightarrow 0$ as $h \rightarrow 0$ and $\alpha \rightarrow 0$ as $h \rightarrow 0$. Suppose also X_h, Q_h, L_H become dense in X, Q and L respectively as $h \rightarrow 0$. Then, there is a $h_0 > 0$ such that for $h \leq h_0$, and $j = 1, 2, \dots, k$, $u_h^j \rightarrow u$ as $h \rightarrow 0$.*

Corollary 3.2. *Suppose X_h, Q_h, L_H consist of piecewise polynomials of degree $k, k-1$ and $k-1$ respectively. Suppose also $u \in H^{k+1}(\Omega) \cap X, p \in H^k(\Omega) \cap Q$. Then,*

$$\|\nabla(u - u_h^1)\| \leq C(u, p)[h^k + \alpha],$$

and in general,

$$\|\nabla(u - u_h^j)\| \leq C(u, p, j)[h^k + \alpha H^k + \alpha^j].$$

This follows by inserting the approximation theoretical orders of convergence into the right hand side of the Proposition 3.4 and keeping only dominant terms. For example,

$$\begin{aligned} \|\nabla(u - u_h^2)\| &\leq C(u, p)[h^k + (h^k + \alpha)^2 + \alpha(h^k + \alpha) + \alpha H^k] \\ &\leq C(u, p)[h^k + \alpha H^k + \alpha^2]. \end{aligned}$$

The error estimate in the corollary explains the typical algorithmic choices:

$$\begin{aligned} j &\geq k : \text{correction step,} \\ \alpha &= \alpha_0 h : \text{regularization parameter,} \\ H &\leq C h^{1-\frac{1}{k}} : \text{lengthscale of large structures.} \end{aligned}$$

4 Numerical Studies

In this section, we consider some numerical experiments for the implementation of defect correction algorithm proposed with (1.6), (1.10). In particular, we present two numerical examples: one is a known analytical solution; and the other is the driven cavity problem.

All computations are carried out in the domain $\Omega = [0, 1] \times [0, 1]$. We divide our domain into triangles. We use Taylor-Hood elements i.e., continuous piecewise quadratic functions for the velocity and continuous piecewise linear functions for the pressure. It is well-known that this conforming pair of finite element spaces satisfies the inf-sup condition (1.3). For every grid, first artificial viscosity system (1.6) is solved, with $\alpha = h$. Then, k (the polynomial degree of the velocity approximation) defect correction steps are performed. Hence, two corrections steps are required for the Taylor-Hood element. All the nonlinear systems are solved by using the Newton method with stopping criteria 10^{-6} . Corollary 3.2 suggests that the algorithmic choices for h and H should be $h \sim H^2$ or equivalently, $H \sim h^{1/2}$ in order to obtain optimal error rates.

As a first numerical illustration, we study a numerical convergence to confirm the error estimate given in Proposition 3.4. The prescribed solution is given by

$$u = -4y^3x^2, \quad v = 2xy^4, \quad p = 2x + 3y - 2.$$

Dirichlet boundary conditions are chosen and the right hand side f is such that (u, v, p) is the solution of (1.1). In this example, our numerical results are performed for $\nu = 1$. For the Taylor-Hood finite element spaces, the theory predicts a convergence rate of $\mathcal{O}(h^2)$ in the energy norm, $\mathcal{O}(h^3)$ in the L^2 norm for the velocity and $\mathcal{O}(h^2)$ for the pressure.

