AN EXPOSITION ON THE TIME-AVERAGED ACCURACY OF APPROXIMATE DE-CONVOLUTION, LARGE EDDY SIMULATION MODELS OF TURBULENCE

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Abstract. The purpose of this report is to give a self-contained and detailed mathematical introduction to the analysis in the report [LL04c] of the accuracy of some predictive model's of turbulence. The models are based on approximate de-convolution methods and were introduced into LES by Stolz and Adams. We recall the development of the models, review the known theory of the models and expand the proof from [LL04c] that the time averaged consistency error of the Nth approximate deconvolution LES model converges to zero uniformly in the kinematic viscosity and in the Reynolds number as the cube root of the averaging radius. We also give a higher order but non-uniform consistency error bound for the zeroth order model directly from the Navier-Stokes equations and study the distribution of consistency errors among length-scales.

Key words. large eddy simulation, approximate deconvolution model, turbulence

1. Introduction.

Direct numerical simulation of turbulent flows of incompressible, viscous fluids is often not computationally economical or even feasible. Thus, various turbulence models are used for simulations seeking to predict flow statistics or averages. In LES (large eddy simulation) the evolution of local, spatial averages is sought. The goal of large eddy simulation is thus to predict reliably the evolution of large scales of turbulent flows. Thus, LES is inherently concerned with dynamics of turbulence. The definition of large scales is done through a local, spacial averaging process associated with an averaging radius δ. The selection of this averaging radius δ is determined typically by three factors: computational resources i.e. δ must be related to the finest computationally feasible mesh, turnaround time needed for the calculation, and estimates of the scales of the persistent eddies needed to be resolved for accurate simulation. Three types of averaging processes are common and in large eddy simulations: filtering by convolution, differential filters and filtering by projection.

Once an averaging radius and a filtering process is selected, an LES model most be developed. Once a model is selected, important and fundamental questions arise concerning whether the model correctly captures the global energy balance of the large scales. Mathematically, this is expressed as the question.

- Does the LES model have unique, strong solutions which depend stably on the problem data?

Broadly, there are two types of LES models of turbulence: *descriptive* or phenomenological models (e.g., eddy viscosity models) and *predictive* models (considered herein). The accuracy of a model measured in a chosen norm, $||\cdot||$, mean

$$||average(NSE\ solution) - LES\ solution||.$$

A model's accuracy can be assessed in several experimental and analytical ways. The strongest test is to obtain an NSE solution from a data base, a DNS or experimental data from a real flow, explicitly average it and compare it to the velocity

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predicted by a large eddy simulation[1]. Another important experimental approach is to use a velocity field from a direct numerical simulation (a DNS) to compute some norm of the model's *consistency error* (defined precisely below) evaluated at the DNS solution (which is regarded as a "truth solution")[2]. Either experimental approach of course limits assessments to Reynolds numbers and geometries for which reliable DNS data is available. A third complementary approach (for which there are currently few results) is to study analytically the model's *consistency error* as a function of the averaging radius δ and the Reynolds number Re. The inherent difficulties in analytical studies are that

- consistency error bounds for infinitely smooth functions hardy address essential features of turbulent flows such as irregularity and richness of scales, and
- worst case bounds for general weak solutions of the Navier Stokes equations are so pessimistic as to yield little insight.

However, it is known that after time or ensemble averaging, turbulent velocity fields are often observed to have an intermediate smoothness as predicted by the Kolmogorov theory (often called the K41 theory), see, for example, [F95],[BIL04],[P00], [S01] , [Les97]. This case is often referred to as homogeneous isotropic turbulence and various norms of flow quantities can be estimated in this case using the K41 theory, Parseval's equality and spectral integration. We mentioned Lilly's famous paper [L67] as an early and important example.

In this report we consider this third way begun in [LL04b] and carried out in [LL04c, LL04c]: consistency error bounds are developed for *time averaged*, fully developed, homogeneous, isotropic turbulence. Such bounds are inherently interesting and they also help answer two important related questions of *accuracy* and *feasibility* of large eddy simulation:

- How small must δ be with respect to Re to have the average consistency error $<< O(1)$?
- Can consistency error $<< O(1)$ be attained for the cutoff length-scale δ within the inertial range?

A related issue is to asses the accuracy of the model in laminar flow regions.

- For smooth u , what is the order of accuracy of the model?

1.1. Summary of results. The notation, formalism, results, proofs and conclusions of this report will be carefully developed in detail in the following sections. However, many interested in de-convolution models and large eddy simulation will be familiar with our notation and the K41 formalism. For those specialists, it is useful to give here a summary of results of the report. We stress that for those not familiar with de-convolution models or the K41 formalism, e.g., as in Lilly's paper, [L67], and for those who are familiar and are interested in the details why of the results are true, we shall develop the results carefully in a complete and self-contained way.

To present the results, let

U = *reference velocity,*

L = *reference global length scale,*

ν = *kinematic viscosity,*

[1] Because of the uncertainties and fluctuations of turbulent flow, often (or usually) "compare" means to compare time-averaged flow statistics rather than compute the time evolution of norms of differences.

[2] These two approaches are known in LES as a priori testing and a posteriori testing of the model under consideration.

$< \cdot >=$ *long time averaging.*

For the generality we consider, a good candidate for U to non-dimensionalize the equations is

$$U =< \frac{1}{L^3}||u(\mathbf{x},t)||^2_{L^2(\Omega)} >^{\frac{1}{2}} .$$

The first question of existence, uniqueness and accuracy of the model was answered in a remarkable paper of Dunca and Epshteyn [DE04] in the following theorem.

THEOREM 1.1. *Consider the Nth approximate de-convolution LES model under periodic boundary conditions and with smooth enough initial condition and body force. For every N this model has a unique strong solution. If the problem data $u_0 \in C^\infty(\Omega)$ and $f \in C^\infty(\Omega \times (0,\infty))$ then the solution is also $C^\infty(\Omega \times (0,\infty))$.*

Let $w = w(\delta)$ denote this solution. Then, there is a subsequence $\delta_j \to 0$ and a weak solution of the Navier-stokes equations such [3]

$$w(\delta_j) \to u, \text{ as } \delta_j \to 0.$$

If additionally the solution to the NSE is a unique strong solution, specifically if $\int_0^T ||\nabla u(t)||^4_{L^2(\Omega)}dt < \infty$, then

$$\max_{0\leq t\leq T} ||\overline{u} - w||^2_{L^2(\Omega)} + \int_0^T \nu||\nabla(\overline{u} - w)||^2_{L^2(\Omega)}dt \leq C \int_0^T ||\tau||^2_{L^2(\Omega)}dt.$$

If additionally $\triangle^{N+1}u \in L^2(\Omega)$, then $\int_0^T ||\tau||^2_{L^2(\Omega)}dt \leq C(u)\delta^{2N+2}$ so the modeling error is $O(\delta^{N+1})$ non-uniformly in the Reynolds number.

Proof. See Dunca and Epshteyn [DE04] for the proof. □

The remaining questions were answered in estimates of averaged consistency errors in [LL04c].Our estimates are based on the following three plausible assumptions on time-averages of solutions of the Navier-Stokes equations and which are implied for homogeneous, isotropic turbulence by the K41 theory.

Assumption 1. The equations are non-dimensionalized by a selection of U and L consistent with

$$< \frac{1}{L^3}||u(\mathbf{x},t)||^2_{L^2(\Omega)} >^{\frac{1}{2}} \leq U.$$

Assumption 2. The time averaged energy dissipation rate $\varepsilon(u)$ satisfies

$$\varepsilon(u) \leq C_1\frac{U^3}{L}.$$

Assumption 3. The time averaged energy spectrum of the flow, defined precisely in section 2, satisfies

$$E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}.$$

Let the filter be a standard differential filter (defined precisely below). Let Re = Reynolds number, G_N = Nth approximate de-convolution operator defined by the van Cittert approximate de-convolution procedure and let

$$\tau_N := G_N\overline{u}G_N\overline{u} - uu$$

[3] As above, the convergence is in the norms of the natural energy spaces for the Navier-Stokes equations.

be the consistency error tensor in the Nth approximate de-convolution model. For homogeneous, isotropic turbulence (i.e., under the above three assumptions) we prove for the whole family of models that the consistency error converges to zero uniformly in the Reynolds number like $O(\delta^{\frac{1}{3}})$.

THEOREM 1.2. *Consider the Cauchy problem or the periodic problem. Suppose Assumptions 1, 2 and 3 hold. Then,*

$$< ||\tau_N||_{L^1(\Omega)} > \leq C_1^{\frac{1}{3}}(N+2)(3+\frac{2}{4N+\frac{10}{3}})^{\frac{1}{2}}\alpha^{\frac{1}{2}}U^2 L^{\frac{7}{6}}\delta^{\frac{1}{3}}.$$

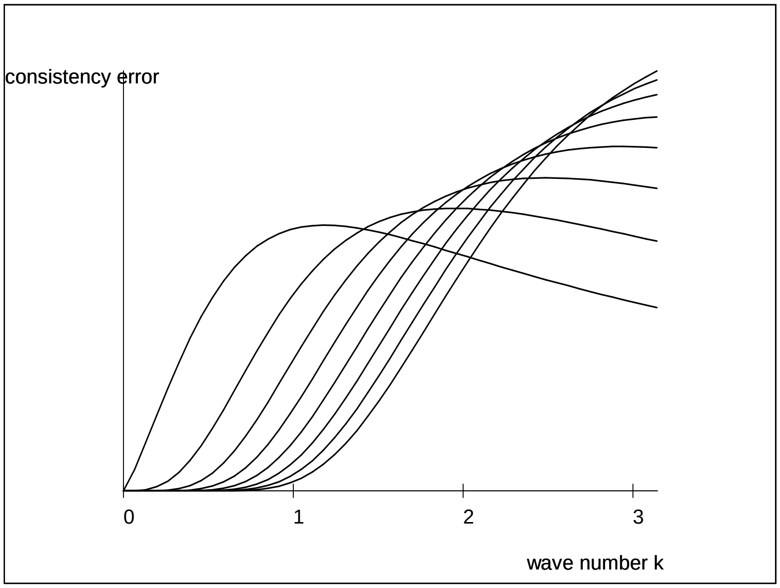
This holds for the periodic problem and the problem without boundaries. For flows in bounded domains with no-slip boundary conditions, the assumption that the flow is homogeneous and isotropic is unrealistic. For flows in bounded domains with no-slip boundary conditions, working directly from the Navier-Stokes equations we prove a result a result for the zeroth order model that is not uniform in Re.

THEOREM 1.3. *Let Ω be a bounded domain with sufficiently smooth boundary with no-slip boundary conditions. Suppose Assumptions 1 and 2 hold. Then,*

$$< ||\tau_0||_{L^1(\Omega)} > \leq C_1^{\frac{1}{2}} L^2 U^2 Re^{\frac{1}{2}}\delta.$$

It is possible to develop such a result without assumptions of homogeneity or isotropy because of the important recent breakthroughs of Doering and Constantin and Wang and Foias and others, [CD92], [W97] , [CKG01], [DF02], in deriving estimates of time-averaging energy dissipation rates directly from the Navier-Stokes equations. Although this result is not uniform in the Reynolds number, this result also implies that large eddy simulation is feasible; in other words, the (time averaged) consistency error is $<< O(1)$ for cutoff length well inside the inertial range.

We also consider the distribution of consistency errors among wave numbers or length scales. This leads to interesting distinctions between the models. For example, the next figure is a plot of the consistency error of the $N = 0$ through 7 approximate de-convolution models over (re-scaled) wave numbers $0 \leq k \leq \pi$. The order of the plots is decreasing consistency error near the origin (i.e., for the largest length scales) corresponds to increasing order.



Consistency errors, N=0 through 7.

Note that the consistency errors decrease rapidly as the order increases over the first third of the resolved scales but decrease very slowly over the next third and actually increases significantly over the remaining third.