Note that, since we try to verify the theory in this simplest setting, the first numerical test problem does not require either subgrid eddy viscosity method or defect correction method for an accurate solution. However, the method (1.6), (1.10) is fully comparable

to the standard finite approach in this laminar case. In Table 1, the error in the usual Galerkin discretization of the Navier-Stokes equations is presented. In particular, we give L^2 and H_0^1 errors and the corresponding convergence rates. As theory predicts, the optimal convergence rates are obtained. In Table 2, we present convergence rates by using the

h	$L^2 - error$	Rate	$H_0^1 - error$	Rate	L^2 pressure	Rate
h=1/2	0.0093		0.2894		0.2934	
h=1/4	0.0011	3.07	0.0710	2.02	0.0786	1.90
h=1/8	1.3181e-004	3.06	0.0177	2.00	0.0200	1.97
h=1/16	1.6321e-005	3.01	0.0044	2.00	0.0051	1.97

Table 1: Convergence rates by using the Galerkin finite element method.

artificial viscosity method (AV) where we perform only initialization step (1.6) to solve Navier-Stokes equations. Since we choose $\alpha = h$, it is expected and observed that the convergence rates for this method are suboptimal. In Table 3, the experimental rates of

h	$L^2 - error$	Rate	$H_0^1 - error$	Rate	L^2 pressure	Rate
h=1/4	0.0061		0.0844		0.4671	
h=1/8	0.0032	0.93	0.0306	1.46	0.2146	1.12
h=1/16	0.0016	1.00	0.0137	1.15	0.1023	1.06

Table 2: Convergence rates by using the artificial viscosity method.

convergence for the subgrid stabilized defect correction method are presented. The scalings between coarse and fine mesh are chosen such that $H \leq h^{1/2}$ satisfied. These numerical results demonstrate that the rates of convergence are optimal, as the theory predicts. Hence, the stabilization used in the method (1.6), (1.10) does not degrade rates of convergence in laminar flows.

H	h	$L^2 - error$	Rate	$H_0^1 - error$	Rate	L^2 pressure	Rate
1/2	1/4	0.0012		0.0701		0.0824	
1/4	1/8	1.4146e-004	3.08	0.0174	2.01	0.0224	1.87
1/8	1/16	1.7565e-005	3.01	0.0043	2.01	0.0059	1.92
1/16	1/32	2.1926e-006	3.00	0.0011	1.96	0.0015	1.97

Table 3: Convergence rates of the subgrid stabilized defect correction method.

Our second example is the benchmark problem of the under-resolved driven cavity at high Reynold numbers. In this example, flow is driven by the tangential velocity field applied to the top boundary in the absence of other body forces. On the segment $\{(x, 1) : 0 < x < 1\}$, the velocity is equal to $u = (1, 0)$. On the rest of the boundary, zero Dirichlet conditions are imposed.

The drawbacks of usual, centered Galerkin methods for convection dominated problems are well known and well documented. Also, the drawbacks of unmodified defect correction method, although less well known, are very well documented since the 1982 work of Hemker

[21, 20]. Thus, the focus of our experiments on the driven cavity problem are to (i) show that subgrid stabilized defect correction method gives high quality, coarse mesh solutions (comparable to the benchmark, fine mesh results of Ghia, Ghia and Shen [13]), (ii) illustrate that stabilization of the finest resolved scales is effective in suppressing the oscillations on the scales typical of unstabilized defect correction method and (iii) illustrate the very substantial improvement in the results produced by the relatively inexpensive correction steps.

We compute an approximate solution for $\nu = 10^{-2}$ and $\nu = 25 \times 10^{-4}$ for the driven cavity flow. In particular, we draw the x component of velocity along the vertical centerline and y component of velocity along the horizontal centerlines. We compare the results obtained by Ghia, Ghia Shin's [13]. The present numerical simulations are considered on very coarse mesh ($h = 1/16, h = 1/32$) and they are compared to very fine mesh ($h = 1/129$) of [13]. Ghia's algorithm is based on the time dependent streamfunction using the coupled implicit and multigrid methods. Their results are used as benchmark data as basis for comparison.

In Figures 1-4, we compare the results obtained by the artificial viscosity method (AV), the subgrid stabilized defect correction method (1.6), (1.10) and the results of [13]. In the case $\nu = 10^{-2}$, there is very little difference between the vertical midlines for all three methods (Fig. 1). For the horizontal midlines, the subgrid stabilized defect correction method is closer to Ghia, Ghia Shin's results than the artificial viscosity (see Fig. 2).