2. Notation and Mathematical Tools. The set of all measurable, square integrable functions, vectors and tensors will be denoted, as usual, by $L^2(\Omega)$ and $L^1(\Omega)$ will denote the integrable functions. We will make no distinction in our notation between the scalar, vector and tensor cases. for a measurable function ϕ defined on the flow domain Ω, the associated norms are defined by

$$||\phi(\mathbf{x})||_{L^2(\Omega)} := \sqrt{\int_\Omega |\phi(\mathbf{x})|^2 d\mathbf{x}},$$
$$||\phi(\mathbf{x})||_{L^1(\Omega)} := \int_\Omega |\phi(\mathbf{x})| d\mathbf{x}.$$

If, for example, the function ϕ represents a fluid velocity then $L^2(\Omega)$ represents the set of all functions with finite kinetic energy.

The function space $H^1(\Omega)$ denotes

$$H^1(\Omega) := \{\phi \in L^2(\Omega) : \nabla\phi \in L^2(\Omega)\}, \text{ and}$$
$$||\phi||^2_{H^1(\Omega)} := ||\nabla\phi||^2_{L^2(\Omega)} + ||\phi||^2_{L^2(\Omega)}.$$

More to come: Fourier transforms. Plancherel's and Parseval's theorems. Basic properties of Fourier transforms.

3. The Space Filtered Navier-Stokes equations. Let the velocity $u(\mathbf{x}, t) = u_j(x_1, x_2, x_3, t)$, $(j = 1, 2, 3)$ and pressure $p(\mathbf{x}, t) = p(x_1, x_2, x_3, t)$ be a weak solution to the underlying Navier Stokes equations (NSE for short)

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = f, \text{ and } \nabla \cdot u = 0, \ in \nleq \times (0, T), \tag{3.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, f is the body force, p is the pressure, and the flow domain Ω is either a bounded region with a smooth (where the no-slip condition is imposed), a cube $\Omega = (0, L)^3$ (with periodic boundary conditions) or all of $\mathbb{R}^3$. The above Navier-Stokes equations are supplemented by the initial condition, the usual pressure normalization condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \ and \quad \int_\Omega p d\mathbf{x} = 0, \tag{3.2}$$

and appropriate boundary conditions. The mathematical theory of the Navier-Stokes equations is based upon the three physical quantities of kinetic energy, energy dissipation rate and power input, given by

$$\text{Kinetic energy: } E(t) := \frac{1}{2} \int_\Omega |u(\mathbf{x}, t)|^2 d\mathbf{x} = \frac{1}{2} ||u||^2_{L^2(\Omega)},$$

$$\text{Energy dissipation rate: } \varepsilon(t) := \frac{1}{L^3} \int_\Omega \nu |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} \ , L = \text{a selected length-scale},$$

$$\text{Power input: } P(t) := \frac{1}{L^3} \int_\Omega f(\mathbf{x}, t) \cdot u(\mathbf{x}, t) d\mathbf{x} \ .$$

Theories of turbulence are typically based on time averages (or other averages) of these same three quantities.

The boundary conditions we consider include the no-slip condition

$$u(\mathbf{x}, t) = 0, \mathbf{x} \in \partial\Omega, \ \Omega \text{ a bounded domain},$$

periodic boundary conditions with zero mean imposed on the solution and all data

$$u(\mathbf{x}+L\mathbf{e}_j, t) = u(\mathbf{x}, t), \int_\Omega u d\mathbf{x} = \int_\Omega p d\mathbf{x} = \int_\Omega f d\mathbf{x} = \int_\Omega u_0 d\mathbf{x} = 0,$$

and the problem on all of $\mathbb{R}^3$ without boundaries (the Cauchy problem). For the Cauchy problem, the role of boundary conditions at infinity is played by the assumption that the solution's kinetic energy, total energy dissipated and power input are all finite

$$\text{for all } t > 0 :, \int_{\mathbb{R}^3} |u(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \tag{3.3}$$

$$\text{for all } T > 0, \int_0^T \int_{\mathbb{R}^3} |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} dt < \infty, \tag{3.4}$$

$$\text{assuming } \int_{\mathbb{R}^3} |u_0(\mathbf{x})|^2 d\mathbf{x} < \infty, \ \int_{\mathbb{R}^3} |f(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \ for \ 0 < t. \tag{3.5}$$

We study a model for spacial averages of the fluid velocity with the following differential filter. Let δ denote the selected averaging radius; given $\phi \in L^2(\Omega)$, its average, denoted $\overline{\phi}$, is the solution in $H^1(\Omega)$ of the following problem [4]:

$$A\overline{\phi} := -\delta^2 \triangle \overline{\phi} + \overline{\phi} = \phi. \tag{3.6}$$

[4] This precise definition of the differential filter is important since we consider dimensional scaling and L. It is obtained by rescaling x to x/L.

The precise scaling in the above with respect to L is important in, for example, geophysical flow problems, [Lew97]. Differential filters are well-established in large eddy simulation, starting with work of Germano[Ger86] and continuing [GL00], [S01], and have many connections to regularization processes such as the Yoshida regularization of semigroups and the very interesting work of Foias, Holm, Titi [FHT01] (and others) on Lagrange averaging of the Navier-Stokes equations.

Averaging the Navier-Stokes equations shows that away from walls in the case of a bounded domain and the no-slip condition and everywhere for the periodic problem and the problem without boundaries, the true flow averages satisfy the (non-closed) equations known as the *space filtered Navier-Stokes equations or SFNSE*

$$\overline{u}_t + \nabla \cdot (\overline{u\ u}) - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f},\ \textit{and}\ \nabla \cdot \overline{u} = 0. \tag{3.7}$$

The SFNSE is not closed. One way (of many) to describe the closure problem is to replace the tensor uu in the SFNSE with a tensor $S(\overline{u}, \overline{u})$ that depends only on $\overline{u}$ not u . Calling w, q the approximations to u,p that result from this model, this leads to the following large eddy simulation model for predicting the averages $\overline{u}, \overline{p}$

$$w_t + \nabla \cdot (\overline{S(w,\ w)}) - \nu \triangle w + \nabla q = \overline{f},\ \textit{and}\ \nabla \cdot w = 0.$$

The consistency error tensor of this approximation is $\tau := S(\overline{u},\ \overline{u}) - u\ u$. For example, the true filtered momentum equation can be rewritten as

$$\overline{u}_t + \nabla \cdot (\overline{S(\overline{u},\ \overline{u})}) - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f} + \overline{\nabla \cdot \tau}.$$

Comparing the last two equations, the deviation of the true flow averages $\overline{u}$ from the model's solution w is driven by the consistency error tensor τ .

Definition 3.1 (Modeling and numerical errors). *The modeling error is the deviation of the model's solution from the true flow averages*

$$e_{modeling} := \overline{u} - w.$$

Let w^h denote a computed approximation of the large eddy simulation model. The numerical error is the deviation of the computed solution from the large eddy simulation model's true solution,

$$e_{numerical} := w - w^h.$$

Subtracting the last two equations (SFNSE-LESmodel) gives the equation for the modeling error, $e(\mathbf{x}, t) = \overline{u} - w$: $e(\mathbf{x}, 0) = 0, \nabla \cdot e = 0$ and

$$(\overline{u} - w)_t + \nabla \cdot (\overline{S(\overline{u},\ \overline{u})} - \overline{S(w,\ w)}) - \nu \triangle (\overline{u} - w) + \nabla (\overline{p} - q) = \overline{\nabla \cdot \tau}, \tag{3.8}$$

which is driven only by the model's consistency error τ . Thus, having a small modeling error depends on (i) having a small model consistency error, and (ii) the model being stable and stable to perturbations.

One approach to finding such a tensor $S(\overline{u}, \overline{u}) \simeq uu$ is approximate de-convolution. An approximate de-convolution operator G_N of order $2N+2$ is an approximate inverse of the filtering process of formal accuracy $O(\delta^{2N+2})$: given $\overline{u}$, $G_N(\overline{u})$ satisfies

$$u = G_N(\overline{u}) + O(\delta^{2N+2}) \text{ for smooth velocity fields } u.$$

Any such approximation de-convolution operator determine a closure model of formal accuracy $O(\delta^{2N+2})$ via $u\ u \simeq S(\overline{u},\overline{u}) := G_N(\overline{u})G_N(\overline{u})$. The associated large eddy simulation approximate de-convolution model is then

$$w_t + \nabla \cdot (\overline{G_N(w)G_N(\ w)}) - \nu \triangle w + \nabla q = \overline{f},\ and\ \nabla \cdot w = 0.$$

As a concrete example, the zeroth order model, studied in [LL03],[LL04], arises from $G_0(\overline{u}) = \overline{u}(= u + O(\delta^2))$, or $S(\overline{u},\overline{u}) = \overline{u}\ \overline{u}$, giving the closure approximation $uu = S(\overline{u},\overline{u}) + O(\delta^2)$ for smooth u . Calling w, q the resulting approximations to $\overline{u}, \overline{p}$, the zeroth order model is given by :

$$w_t + \nabla \cdot (\overline{w\ w}) - \nu \triangle w + \nabla q = \overline{f},\ and\ \nabla \cdot w = 0. \tag{3.9}$$

Subtracting the model from the SFNSE, the model's error, $e := \overline{u} - w$, satisfies $e(\mathbf{x}, 0) = 0, \nabla \cdot e = 0$ and

$$(\overline{u} - w)_t + \nabla \cdot (\overline{u\ u} - \overline{w\ w}) - \nu \triangle(\overline{u} - w) + \nabla(\overline{p} - q) = \overline{\nabla \cdot (\overline{uu} - uu)}, \tag{3.10}$$

which is driven only by the term $\overline{\nabla \cdot (\overline{uu} - uu)}$. The consistency error of the zeroth order model is thus defined to be $\tau_0 := \overline{u}\ \overline{u} - u\ u$. Since the model is stable and stable to perturbations, [LL04], the *accuracy* of the model is governed by the size of various norms of its consistency error tensor τ_0[5].

4. More about differential filters. It is useful is provide some background information about filtering in general and the simple differential filter we focus on in particular. Recall that L denotes the global length-scale and δ the selected averaging radius. For the periodic problem or the problem on all of $\mathbb{R}^3$ (the Cauchy problem), the differential filter can be defined very simply as follows: given $\phi \in L^2(\Omega)$, its average, denoted $\overline{\phi}$, is the solution with $\overline{\phi} \in L^2(\Omega)$ and $\nabla\overline{\phi} \in L^2(\Omega)$ of the following problem

$$A\overline{\phi} := -\delta^2 \triangle \overline{\phi} + \overline{\phi} = \phi. \tag{4.1}$$

The parameter δ has units of length. It represents the cut-off length scale in the filter or, equivalently, the filter's averaging radius. The exact interpretation of δ as a physical length, rather than a generic, nondimensional parameter, in the above is important in, for example, geophysical flow problems, [Lew97].Differential filters are well-established in large eddy simulation, starting with work of Germano[Ger86] and continuing [GL00], [S01], and have many connections to regularization processes such as the Yoshida regularization of semigroups and the very interesting work of Foias, Holm, Titi [FHT01] (and others) on Lagrange averaging of the Navier-Stokes equations.

The definition of the differential filter must be modified for the problem with no-slip boundary conditions. Briefly, the above definition preserves incompressibility for periodic boundary conditions and for the Cauchy problem but not for the problem with no-slip boundary conditions. The correct modification is to solve a shifted Stokes problem. Given $\phi \in L^2(\Omega)$, with $\nabla\phi \in L^2(\Omega)$ its average, denoted $\overline{\phi}$, is the solution with $\overline{\phi} \in L^2(\Omega)$, and $\nabla\ \overline{\phi} \in L^2(\Omega)$ of the following problem: find $\overline{\phi}, \lambda$ satisfying

$$\begin{aligned} -\delta^2\{\triangle\overline{\phi} + \nabla\lambda\} + \overline{\phi} &= \phi, \text{ in } \Omega, \\ \nabla \cdot \overline{\phi} &= \nabla \cdot \phi \text{ , in } \Omega, \\ \overline{\phi} &= \phi \text{ , on } \partial\Omega. \end{aligned}$$

[5] As above, analysis of the dynamics of this error equation in [LL03], [LL04] showed that the modeling error is actually driven by τ_0 rather than $\nabla \cdot \overline{\tau_0}$.

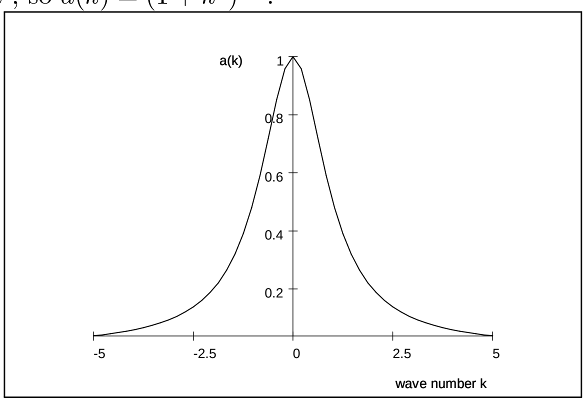
Generally, a filtering operator is shares many features of an averaging process; it is a positive semi-definite operator which is smoothing, attenuates high frequencies and converges to the identity operator as $\delta \to 0$, i.e., $\overline{\phi} \to \phi$ as $\delta \to 0$. Much of the mathematical foundation of filtering was laid down by the famous applied mathematician Norbert Wiener (who studied the de-convolution problem), e.g.[W49] . Much of this theory is phrased in terms of the transfer function or symbol of the filtering operator under Fourier transform, [W33]. The Fourier transform of the above equation (4.1) which defines the filtering operation is

$$(\delta^2(k_1^2 + k_2^2 + k_3^2) + 1)\widehat{\overline{\phi}}(k_1, k_2, k_3) = \widehat{\phi}(k_1, k_2, k_3).$$

Here $\mathbf{k} = (k_1, k_2, k_3)$ is the dual variable of the Fourier transform. The dual variable (k_1, k_2, k_3) is also called the wave number vector and its magnitude $k := \sqrt{k_1^2 + k_2^2 + k_3^2} = |\mathbf{k}|$ the wave number. The last equation can be rewritten as

$$\widehat{\overline{\phi}}(k_1, k_2, k_3) = \frac{1}{\delta^2(k_1^2 + k_2^2 + k_3^2) + 1}\widehat{\phi}(k_1, k_2, k_3),$$

so the transfer function or symbol of the filter is $a(\delta k) := (\delta^2(k_1^2 + k_2^2 + k_3^2) + 1)^{-1} = [1 + |\delta\mathbf{k}|^2]^{-1} = [1 + (\delta k)^2]^{-1}$. Normally this is represented graphically by the obvious re-scaling $\delta k \to k$, so $a(k) = (1 + k^2)^{-1}$.



Transfer function of the differential filter.

The smoothing property of the differential filter is reflected in the decay at infinity of its transfer function.

It follows readily that the averaging process is stable and smoothing in the sense below and the filtered function is a good approximation of the true function, if it is smooth. Detailed proofs of such estimates in these and other norms are given in [LL04].

Lemma 4.1 (Stability and smoothing of differential filters). *For any* $\phi \in L^2(\Omega)$,

$$||\overline{\phi}||_{L^2(\Omega)} \leq ||\phi||_{L^2(\Omega)}, \qquad (4.2)$$

$$2\delta||\nabla\overline{\phi}||_{L^2(\Omega)} \leq ||\phi||_{L^2(\Omega)}, \ \text{and} \qquad (4.3)$$

$$\frac{1}{2}\delta^2||\triangle\overline{\phi}||_{L^2(\Omega)} \leq ||\phi||_{L^2(\Omega)}. \qquad (4.4)$$

Proof. Multiply (1.5) by $\overline{\phi}$, integrate over Ω and integrate by parts. This gives

$$\int_\Omega \delta^2|\nabla\overline{\phi}|^2 + |\overline{\phi}|^2 d\mathbf{x} \leq \int_\Omega \phi\overline{\phi} d\mathbf{x}$$

Use the Cauchy-Schwarz inequality on the right hand side gives the first two. For the third, note that the definition of the differential filter implies $-\delta^2\triangle\overline{\phi} = \phi - \overline{\phi}$. thus,

$$\delta^2||\triangle\overline{\phi}||_{L^2(\Omega)} = ||\phi - \overline{\phi}||_{L^2(\Omega)} \leq \\ \leq 2||\phi||_{L^2(\Omega)}.$$

□

LEMMA 4.2 (Accuracy of approximation by filtering). *For any* $\phi \in L^2(\Omega)$,

$$||\phi - \overline{\phi}||_{L^2(\Omega)} \leq \frac{1}{2}\delta||\nabla\phi||_{L^2(\Omega)}, \tag{4.5}$$

$$||\nabla(\phi - \overline{\phi})||_{L^2(\Omega)} \leq \frac{1}{\sqrt{2}}\delta||\triangle\phi||_{L^2(\Omega)}, \text{ and} \tag{4.6}$$

$$||\phi - \overline{\phi}||_{L^2(\Omega)} \leq \delta^2||\triangle\phi||_{L^2(\Omega)}. \tag{4.7}$$

Proof. The averaging error by $\Phi = (\phi - \overline{\phi})$ satisfies the equation

$$-\delta^2\triangle\Phi + \Phi = -\delta^2\triangle\phi.$$

The above error bounds for Φ follow in much the same ways as the above stability bounds. □

5. The K-41 formalism. The most important components of the K-41 theory are the time (or ensemble) averaged energy dissipation rate, ε, and the distribution of the flows averaged kinetic energy across wave numbers, $E(k)$. Let $< \cdot >$ denote long time averaging

$$< \phi > (\mathbf{x}) := \overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T \phi(\mathbf{x}, t)dt. \tag{5.1}$$

Time averaging is the original approach to turbulence of Reynolds, [R95]. It satisfies the following Cauchy-Schwarz inequality.

LEMMA 5.1 (Time-averaged Cauchy-Schwartz inequality). *For all*

$$\phi, \psi : (0, \infty) \to L^2(\Omega),$$

with the right hand side finite

$$< (\phi, \psi)_{L^2(\Omega)} > \quad \leq \quad < ||\phi||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||\psi||^2_{L^2(\Omega)} >^{\frac{1}{2}} . \tag{5.2}$$

Proof. This follows, for example, by applying the usual Cauchy-Schwarz inequality on $\Omega \times (0, T)$ followed by taking limits or from the connection with the inner product on the space of Besicovitch almost periodic functions, e.g., [Z85],[L84], [CB89]. □

Given the velocity field of a particular flow, $u(\mathbf{x}, t)$, the (time averaged) energy dissipation rate of that flow is defined to be

$$\varepsilon(u) := < \frac{1}{L^3}\int_\Omega \nu|\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} \ > . \tag{5.3}$$

It is known for many turbulent flows that the energy dissipation rate ε scales like $\frac{U^3}{L}$. This estimate, which is exactly Assumption 2 below, follows for homogeneous, isotropic turbulence from the K41 formalism, [F95], [Les97], [P00], [S84], [S98] and was first proven as an upper bound directly from the Navier Stokes equations for turbulent flows in bounded domains driven by persistent shearing of a moving boundary (rather than a body force), [CD92] ,[W97], . The same estimate has been proven, [F97], [CKG01], [DF02] when the flow is driven by a persistent body force, the boundary conditions are periodic and the forcing acts on the largest modes.

The total kinetic energy in the flow (assuming unit density) at time t is, as above, $E(t) := \int_\Omega \frac{1}{2}|u(\mathbf{x},t)|^2 d\mathbf{x}$.Thus, the kinetic energy at the point $\mathbf{x}$ in space at time t is given by $\frac{1}{2}|u(\mathbf{x},t)|^2$. For the rest of this section, let Ω denote $\mathbf{R}^3$.[6] If $\widehat{u}(\mathbf{k},t)$ denotes the Fourier transform of $u(\mathbf{x},t)$ where $\mathbf{k}$ is the wave-number vector and $k = |\mathbf{k}|$ is its magnitude, then Parseval's equality states that $\int_\Omega |u(\mathbf{x},t)|^2 d\mathbf{x} = \int_{\mathbf{R}^3} |\widehat{u}(\mathbf{k},t)|^2 d\mathbf{k}$. This implies that the kinetic energy in the flow can be evaluated in physical space or in wave number space using the Fourier transform $\widehat{u}$ of u

$$E(t) := \frac{1}{2}||u||^2_{L^2(\Omega)} = \int_\Omega \frac{1}{2}|u(\mathbf{x},t)|^2 d\mathbf{x} = \int_{\mathbf{R}^3} \frac{1}{2}|\widehat{u}(\mathbf{k},t)|^2 d\mathbf{k}. \tag{5.4}$$

Thus $\frac{1}{2}|\widehat{u}(\mathbf{k},t)|^2$ represents the distribution of kinetic energy at time t among waves with associated wave-number vector $\mathbf{k}$. Time averaging and rewriting the last integral in spherical coordinates gives

$$< E(t) > = < \frac{1}{2}||u||^2_{L^2(\mathbb{R}^3)} > = \int_0^\infty E(k)dk, \tag{5.5}$$

$$\text{where } E(k) \ : = \int_{|\mathbf{k}|=k} \frac{1}{2}|\widehat{< u >}(\mathbf{k},t)|^2 d\sigma. \tag{5.6}$$

Thus, $E(k)$ represents the total (time averaged) kinetic energy in all waves with frequency k. The case of homogeneous, isotropic turbulence includes the assumption that after time or ensemble averaging $\widehat{< u >}(\mathbf{k})$ depends only on k and not the angles θ or φ. Thus, in this case,

$$E(k) = 2\pi k^2 |\widehat{< u >}(k)|^2. \tag{5.7}$$

Further, the K-41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < k_{\min} := U\nu^{-1} \leq k \leq \varepsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}} =: k_{\max} < \infty, \tag{5.8}$$

known as the inertial range, beyond which the kinetic energy in u is negligible, and in this range

$$E(k) \doteq \alpha\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{5.9}$$

where α is the universal Kolmogorov constant whose value is generally believed to be between 1.4 and 1.7, k is the wave number and ε is the particular flow's energy dissipation rate. The energy dissipation rate ε is the only parameter which differs from

[6] All the results hold as well for $\Omega = (0, L)^3$ with periodic boundary conditions if integrals are replaced by sums, Fourier transforms by Fourier series and Parseval's equality for Fourier transforms by Parseval's equality for Fourier series.