In the case of $\nu = 25 \times 10^{-4}$, namely for higher Reynolds number, Fig. 3 and Fig. 4 clearly show that the subgrid stabilized defect correction method performs much better than the artificial viscosity method; and is comparable to the results obtained by Ghia, Ghia Shin on a more refined mesh. We observe that artificial viscosity method is overly diffusive.

5 Conclusion

The natural combination of defect correction with multiscale stabilization retains the best features of both methods and overcomes many of their deficits. The combination is accurate, efficient and a convergence theory of the combination is developed. This latter theory shows that the good accuracy and stability properties are no accident: they are general features of the method.

This combination has strong promise but many open questions remain including the correct extension of the method to time dependent problems, further numerical analysis (other norms, error functionals, ...) and more extensive testing.

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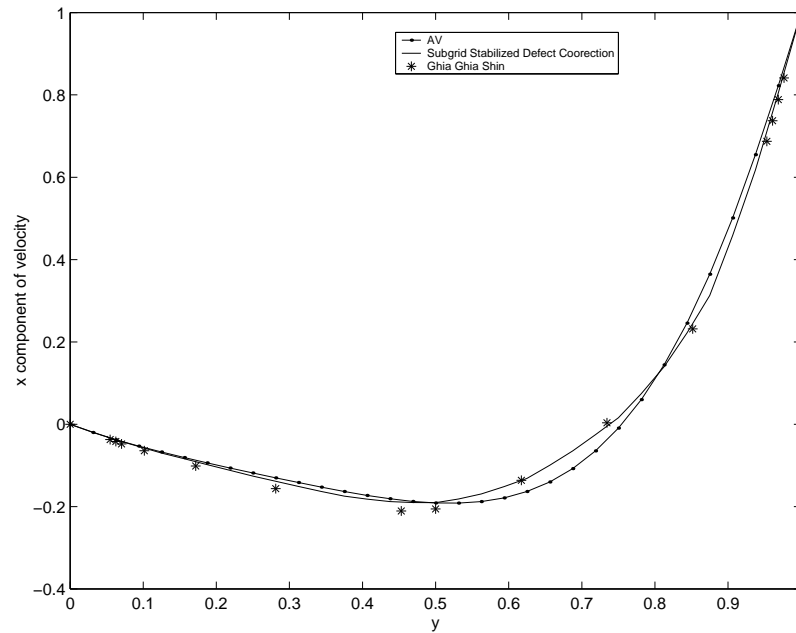


Figure 1: Vertical midlines for $\nu = 10^{-2}$ for $H = 1/8, h = 1/16$.

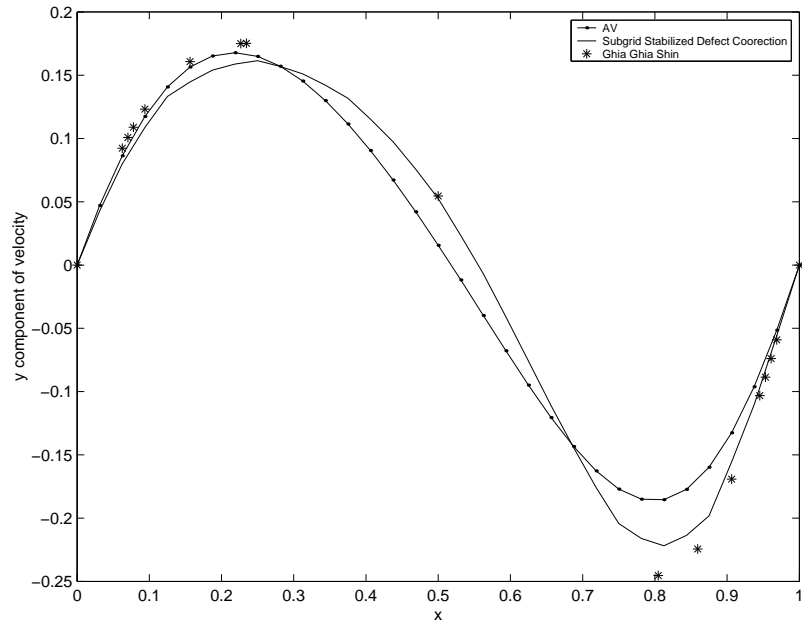


Figure 2: Horizontal midlines for $\nu = 10^{-2}$ for $H = 1/8$, $h = 1/16$.