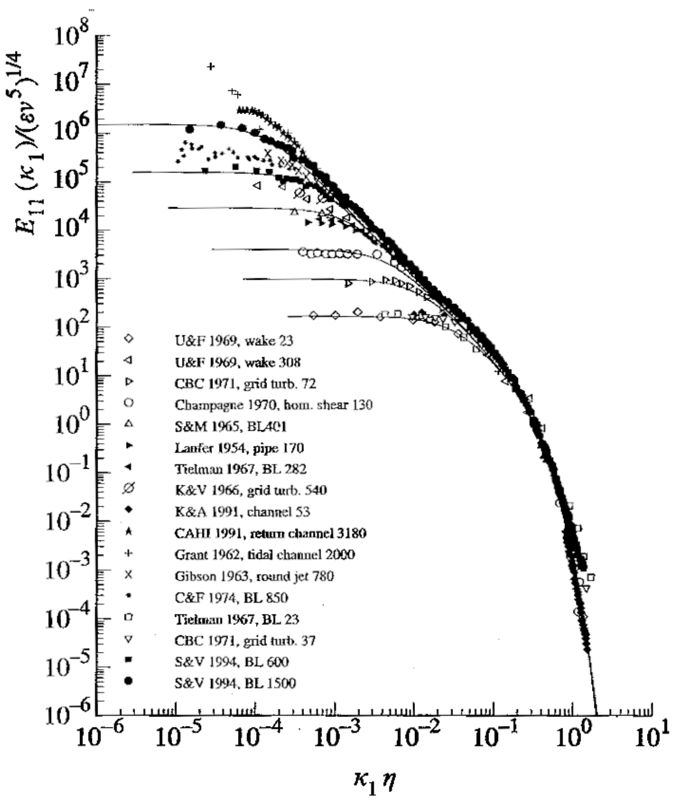


FIG. 5.1. *log-log plot reveals $k^{-\frac{5}{3}}$ law, figure 6.14 page 235 in [P00]*

one flow to another. Outside the inertial range the kinetic energy in the small scales decays exponentially. Thus, we still have $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ since, after time averaging the energy in those scales is negligible, $E(k) \simeq 0$ for $k \geq k_{\max}$ and $E(k) \leq E(k_{\min})$ for $k \leq k_{\min}$. The fundamental assumption underlying our consistency error estimates is Assumption 3 that over all wave numbers

$$E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}.$$

Indeed, in the next figure, which is taken from Pope [P00], figure 6.14 page 235 in [P00], the power spectrums of 17 different turbulent flows taken from Saddoughi and Veeravalli [SV94, SV94] (which also contains the references to the particular experiments) are plotted on log-log plots. The slope of the linear region in this plot has the universal value of $-\frac{5}{3}$ for all 17 turbulent flows, exactly corresponding to the $k^{-\frac{5}{3}}$ law. Note that for all 17 flows the actual energy spectrum outside the inertial range is below this line with slope $-\frac{5}{3}$, verifying that for all wave numbers $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$.

Our estimates are based on the following three plausible assumptions on time-averages of solutions of the Navier-Stokes equations and which are implied for homogeneous, isotropic turbulence by the K41 theory. Assumptions 1 is a choice of length and velocity scales. Assumption 2 is that the energy dissipation rate ε scales like $C_1\frac{U^3}{L}$. This assumption is consistent with (and predicted by) the K41 formalism,

[F95], [Les97, Lesieur's book] . The estimate $\varepsilon \leq C_1 \frac{U^3}{L}$ has also been proven directly from the Navier-Stokes equations for turbulent shear flows in bounded domains by Constantin and Doering [CD92] , and Wang [W97] without any assumptions of homogeneity or isotropy. Assumption 2 has been proven directly from the Navier-Stokes equations for flows driven by body forces by Foias [F97], Doering and Foias [DF02], Childress, Kerswell and Gilbert [CKG01] (others have also contributed to this important theory as well) provided the forcing function inputs energy into the largest scales[7]. Thus, a consistency error estimate using Assumptions 1 and 2 is one derived from the NSE and is independent of any assumptions of homogeneity and isotropy (and the K41 theory). An estimate using Assumption 3 depends through Assumption 3 on K41.

Assumption 1. The equations are non-dimensionalized by a selection of U and L consistent with

$$< \frac{1}{L^3} ||u(\mathbf{x},t)||^2_{L^2(\Omega)} >^{\frac{1}{2}} \leq U.$$

Assumption 2. The time averaged energy dissipation rate $\varepsilon(u)$ satisfies

$$\varepsilon(u) \leq C_1 \frac{U^3}{L}.$$

Assumption 3. The time averaged energy spectrum of the flow satisfies

$$E(k) \leq \alpha \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}.$$

6. Approximate de-convolution. The de-convolution problem is central in

image processing, [BB98]. Thus, there are very many algorithms that can be adapted from image processing to give a possible large eddy simulation closure model. However, the challenges of turbulence are substantially different from those of image processing. The noise occurring in images stem from sources whose statistics are at least predictable. The "noise" in a turbulent flow simulation comes from the nonlinear interactions of the large eddy simulation closure model, boundary conditions, discretization procedure etc. The goal of de-convolution in large eddy simulation is to recover accurately the resolved scales *asymptotically* as $\delta \to 0$. The de-convolution closure must result in an LES model with lucid energy balance and favorable properties for its approximate solution. In particular, the de-convolution LES model should have a unique, strong smooth solution which has a negligible amount of energy is the solution scales smaller than $O(\delta)$ so that the amount of information in the model's solution is substantially reduced over that of the Navier-Stokes equations. With these constraints in mind, we review de-convolution methods. First, we note that with a differential filter, exact de-convolution is possible but not useful (an observation of M. Germano). Indeed, since $\overline{\phi} = (-\delta^2 \triangle + 1)^{-1}\phi$, the solution of the de-convolution problem is $\phi = (-\delta^2 \triangle + 1)\overline{\phi}$. Let the operator A denote this exact de-convolution operator $A := (-\delta^2 \triangle + 1)$. Clearly A is not a bounded operator on $L^2(\Omega)$ so exact de-convolution is unstable to high frequency perturbations (this is known as the

[7] This is exactly the case described in the Richardson-Kolmogorov energy cascade. Thus, the restriction on the body forces in these results is both consistent with the K41 theory and a mathematical formalization of the Richardson's description of the energy cascade.

small divisor problem). Further, if exact de-convolution is used to obtain the (exact) equation for the flow averages, we obtain

$$\overline{u}_t + \nabla \cdot (\overline{A\overline{u}\ A\overline{u}}) - \nu \triangle \overline{u} + \nabla \overline{p} = \overline{f},\ \textit{and}\ \nabla \cdot \overline{u} = 0.$$

The important observation is that this is just a change of variables in the Navier-Stokes equations. Thus, it does not reduce the amount of information contained in the model's solution and any theory of the model will share all the gaps in its mathematical theory with that of the Navier-Stokes equations. Thus, only approximate de-convolution can produce a useful model.[8]

The basic problem in approximate de-convolution is thus: given $\overline{u}$ find useful approximations of u. In other words, solve the equation

$$Gu = \overline{u},\ \text{solve for } u.$$

For most averaging operators, G is symmetric and positive semi-definite. Typically, G is not invertible or at least not stably invertible due to small divisor problems. Thus, this de-convolution problem is ill-posed. Also typically, G is invertible on a subspace which becomes dense in $L^2(R^3)$ as $\delta \to 0$.

The de-convolution algorithm we consider was studied by van Cittert in 1931. For each $N = 0, 1, ...$ it computes an approximate solution u_N to the above de-convolution equation by N steps of a fixed point iteration, [BB98]. Rewrite the above de-convolution equation as the fixed point problem:

$$\text{given } \overline{u} \text{ solve } u = u + \{\overline{u} - Gu\} \text{ for } u.$$

The de-convolution approximation is then computed as follows.

Algorithm 6.1 (van Cittert approximate de-convolution algorithm). $u_0 = \overline{u}$,
for n=1,2,...,N-1, perform
$u_{n+1} = u_n + \{\overline{u} - Gu_n\}$

Clearly, this is nothing but the first order Richardson iteration for the operator equation $Gu = \overline{u}$ involving a possibly non-invertible operator G. Since the de-convolution problem is ill posed, convergence as $N \to \infty$ is not expected. Call $u_N = G_N \overline{u}$. By eliminating the intermediate steps, it is easy to find an explicit formula for the N^{th} de-convolution operator G_N :

$$G_N \phi := \sum_{n=0}^{N} (I - G)^n \phi. \tag{6.1}$$

For example, the approximate de-convolution operator corresponding to $N = 0, 1, 2$ are $G_0 \overline{u} = \overline{u}$, and $G_1 \overline{u} = 2\overline{u} - \overline{\overline{u}}$, and $G_2 \overline{u} = 3\overline{u} - 3\overline{\overline{u}} + \overline{\overline{\overline{u}}}$.

Lemma 6.2 (Stability of approximate de-convolution). *Let averaging be defined by the above differential filter. Let $||\cdot||$ denote the $L^2(\Omega)$ operator norm. Then G_N is a self-adjoint, positive semi-definite operator on $L^2(\Omega)$ and*

$$||G_N|| = N + 1.$$

[8] One, completely different, way to generate LES models is by asymptotic approximations of this equation. For example, if the nonlinear term is expanded and truncated st $O(\delta^4)$ terms the Rational Model of [GL98] results after simplifications.

Proof. First, note that since $A^{-1}=G$ is a self-adjoint positive definite operator with eigenvalues between zero and one and $G_N := \sum_{n=0}^{N}(I-G)^n$, G_N is also self-adjoint. By the spectral mapping theorem

$$\lambda(G_N)=\sum_{n=0}^{N}\lambda(I-G)^n=\sum_{n=0}^{N}(I-\lambda(G))^n.$$

Thus, the eigenvalues of G_N are non-negative and G_N is also positive semi-definite. Since G_N is self-adjoint, the operator norm $||G_N||$ is also easily bounded by the spectral mapping theorem by

$$||G_N||=\sum_{n=0}^{N}\lambda_{\max}(I-A^{-1})^n=\sum_{n=0}^{N}(I-\frac{1}{\lambda_{\max}})^n=N+1. \tag{6.2}$$

□

The point-wise accuracy of the van Cittert de-convolution operator was estimated in Dunca and Epshteyn [DE04]by exploiting the fact that G_N is a truncated geometric series. We give the argument next for completeness and to highlight the simple and elegant mathematical structure of these models.

LEMMA 6.3 (Error in approximate de-convolution). *For any* $\phi\in L^2(\Omega)$,

$$\begin{aligned}\phi-G_N\overline{\phi}&=(I-A^{-1})^{N+1}\phi\\&=(-1)^{N+1}\delta^{2N+2}\triangle^{N+1}A^{-(N+1)}\phi.\end{aligned}$$

Proof. Let $B=I-A^{-1}$. *Since* $\overline{\phi}=A^{-1}\phi,\overline{\phi}=(I-B)\phi$.
Since $G_N:=\sum_{n=0}^{N}B^n$,*a geometric series calculation gives*

$$(I-B)G_N\overline{\phi}=(I-B^{N+1})\overline{\phi}.$$

Subtraction gives

$$\phi-G_N\phi=AB^{N+1}\overline{\phi}=B^{N+1}A\overline{\phi}=B^{N+1}\phi.$$

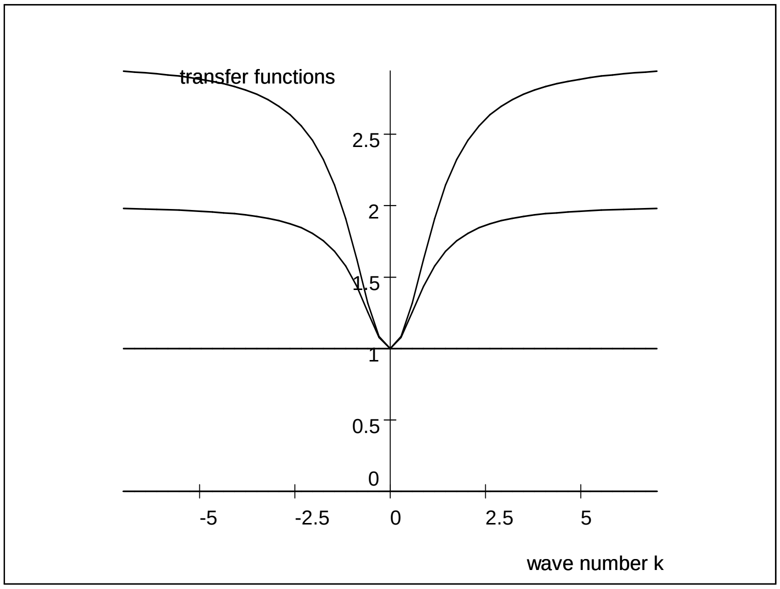
Finally, $B=I-A^{-1}$, *so rearranging terms gives*

$$\begin{aligned}\phi-G_N\phi&=(A-I)^{N+1}A^{-(N+1)}\phi\\&=A^{-(N+1)}((-1)^{N+1}\delta^{2N+2}\triangle^{N+1})\phi,\end{aligned}$$

which are the claimed results. □

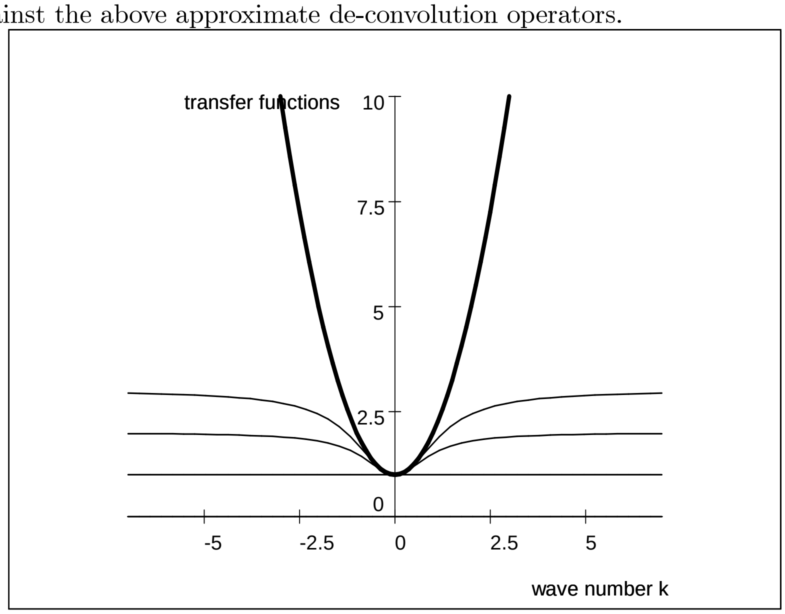
It is insightful to consider the Cauchy problem or the periodic problem and visualize the approximate de-convolution operators G_N in wave number space (re-scaled by $k\leftarrow\delta k$). The transfer function or symbol of the first three are

$$\begin{aligned}\widehat{G_0}&=1,\\ \widehat{G_1}&=2-\frac{1}{k^2+1}=\frac{2k^2+1}{k^2+1},\ \text{and}\\ \widehat{G_2}&=1+\frac{k^2}{k^2+1}+(\frac{k^2}{k^2+1})^2.\end{aligned}$$



Approximate de-convolution operators

Note that in the above plot of the transfer functions, each is strictly bounded away from zero, positive and uniformly bounded. Thus, each is a bounded, SPD operator on $L^2(\Omega)$ and the quadratic form with each gives an equivalent norm. These three are plotted in the next figure together with the transfer function of exact de-convolution. The transfer function of exact deconvolution is $k^2 + 1$. The next figure plots this in bold against the above approximate de-convolution operators.



Exact and approximate de-convolution operators

The large scales are associated with the smooth components and with the wave numbers near zero (i.e., $|k|$ small). Thus, the fact that G_N is a very accurate solution of the de-convolution problem for the large scales is reflected in the above graph in that the transfer functions have high order contact near $k = 0$.

7. Approximate de-convolution LES models. The family of *Approximate De-convolution Models* (or ADM's) we study was pioneered in large eddy simulation by Stolz and Adams in a series of papers,[AS01], [SA99]. Let G_N $(N = 0, 1, 2, ...)$ denote the van Cittert, [BB98], approximate de-convolution operator from the last section. This de-convolution approximation satisfies, [DE04],

$$u = G_N \overline{u} + O(\delta^{2N+2}), \text{ for smooth } u. \tag{7.1}$$

The models studied by Adams and Stolz are given by

$$w_t + \nabla \cdot (\overline{G_N w \, G_N w}) - \nu \triangle w + \nabla q + w' = \overline{f}, \; and \; \nabla \cdot w = 0. \tag{7.2}$$

The w'term is included to damp strongly the temporal growth of the fluctuating component of w driven by noise, numerical errors, inexact boundary conditions and so on[9]. The simplest example of such a model arises when $N = 0$ and the w' term is dropped. This zeroth order model also arises as the zeroth order model in many different families of LES models. It is given by

$$w_t + \nabla \cdot (\overline{w \, w}) - \nu \triangle w + \nabla q = \overline{f}, \; and \; \nabla \cdot w = 0. \tag{7.3}$$

The theory of the whole family of models begins, like the Leray theory of the Navier-Stokes equations, with a clear global energy balance. This is the key to the development of this family of models (or any model for that matter). It is clearest for the periodic problem and for the zeroth order model.

Proceeding formally for the moment, let (w, q) denote a periodic solution of the zeroth order model. Multiplying by Aw and integrating over the flow domain gives

$$\int_\Omega w_t \cdot Aw + \nabla \cdot (\overline{w \, w}) \cdot Aw - \nu \triangle w \cdot Aw + \nabla q \cdot Awdx = \int_\Omega \overline{f} \cdot Awdx.$$

The nonlinear term exactly vanishes because

$$\begin{aligned} \int_\Omega \nabla \cdot (\overline{w \, w}) \cdot Awdx &= \int_\Omega A^{-1}(\nabla \cdot (w \; w)) \cdot Awdx = \\ &= \int_\Omega \nabla \cdot (w \; w) \cdot (A^{-1}Aw)dx = \int_\Omega \nabla \cdot (w \; w) \cdot wdx = \\ &= \int_\Omega w \; \cdot \nabla w \cdot wdx = 0. \end{aligned}$$

Integrating by parts the remaining terms gives

$$\begin{aligned} &\frac{d}{dt}\{||w(\cdot,t)||^2_{L^2(\Omega)} + \delta^2 ||\nabla w(\cdot,t)||^2_{L^2(\Omega)}\} + \\ &\qquad + \nu\{||\nabla w(\cdot,t)||^2_{L^2(\Omega)} + \delta^2 ||\triangle w(\cdot,t)||^2_{L^2(\Omega)}\} = \int_\Omega f \cdot wdx. \end{aligned}$$

[9] The consistency error induced by adding the $w\prime$ term is smaller than that of the nonlinear term. While this term does affect the model's dynamics, especially over longer time intervals, it does not affect the overall consistency error estimate. Thus, in our study of consistency errors in the next section we will drop the w' term.

Integrating this gives an energy equality which has parallels to that of the NSE. In particular, we can clearly identify three physical quantities of kinetic energy, energy dissipation rate and power input. Let L =the selected length-scale; then these are given by

$$\text{Model's Kinetic energy: } E_{model}(t) := \frac{1}{2}\int_\Omega |w(\mathbf{x},t)|^2 d\mathbf{x} = \frac{1}{2}||w(\cdot,t)||^2_{L^2(\Omega)},$$

$$\text{Model's Energy dissipation rate:} \qquad \varepsilon_{model}(t) := \frac{1}{L^3}\int_\Omega \nu|\nabla w(\mathbf{x},t)|^2 d\mathbf{x}\ ,$$

$$\text{Model's Power input:} \qquad P_{model}(t) := \frac{1}{L^3}\int_\Omega f(\mathbf{x},t)\cdot w(\mathbf{x},t)d\mathbf{x}\ .$$

This estimate, combined with a construction now standard for the Navier-Stokes equations, is strong enough to prove existence, uniqueness and regularity of strong solution of the zeroth order model, [LL04b], [LL04].

THEOREM 7.1. *Consider the zeroth order model under periodic boundary conditions and with smooth enough initial condition and body force. The zeroth order model has a unique strong solution. If the problem data* $u_0 \in C^\infty(\Omega)$ *and* $f \in C^\infty(\Omega\times(0,\infty))$ *then the solution is also* $C^\infty(\Omega\times(0,\infty))$.

Let $w = w(\delta)$ *denote this solution. Then, there is a subsequence* $\delta_j \to 0$ *and a weak solution of the Navier-stokes equations such* [10]

$$w(\delta_j) \to u, \ \ as\ \delta_j \to 0.$$

Proof. See [LL04b], [LL04] for proofs. □

These results are remarkable because these are exactly properties which any reasonably derived sub model for the Navier Stokes equations should satisfy but which either have not been proven or likely are not true for most models in large eddy simulation. Again, the key idea is a simple and direct connection between the models kinetic energy hand the kinetic energy of the Navier Stokes equations through the deconvolution operator. The main difficulty with this model is that it is a mathematical toy: It is not sufficient the accurate. Extension of these results to the entire family of higher-order approximate de-convolution LES models is filled with technical intricacies but was accomplished successfully in a beautiful paper of Dunca and Epshteyn [DE04]. The first step in such an extension is to verify that the model satisfies an energy inequality. Paralleling the case of the zeroth order model, the natural idea is to make the nonlinear term disappear in the energy inequality by multiplying the model by $AG_N w$ then integrating over the flow domain and integrating by parts. To give a hint of why this procedure is hopeful, we note that, in effect, this is simply re-norming the space $L^2(\Omega)$.

LEMMA 7.2. *The norm* $||v||_{G_N} := (v, G_N v)^{\frac{1}{2}}$ *is a norm on* $L^2(\Omega)$ *which is equivalent to the* $L^2(\Omega)$ *norm.*

Proof. This follows from the fact that the transfer function of G_N , $\widehat{G_N}$, is bounded and bounded away from zero. □

By this approach, Dunca and Epshteyn proved the following.

[10] The convergence is in the norms of the natural energy spaces for the Navier-Stokes equations.

Theorem 7.3. *Consider the Nth approximate de-convolution LES model under periodic boundary conditions and with smooth enough initial condition and body force. For every N this model has a unique strong solution. If the problem data* $u_0 \in C^\infty(\Omega)$ *and* $f \in C^\infty(\Omega \times (0,\infty))$ *then the solution is also* $C^\infty(\Omega \times (0,\infty))$ *.*

Let $w = w(\delta)$ *denote this solution. Then, there is a subsequence* $\delta_j \to 0$ *and a weak solution of the Navier-stokes equations such* [11]

$$w(\delta_j) \to u, \text{ as } \delta_j \to 0.$$

If additionally the solution to the NSE is a unique strong solution, specifically if $\int_0^T ||\nabla u(t)||^4_{L^2(\Omega)} dt < \infty$*, then*

$$\max_{0\le t\le T} ||\overline{u} - w||^2_{L^2(\Omega)} + \int_0^T \nu ||\nabla(\overline{u} - w)||^2_{L^2(\Omega)} dt \le C \int_0^T ||\tau||^2_{L^2(\Omega)} dt.$$

If additionally $\triangle^{N+1} u \in L^2(\Omega)$ *, then* $\int_0^T ||\tau||^2_{L^2(\Omega)} dt \le C(u)\delta^{2N+2}$ *so the modeling error is* $O(\delta^{N+1})$ *non-uniformly in the Reynolds number.*

Proof. See Dunca and Epshteyn [DE04] for the proof. □

This is an amazing result; there are few other LES models with a fully rigorous, mathematical proof that the modeling error is of a high order of accuracy. However, the smoothness required of the solution restricts the result to laminar flow regions and the non-uniformity in the Reynolds number can hide the most significant part of the error. For our purposes, it is important to note that the modeling error is controlled by τ rather than $\nabla \cdot \tau$ or $\overline{\nabla \cdot \tau}$. Thus, the appropriate quantity to estimate is τ rather than $\nabla \cdot \tau$ or $\overline{\nabla \cdot \tau}$.

8. Consistency error estimates. Consider the family of *Approximate De-convolution Models* (or ADM's) presented in the last section. The size of the N^{th} models consistency error tensor directly determines the model's accuracy for the N^{th} approximate de-convolution LES models, [DE04]. Let G_N $(N = 0, 1, 2, ...)$ denote the above van Cittert, [BB98], approximate de-convolutionoperator. By the above lemma, this de-convolution approximation satisfies, [DE04],

$$u = G_N \overline{u} + O(\delta^{2N+2}), \text{ for smooth } u. \tag{8.1}$$

The models studied by Adams and Stolz are given by

$$w_t + \nabla \cdot (\overline{G_N w \, G_N w}) - \nu \triangle w + \nabla q + w' = \overline{f}, \text{ and } \nabla \cdot w = 0. \tag{8.2}$$

The w'term is included to damp strongly the temporal growth of the fluctuating component of w driven by noise, numerical errors, inexact boundary conditions and so on. The consistency error induced by adding the $w\prime$ term is smaller than that of the nonlinear term. While it does affect the model's dynamics, it does not affect the overall consistency error estimate. Thus, herein we drop the w' term.