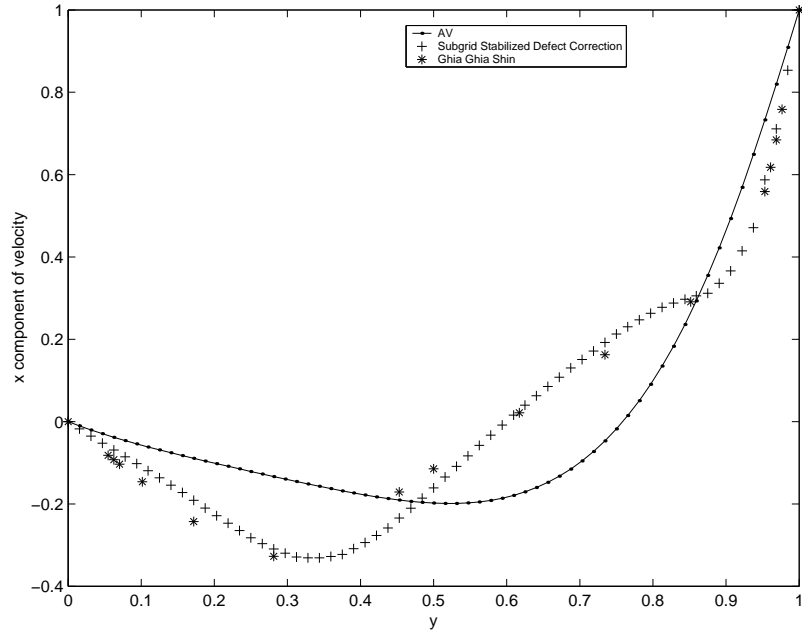


Figure 3: Vertical midlines for $\nu = 25 \times 10^{-4}$ for $H = 1/16$, $h = 1/32$.

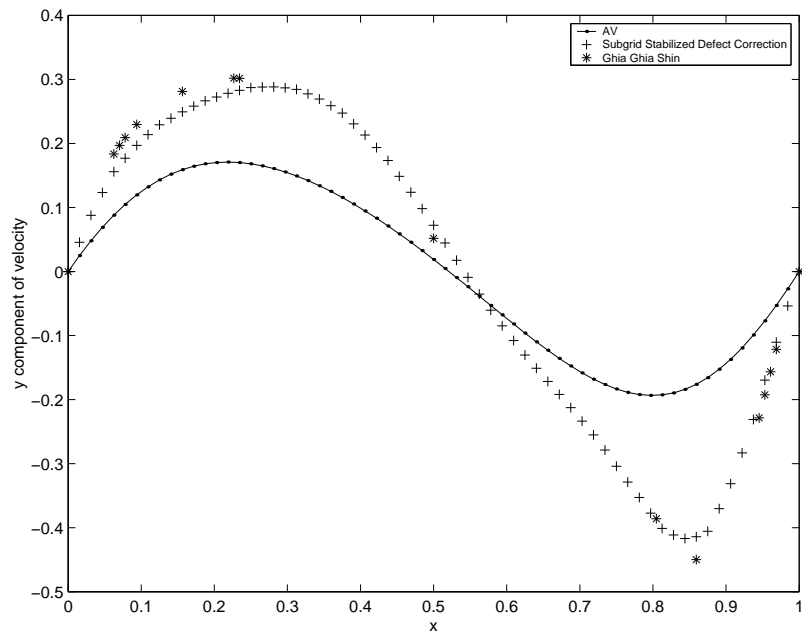


Figure 4: Horizontal midlines for $\nu = 25 \times 10^{-4}$ for $H = 1/16, h = 1/32$.