First note in all cases, estimates for the consistency error depends upon estimates of $u - G_N \overline{u}$ because

$$\tau_N = G_N \overline{u} \, G_N \overline{u} - u \, u = (G_N \overline{u} - u) G_N \overline{u} + u(G_N \overline{u} - u), N = 0, 1, 2, \cdots. \tag{8.3}$$

[11] As above, the convergence is in the norms of the natural energy spaces for the Navier-Stokes equations.

Thus, by the time averaged Cauchy-Schwarz inequality, we have

$$< ||\tau_N||_{L^1(\Omega)} >\leq (1+||G_N||) < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||u - G_N\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \tag{8.4}$$

Thus, estimates for the consistency error in $L^1(\Omega)$ flow from estimates of $||u - G_N\overline{u}||_{L^2(\Omega)}$, estimates of $< ||u - G_N\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}}$and and stability bounds for G_N.

8.1. The Zeroth Order Model. Consider first the case of the zeroth order model, (1.7) above. We prove an estimate for the modeling error for the zeroth order model directly from the Navier-Stokes equations.

THEOREM 8.1. *Let Ω be a bounded domain with sufficiently smooth boundary with no-slip boundary conditions. Suppose Assumptions 1 and 2 hold. Then,*

$$< ||\tau_0||_{L^1(\Omega)} >\leq C_1^{\frac{1}{2}} L^2 U^2 Re^{\frac{1}{2}} \delta.$$

Proof. Consider τ_0. By the time-averaged Cauchy-Schwarz inequality we have

$$< ||\tau_0||_{L^1(\Omega)} >\leq 2 < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||u - \overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \tag{8.5}$$

Estimates for τ thus follow from estimates for $||u - \overline{u}||_{L^2(\Omega)}$ and $< ||u - \overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}}$. We estimate $< ||\tau_0||_{L^1(\Omega)} >$ by scaling, simple inequalities and Assumptions 1 and 2 as follows. Holder's inequality and the estimate

$$||u - \overline{u}||_{L^2(\Omega)} \leq \frac{1}{2}\delta||\nabla\overline{u}|| \leq \frac{1}{2}\delta||\nabla u||$$

give

$$< ||\tau_0||_{L^1(\Omega)} >\leq 2 < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}} < ||u - \overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \leq \tag{8.6}$$

$$\leq UL^{\frac{3}{2}}\delta < ||\nabla u||^2 >^{\frac{1}{2}} \leq \tag{8.7}$$

$$\leq L^3 U\delta\nu^{-\frac{1}{2}} < \frac{1}{L^3}\nu||\nabla u||^2_{L^2(\Omega)} >^{\frac{1}{2}}, \text{ or} \tag{8.8}$$

$$< ||\tau_0||_{L^1(\Omega)} >\leq \nu^{-\frac{1}{2}} L^3 U\delta\varepsilon^{\frac{1}{2}}. \tag{8.9}$$

Assumption 2 is that the time averaged energy dissipation rate ε is bounded by $C_1\frac{U^3}{L}$. As noted in above, this assumption is consistent with the K41 formalism, [F95], [Les97, Lesieur's book] and has also been proven directly from the Navier-Stokes equations for turbulent flows in bounded domains by, e.g., [CD92] ,[W97] . Using this upper bound for ε and rewriting the result in term of the Reynolds number gives the claimed bound

$$< ||\tau_0||_{L^1(\Omega)} >\leq C_1^{\frac{1}{2}} L^2 U^2 Re^{\frac{1}{2}} \delta. \tag{8.10}$$

□

REMARK 8.2. *How sharp is the estimate $< ||\tau_0||_{L^1(\Omega)} >\leq C_1^{\frac{1}{2}} L^2 U^2 Re^{\frac{1}{2}} \delta$? And, what does "sharp" mean? Since the above inequality contains a generic constant, generally such an estimate is considered sharp if a lower inequality holds in some cases with the same scaling in the important variables but possibly a different multiplicative constant. Thus, we would like to prove a (much harder) estimate of the following* *form for some velocity fields* $< ||\tau_0||_{L^1(\Omega)} >\geq C_2(L,U)Re^{\frac{1}{2}}\delta$. *Such an estimate is likely not true. To see why, note that the upper estimate is an inequality instead of an equality due to three steps. It is useful (even when proving a lower estimate is not hoped for with current mathematical techniques) to consider these steps to assess how sharp the final result will be. The three steps are Holder's inequality, the stability bound* $||\nabla\overline{u}|| \leq ||\nabla u||$ *and the bound on the energy dissipation rate in Assumption 3. Holders inequality is not "sharp" in the above sense only when the terms involved are orthogonal. Thus it is generically sharp. The bound in assumption 3 is dimensionally correct and consistent with the K41 theory. Thus, it is widely believed to be sharp although there is a gap between what is believed and what has been proven in the literature on the subject. The main step in which sharpness is lost is in the stability bound* $||\nabla\overline{u}|| \leq ||\nabla u||$. *Indeed, this is a sharp etimate for the large scales of the velocity but a very crude estimate of the small scales. It does not take into account any of the smoothing of the filtering. Thus, this is the main ingredient which we shall try to improve to obtain an estimate that (we believe) better reflects the actual behavior of the consistency error.*

The bound in this theorem implies the model becomes accurate already in the inertial range as certain flow features begin to resolve. However, it is also pessimistic since it requires $\delta << Re^{-\frac{1}{2}}$ for accuracy, see the conclusions section. This estimate comes directly from the Navier Stokes equations and is thus independent of the K41 theory. Remarkably, using K41 it is improvable to one uniform in the Reynolds number in the case of homogeneous, isotropic turbulence, i.e., using Assumption 3. We shall prove this uniform estimate for the entire family of models. Before doing so, we shall prove another non-uniform estimate for the zeroth order model that is higher order in δ .

For the case $N = 0$ and for smooth u, it is easy to show that $\tau_0 = O((\ \delta)^2)$.Indeed, simple estimates give

Lemma 8.3. *For any* u *with* $\triangle\overline{u} \in L^2(\Omega)$

$$||u-\overline{u}||_{L^2(\Omega)} \leq \delta^2 ||\triangle\overline{u}||_{L^2(\Omega)}.$$

Proof. Since $-\delta^2\triangle\overline{u}+\overline{u} = u$,it follows that $u-\overline{u} = -\delta^2\triangle\overline{u}$. Taking norms of both sides gives the result. □

Since $\tau = \overline{u}\ (\overline{u}-u)+(\overline{u}-u)u$, it follows immediately that

$$||\tau_0||_{L^1(\Omega)} \leq 2||u||_{L^2(\Omega)}\delta^2||\triangle\overline{u}||_{L^2(\Omega)}.$$

Next we show that under Assumptions 1, 2 and 3, the time average of this yields an $O(\ \delta^2)$ bound.

Theorem 8.4. *Consider either the Cauchy problem or the periodic problem. Suppose Assumptions 1, 2 and 3 hold. Then,*

$$< ||\tau_0||_{L^1(\Omega)} >\leq \sqrt{\frac{6}{5}} C_1^{\frac{3}{4}} \alpha^{\frac{1}{2}} \frac{U^2}{L^{\frac{1}{2}}} Re^{\frac{5}{4}} \delta^2 \tag{8.11}$$

Proof. Taking the time average of the previous inequality and using the time-averaged Cauchy-Schwarz inequality gives

$$< ||\tau_0||_{L^1(\Omega)} >\leq 2 < ||u||^2_{L^2(\Omega)} >^{\frac{1}{2}}\ \delta^2 < ||\triangle\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \tag{8.12}$$

$$\leq 2UL^{\frac{3}{2}}\delta^2 < ||\triangle\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} \tag{8.13}$$

$$\leq 2UL^{\frac{3}{2}}\delta^2 < ||\triangle u||^2_{L^2(\Omega)} >^{\frac{1}{2}}, \tag{8.14}$$

using Assumption 1 and stability of averaging. The term $< ||\triangle u||^2_{L^2(\Omega)} >^{\frac{1}{2}}$can be estimated in the case of homogeneous, isotropic turbulence (i.e., under Assumption 3) using spectral integration as follows

$$< ||\triangle u||^2_{L^2(\Omega)} >= \int_{k_0}^{k_{\max}} k^4 E(k) dk \tag{8.15}$$

$$\leq \alpha \varepsilon^{\frac{2}{3}} \int_0^{k_{\max}} k^{\frac{7}{3}} dk = .3\alpha\varepsilon^{\frac{2}{3}} (\varepsilon^{\frac{1}{4}} \nu^{-\frac{3}{4}})^{\frac{10}{3}}. \tag{8.16}$$

Using the estimate $\varepsilon \leq C_1 \frac{U^3}{L}$ and rearranging the resulting right hand side into terms involving the Reynolds number gives

$$< ||\triangle u||^2_{L^2(\Omega)} >^{\frac{1}{2}} \leq \sqrt{\frac{3}{10}} C_1^{\frac{3}{4}} \alpha^{\frac{1}{2}} \frac{U}{L^2} Re^{\frac{5}{4}} \delta^2. \tag{8.17}$$

Inserting this estimate into the bound for the consistency error gives

$$< ||\tau_0||_{L^1(\Omega)} > \leq \sqrt{\frac{6}{5}} C_1^{\frac{3}{4}} \alpha^{\frac{1}{2}} \frac{U^2}{L^{\frac{1}{2}}} Re^{\frac{5}{4}} \delta^2, \tag{8.18}$$

which is the claimed result. □

This is an asymptotically higher power of δ for moderate Reynolds numbers but it yields the consistency condition $\delta << Re^{-\frac{5}{8}}$ which is worse than the preceding one. While this last bound is relevant in smooth regions of transitional flows, this smoothness, $\triangle u \in L^2(\Omega)$, needed does not describe the typical case of turbulent flows and is the cause of the more singular behavior as $\rightarrow \infty$. These two estimates, are not sufficiently sharp to draw useful conclusions at higher Reynolds numbers. For example, this estimate suggests the zeroth order model is $O(\ \delta)$ accurate only for $\delta << Re^{-\frac{1}{2}}$. In our third estimate (a special case of the next theorem), using the K41 phenomenology and spectral integration, we show, remarkably, the time averaged modeling consistency error is $O(\delta^{\frac{1}{3}})$ *uniformly in the Reynolds number Re and the kinematic viscosity* ν:

$$< ||\tau_0||_{L^1(\Omega)} > \leq C_1^{\frac{1}{3}} \sqrt{\frac{72}{5}} U^2 L^{\frac{7}{6}} \varepsilon^{\frac{1}{3}} \delta^{\frac{1}{3}}. \tag{8.19}$$

This bound is, we believe, quite sharp. It is a special case of the next theorem (which gives the analogous consistency error estimate for the entire family of de-convolution models).

To illustrate the improvement of this result over the previous two, suppressing all parameters except δ and Re, these two bounds together imply

$$< ||\tau_0||_{L^1(\Omega)} > \simeq C \min\{\delta^{\frac{1}{3}}, Re^{\frac{1}{2}}\delta, Re^{\frac{5}{4}}\delta^2\}. \tag{8.20}$$

The third upper bound in the minimum is never smaller than the first two for unresolved flows. The crossover point when the first becomes sharper than the second in the above minimum is when $\delta^{\frac{1}{3}} \simeq Re^{\frac{1}{2}}\delta$, or equivalently $\delta \simeq Re^{-\frac{3}{4}}$, i.e., also only when the flow is fully resolved according to the classical estimates of the numbers of degrees of freedom in a turbulent flow!

8.2. The General Approximate de-convolution Model. The sub-section gives a proof of consistency error estimates for the general de-convolution model. Thus consider the consistency error, τ_N , where τ_N is defined by

$$\tau_N := G_N\overline{u}G_N\overline{u} - u\ u. \tag{8.21}$$

THEOREM 8.5. *Consider the Cauchy problem or the periodic problem. Suppose Assumptions 1, 2 and 3 hold. Then, the model's consistency error satisfies*

$$< ||\tau_N||_{L^1(\Omega)} > \leq \frac{2\alpha^{\frac{1}{2}}}{(2N+\frac{4}{5})^{\frac{1}{2}}}\frac{U^2}{L^{N-\frac{1}{2}}}\delta^{2N+2}Re^{\frac{3}{4}N+\frac{1}{2}},$$

non uniformly in the Reynolds number and

$$< ||\tau_N||_{L^1(\Omega)} > \leq C_1^{\frac{1}{3}}(N+2)(3+\frac{2}{4N+\frac{10}{3}})^{\frac{1}{2}}\alpha^{\frac{1}{2}}U^2L^{\frac{7}{6}}\delta^{\frac{1}{3}},$$

uniformly in Re.

Proof. The analysis in the case $N = 1, 2, 3, \cdots$ follows the zeroth order case. Using stability of G_N and the estimates

$$< ||\tau_N||_{L^1} > \leq (1 + ||G_N||) < ||u||^2_{L^2} >^{\frac{1}{2}} < ||u - G_N\overline{u}||^2_{L^2} >^{\frac{1}{2}} . \tag{8.22}$$

Indeed, by Assumption 1 we can begin with $< ||u||^2_{L^2} >^{\frac{1}{2}} \leq UL^{\frac{3}{2}}$, yielding

$$< ||\tau_N||_{L^1(\Omega)} > \leq (1 + ||G_N||)UL^{\frac{3}{2}} < ||u - G_N\overline{u}||^2_{L^2(\Omega)} >^{\frac{1}{2}} . \tag{8.23}$$

As for the zeroth order model, we use spectral integration to evaluate the error in $u - G_N\overline{u}$. Lemma 3.1 implies

$$I := < ||u - G_N\overline{u}||^2_{L^2(\Omega)} > = 2\int_{k_0}^{k_{\max}}(\frac{\delta^2k^2}{1+\delta^2k^2})^{2N+2}E(k)dk. \tag{8.24}$$

Since $E(k) \leq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ holds by Assumption 3, we have

$$I \leq 2\alpha\varepsilon^{\frac{2}{3}}\int_{k_0}^{k_{\max}}(\frac{\delta^2k^2}{1+\delta^2k^2})^{2N+2}k^{-\frac{5}{3}}dk.$$

The remainder of the work in the proof is direct estimation of the above integral. The integral I requires different treatments for small and large wave numbers. We shall thus estimate the two cases separately depending on which term in the denominator (δ^2k^2 or 1) is dominant. The transition point is the cutoff wave number $\frac{1}{\delta}$; thus we break it into two integrals at this point

$$I := I_{low} + I_{high}, \text{where } I_{low} = \int_{k_0}^{\frac{1}{\delta}}...dk, \text{and } I_{high} = \int_{\frac{1}{\delta}}^{k_{high}}...dk. \tag{8.25}$$

For the low frequency components we have

$$\frac{1}{2}\delta^2k^2 \leq \frac{\delta^2k^2}{\delta^2k^2+1} \leq \delta^2k^2, \text{for } 0 \leq k \leq \frac{1}{\delta}, \tag{8.26}$$

For the low frequencies thus we have

$$I_{low} \leq \delta^{4N+4} 2\alpha\varepsilon^{\frac{2}{3}} \int_0^{\frac{1}{\delta}} k^{4N+\frac{7}{3}} dk = \tag{8.27}$$

$$= \frac{2\alpha\varepsilon^{\frac{2}{3}}}{4N+\frac{10}{3}} \delta^{\frac{2}{3}}. \tag{8.28}$$

Thus,

$$I_{low} \leq \frac{2\alpha\varepsilon^{\frac{2}{3}}}{4N+\frac{10}{3}} \delta^{\frac{2}{3}}. \tag{8.29}$$

For the high frequency components the dominant term in the denominator is the k^2 term. Thus, we have

$$\frac{1}{2} \leq \frac{\delta^2 k^2}{\delta^2 k^2 + 1} \leq 1, \text{for } \frac{1}{\delta} \leq k \leq \infty. \tag{8.30}$$

We thus have

$$I_{high} \leq 2\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{1}{\delta}}^{k_{\max}} k^{-\frac{5}{3}} dk \tag{8.31}$$

$$\leq 2\alpha\varepsilon^{\frac{2}{3}} \int_{\frac{1}{\delta}}^{\infty} k^{-\frac{5}{3}} dk \leq 3\alpha\varepsilon^{\frac{2}{3}} \delta^{\frac{2}{3}}, \tag{8.32}$$

or, collecting these two estimates,

$$I \leq (3 + \frac{2}{4N+\frac{10}{3}}) \alpha\varepsilon^{\frac{2}{3}} \delta^{\frac{2}{3}}. \tag{8.33}$$

Using the bound $\varepsilon \leq C_1 \frac{U^3}{L}$ from Assumption 2 gives the second estimate

$$< ||\tau_N||_{L^1(\Omega)} > \leq C_1^{\frac{1}{3}} (N+2)(3 + \frac{2}{4N+\frac{10}{3}})^{\frac{1}{2}} \alpha^{\frac{1}{2}} U^2 L^{\frac{7}{6}} \delta^{\frac{1}{3}}. \tag{8.34}$$

□

The impact of these estimates on practical issues in large eddy simulation is considered in the conclusion section which follows last.

9. The distribution of consistency errors among wave numbers. The above analysis is quite sharp and shows a universal and Reynolds number uniform convergence to zero of the total, global consistency error as $O((\ \delta)^{\frac{1}{3}})$ for the entire family of LES ADMs. Naturally, this raises the question of how to differentiate one model from another (e.g., a low order model from a high order one). The only possible answer is to examine the distribution of consistency errors among wave numbers. First note that the above bounds for the consistency errors can be rewritten as

$$< ||\tau_N||_{L^1(\Omega)} >^2 \leq 8 C_1^{\frac{2}{3}} \alpha U^4 L^{\frac{7}{3}} \delta^{\frac{2}{3}} \int_{k_0}^{k_{\max}} (N+2)^2 (\frac{(\delta k)^2}{1+(\delta k)^2})^{2N+2} (\delta k)^{-\frac{5}{3}} \delta dk.$$

With the obvious change of variables $k \leftarrow \delta$ this bound can be rewritten as

$$< ||\tau_N||_{L^1(\Omega)} >^2 \leq 8 C_1^{\frac{2}{3}} \alpha U^4 L^{\frac{7}{3}} \delta^{\frac{2}{3}} \int_{\frac{1}{\delta}k_0}^{\frac{1}{\delta}k_{\max}} (N+2)^2 (\frac{k^2}{1+k^2})^{2N+2} k^{-\frac{5}{3}} dk.$$

The above is an inequality but each step either arises from assumptions connected with the K41 theory (which is observed sharp at very high Reynolds numbers) or mathematical bounds which are either two sided with generic constants or for which equality is attainable. Thus, we believe that the above inequality is a quite accurate description of the true consistency error behavior possibly apart from a generic multiplicative constant. Thus, we shall view the function on the right hand side as the distribution of consistency errors among wave numbers

$$< |\widehat{\tau_N}| > (k) \leq \{8C_1^{\frac{2}{3}}\alpha U^4 L^{\frac{7}{3}}\delta^{\frac{2}{3}}(N+2)^2(\frac{\delta^2 k^2}{1+\delta^2 k^2})^{2N+2}k^{-\frac{5}{3}}\delta^{\frac{5}{3}}\delta\}^{\frac{1}{2}};$$

Re-scaling the wave numbers by $k \leftarrow \frac{k}{\delta}$ gives a more tractable formula in the re-scaled wave numbers

$$< |\widehat{\tau_N}| > (k) \leq \{8C_1^{\frac{2}{3}}\alpha U^4 L^{\frac{7}{3}}\delta^{\frac{2}{3}}(N+2)^2(\frac{k^2}{1+k^2})^{2N+2}k^{-\frac{5}{3}}\}^{\frac{1}{2}}$$

The above estimates show that, over the resolved scales, $0 \leq k \leq \pi$, the consistency errors distribution among wave numbers is given by

$$< |\widehat{\tau_N}| > (k) \leq C(\alpha, U, L, \delta)\{(N+2)(\frac{k^2}{1+k^2})^{N+1}k^{-\frac{5}{6}}\}, \text{ for } 0 \leq k \leq \pi.$$

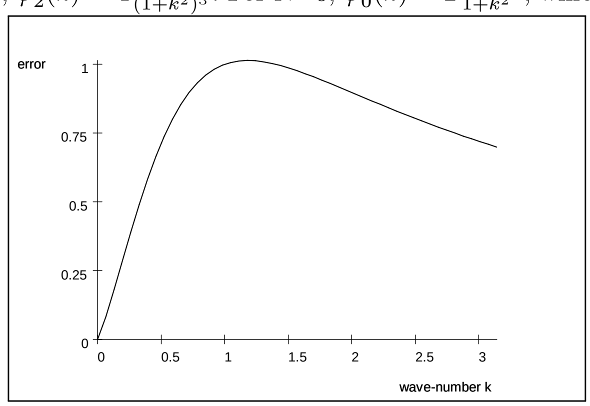
The constant multiplier on the right hand side is independent of N, so the difference in the consistency errors among the de-convolution models is contained in the other multiplier $\{(N+2)(\frac{k^2}{1+k^2})^{N+1}k^{-\frac{5}{6}}\} \simeq < |\widehat{\tau_N}| > (k)/C(\alpha, U, L, \delta)$. Thus define

$$\varphi_N(k) := (N+2)(\frac{k^2}{1+k^2})^{N+1}k^{-\frac{5}{6}}, \text{ for } 0 \leq k \leq \pi.$$

We then are led to comparing for varying N, the plots of

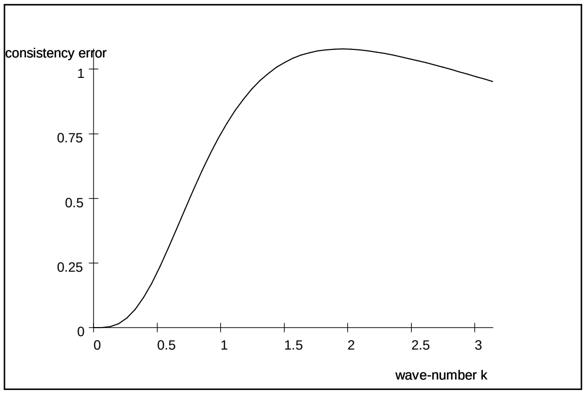
$$\varphi_N(k) := (N+2)(\frac{k^2}{1+k^2})^{N+1}k^{-\frac{5}{6}}, \text{ for } 0 \leq k \leq \pi.$$

For $N = 0, 1, 2, \cdots, 7$, for example, we have (dropping the primes) $\varphi_0(k) = 2\frac{k^{2-\frac{5}{6}}}{1+k^2}$, $\varphi_1(k) = 3\frac{k^{4-\frac{5}{6}}}{(1+k^2)^2}$, $\varphi_2(k) = 4\frac{k^{6-\frac{5}{6}}}{(1+k^2)^3}$. For N=0, $\varphi_0(k) = 2\frac{k^{2-\frac{5}{6}}}{1+k^2}$, which is plotted next.



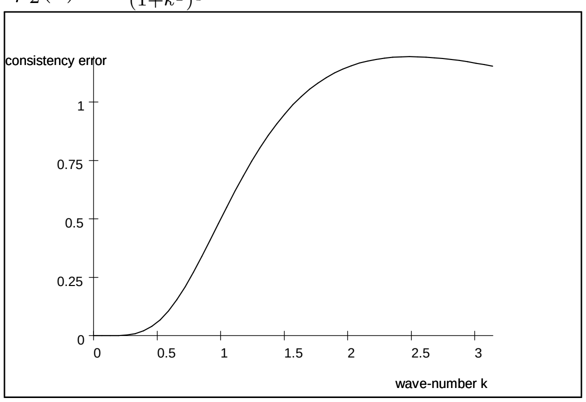
Consistency errors, N=0 LES-ADM..

For N=1 we have, $\varphi_1(k) = 3\frac{k^{4-\frac{5}{6}}}{(1+k^2)^2}$



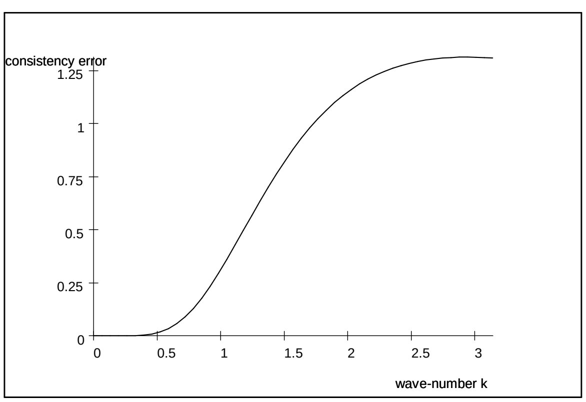
Consistency errors, N=1 LES-ADM.

For N=2 we have $\varphi_2(k) = 4\frac{k^{6-\frac{5}{6}}}{(1+k^2)^3}$



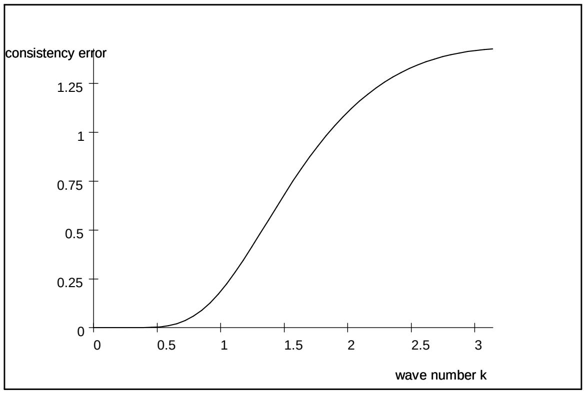
Consistency errors, N=2 LES-ADM.

For N=3, we have $\varphi_3(k) = 5(\frac{k^2}{1+k^2})^4 k^{-\frac{5}{6}}$



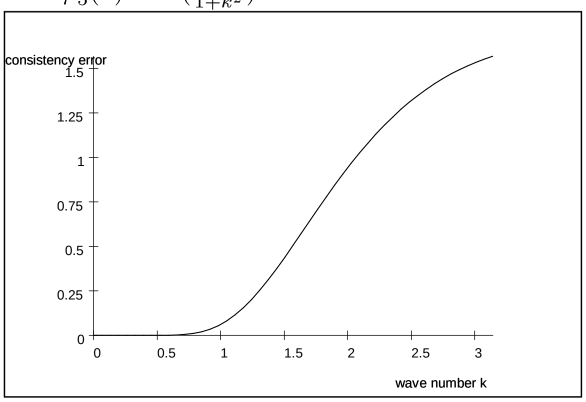
Consistency errors, N=3 LES-ADM.

For N=4 we have $\varphi_4(k) := 6(\frac{k^2}{1+k^2})^5 k^{-\frac{5}{6}}$.



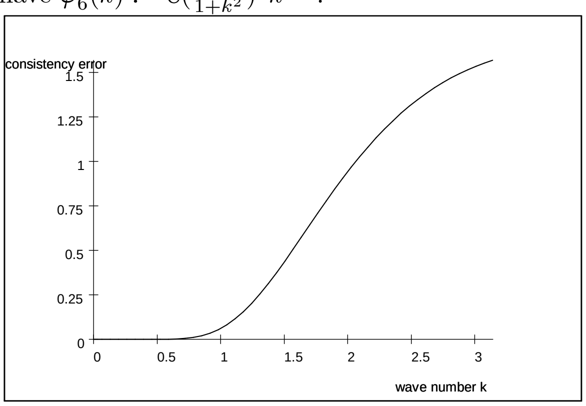
Consistency errors, N=4 LES-ADM.

For N=5 we have $\varphi_5(k) := 7(\frac{k^2}{1+k^2})^6 k^{-\frac{5}{6}}$.



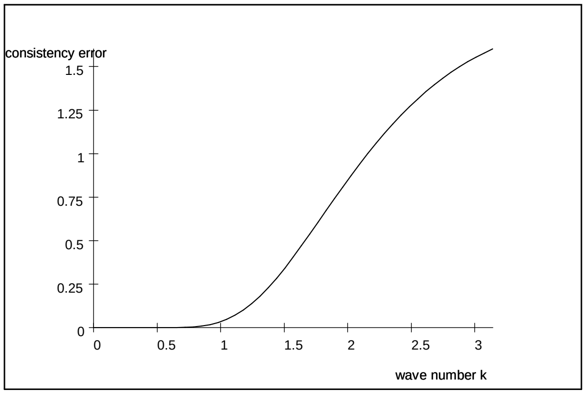
Consistency errors, N=5 LES-ADM.

For N=6 we have $\varphi_6(k) := 8(\frac{k^2}{1+k^2})^7 k^{-\frac{5}{6}}$.



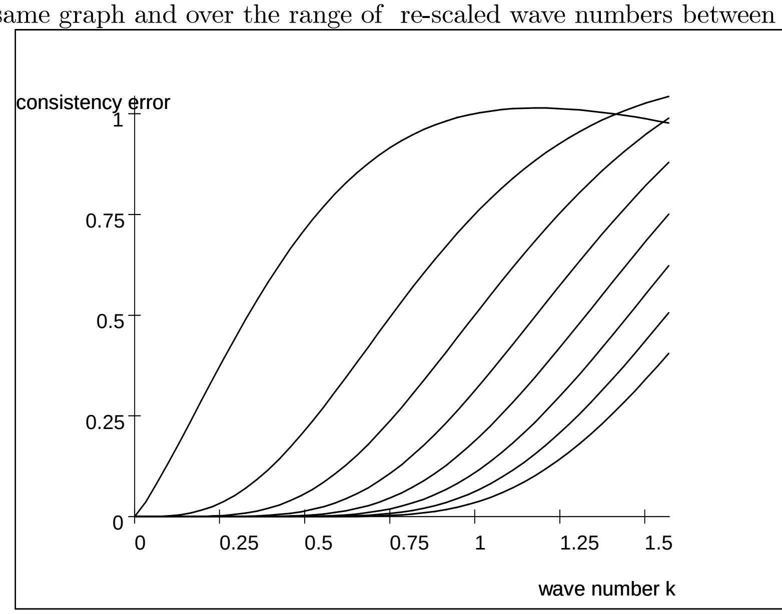
Consistency errors, N=6 LES-ADM.

For N=7 we have $\varphi_7(k) := 9(\frac{k^2}{1+k^2})^8 k^{-\frac{5}{6}}$



Consistency errors, N=7 LES-ADM.

The pattern is clear: higher order models quickly reduce the consistency errors in the largest scales and increase the consistency errors around the cutoff frequency. The decrease of the first and the increase of the second are done in a manner compatible with the overall global consistency error of $O((\ \delta)^{\frac{1}{3}})$. In particular, the error associated with re-scaled wave numbers between zero and one is very small in the models of orders 4,5 and 6. Very little improvement in the re-scaled frequencies between 1 and π for even high order models. Indeed, the next figure gives a plot of the above , all on the same graph and over the range of re-scaled wave numbers between 0 and 1.



Consistency errors, N=0 through 7.

Converting re-scaled wave numbers to length scales shows that with filter length-scale δ , the higher order models have very small consistency errors associated with the length scales $l \geq \pi\ \delta$ and consistency errors associated with length scales l satisfying $\delta \leq l \leq \pi\ \delta$ that are large and actually increase as the order of the model increases. This suggests that LES-ADMs can provide great accuracy for the largest third of the scales and that predictions near the cutoff length scale are tenuous at best.

10. Conclusions. For large eddy simulation with de-convolution models to be feasible for fully developed turbulence two competing restrictions on the averaging radius must simultaneously be satisfied. First, δ must be well inside the inertial range, giving a lower bound on the averaging radius, $\delta >> \varepsilon^{-\frac{1}{4}}\nu^{\frac{3}{4}}$. Second, the models consistency error must be small: $< ||\tau|| ><< 1$. We have seem that this gives an upper bound on δ which decreases as Re increases. For large eddy simulation to be useful, these two constraints must be satisfied simultaneously.

To illustrate the competition between these two constraints, consider the zeroth order model first and suppress all constants except δ and Re. Using the consistency error bound in Theorem 1.2 yields a narrow band of possible values of the averaging radius

$$CRe^{-\frac{3}{4}} << \frac{\delta}{L} << CRe^{-\frac{1}{2}}. \tag{10.1}$$

Thus, the extra analysis required is important for giving an accurate analytical assessment of large eddy simulation. Indeed, using instead the sharper estimate of Theorem 1.1, $< ||\tau_0|| >\leq C\delta^{\frac{1}{3}}$, predicts feasibility of any of the approximate de-convolution models provided

$$CRe^{-\frac{3}{4}} << \frac{\delta}{L} << O(1). \tag{10.2}$$

In most applications, turbulent flows simulations require a (more) universal model which is accuracy for the application's heterogeneous mix of laminar and transitional regions, boundary layers and fully developed turbulence. The entire family of approximate de-convolution models shares a remarkable uniform in Re, time-averaged accuracy for fully developed turbulence. On the other hand, the higher order models are significantly more accurate in the laminar and transitional regions. The overall analytic conclusion is that higher order models are preferable to lower order models up to the point where their computational cost become prohibitive. This observation, while surprising from the point of view of traditional error analysis, is consistent with the extensive experiments in the work of Stolz and Adams with the models.

The global consistency error is interesting but the distribution of those errors among scales might be essential. Examination of the distribution of consistency errors among scales in the appendix reveals an interesting pattern. First, the zeroth order model, which was the critical case for understanding the analysis of the entire family, has much greater consistency error for the large scales than any of the higher order models but has less consistency error for the finest resolved scales than the higher order models. Second, with filter length-scale δ , the higher order models have very small consistency errors associated with the largest length scales $l \geq \pi\ \delta$. Consistency errors associated with intermediate, resolved length scales l satisfying $\delta \leq l \leq \pi\ \delta$ are large and essentially undiminished as the order of the model increases. This suggests that LES-ADMs can provide great accuracy for the largest third of the scales and that predictions near the cutoff length scale are tenuous at best.

Our results raise (at least) three questions worthy of further study. The uniform-in-Re global accuracy of $O(\delta^{\frac{1}{3}})$ seems to be an essential feature of scale invariant models of fully developed turbulence. To improve it seems to require scale dependent models. The first question is how to formulate such models so that they are computationally agreeable. Second, a broader understanding of an LES model's consistency errors requires developing estimates directly from the Navier-Stokes equations- a very hard analytical problem. We have given one such estimate but this is not uniform in the Reynolds number. Third, understanding how an LES model's dynamics redistributes consistency errors into modeling errors is also a critical question for large eddy simulation. This is admittedly an extremely broad and difficult problem since the corresponding one for the Navier-Stokes equations is open. However, this problem has two interesting features not shared by the Navier-Stokes equations. First, the LES-ADM has a smooth, globally unique, strong solution. Thus, many technical problems which occur for the Navier-Stokes equations may not be present. Second, examining the plots in the appendix shows that a useful approximation is to assume that the LES-ADM's error equation is forced only at the moderately small scales and the result of this forcing on the largest scales is sought. Finally, we note that the results herein are extendable from the Cauchy problem using Fourier transforms (herein) to L-periodic problems using Fourier series.

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