A family of new, high order NS-α models arising from helicity correction in Leray turbulence models

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November 8, 2006

Abstract

This report (i) gives a reinterpretation of the NS-α model which leads to (ii) a family of high order NS-α-deconvolution models with the NS-α model as the zeroth order case. First, we show that the Navier-Stokes-α model arises by adding a helicity correction term to the Leray-α model. Higher order Leray models have recently been proposed in [1],[2]: the so-called Leray-deconvolution models. Second, we use an analogous idea to develop a family of higher order accurate NS-α type models, the rotational Leray-deconvolution models. This formulation also gives insight into the design of efficient algorithms for NS-α models.
keywords : NS-alpha, Leray-alpha, helicity, turbulence, approximate deconvolution
AMS subject classifications: 76D05, 76F65, 35L65.

1 Introduction

The Navier-Stokes-alpha (NS-α) model of fluid turbulence has recently been developed by Foias, Holm, Titi, Chen, Olson, and Wynne [3], and has experienced an explosion of interest by mathematicians and fluid dynamicists thereafter. It has been shown that this model has excellent mathematical and physical properties, including uniqueness of strong solutions, model energy and helicity conservation, and an accurate energy spectrum up to a (filtering radius α dependent) cut-off length scale [4],[5]. The NS-α model is given by

$$v_t + \overline{v}\cdot\nabla v + (\nabla\overline{v})^T\cdot v + \nabla p = \nu\Delta v + f, \quad \nabla\cdot\overline{v} = 0, \tag{1}$$

where *overbar* denotes the so-called alpha (α) spatial filter:

$$\overline{v} = (-\alpha^2\Delta + I)^{-1}v, \tag{2}$$

The model (1)-(2) has been derived as a closure approximation to the Reynolds equations [3], from the viewpoint of Kelvin's circulation theorem [4], and as a correction restoring frame invariance to the Leray model (3) below (Guermond, Oden, and Prudhomme [6]). Motivated by [6] and our study of helicity conservation in Large Eddy Simulation (LES) models [7], we show the NS-α model also arises as a helicity-corrected Leray-α model of turbulence.

Next, we give a new family ($N = 0, 1, 2, ...$) of models, of arbitrarily high order of accuracy (in terms of consistency error, defined in Section 2) which includes NS-α as the zeroth order case. This new family of models shows how to combine the good mathematical and physical properties of NS-α with the high accuracy of deconvolution models.

[1] Partially supported by NSF Grant DMS 0508260

2 Derivation of NS-α as a helicity-corrected Leray-α

Consider the Leray-α regularization of the Navier-Stokes equations (NSE) on the periodic box $\Omega = (0, L)^3$, given by

$$w_t + \overline{w} \cdot \nabla w + \nabla q = \nu \Delta w + f, \quad \nabla \cdot w = 0 \tag{3}$$

The model (3), although with a filter different than the α filter (2) used here, was studied by J. Leray in 1934 [8]. Although this model has some nice theoretical properties [9], as a model for simulations of turbulent flows, it can have problems with accuracy and physical fidelity. One cause of this is it does not treat rotation accurately in three dimensional turbulence.

Under periodic boundary conditions, $\nabla \cdot w = 0$ and $\nabla \cdot \overline{w} = 0$ are equivalent. Thus, in the periodic case, there is only one difference between the NS-α model (1) and the Leray-α model (3): the extra term $(\nabla \overline{v})^T \cdot v$ in the NS-α momentum equation.

On a periodic domain, the 3d Euler equations conserve both energy ($E = \frac{1}{2}\int_\Omega u \cdot u$) and helicity ($H = \int_\Omega u \cdot (\nabla \times u)$) [10]. Thus the nonlinearity in the Navier-Stokes equations (NSE) neither adds nor dissipates helicity. In fact, just as 2d turbulent flows have an energy and enstrophy cascade, it has recently been shown that in 3d, the nonlinearity in the NSE is responsible for *cascading* helicity as well as energy through the inertial range [11],[12]. Hence for a turbulence model to even begin to treat this rotational quantity correctly, its nonlinearity should neither create nor destroy helicity.

We begin by recalling a calculation from [7]. Let $(\cdot, \cdot)$ and $||\cdot||$ denote the usual $L^2(\Omega)$ inner product and norm.

Lemma 2.1. *Under periodic boundary conditions and in the absence of viscosity and external force, a solution w to the Leray-α model (3) satisfies*

$$\frac{d}{dt}H(t) = \frac{d}{dt}(w, \nabla \times w) = 2((\nabla \overline{w})^T \cdot w, \nabla \times w). \tag{4}$$

Proof. Set $\nu = f = 0$ in (3), multiply the momentum equation in (3) by $(\nabla \times w)$ and integrate over Ω. This gives

$$(w_t, \nabla \times w) + (\overline{w} \cdot w, \nabla \times w) + (\nabla q, \nabla \times w) = 0. \tag{5}$$

We rewrite (5) using the identity

$$a \times (\nabla \times b) = -a \cdot \nabla b - (\nabla a)^T \cdot b + \nabla(a \cdot b), \tag{6}$$

which gives

$$(w_t, \nabla \times w) - (\overline{w} \times (\nabla \times w), \nabla \times w) - ((\nabla \overline{w})^T \cdot w, \nabla \times w) + (\nabla(q + \overline{w} \cdot w), \nabla \times w) = 0. \tag{7}$$

Using integration by parts, the fact the the curl of two vectors is perpendicular to each of them, and the curl of the gradient of a scalar function vanishes, (7) reduces to

$$\frac{1}{2}\frac{d}{dt}(w, \nabla \times w) - ((\nabla \overline{w})^T \cdot w, \nabla \times w) = 0. \tag{8}$$

Integrating over time completes the proof. □

Remark 2.2. *Since in general $((\nabla \overline{w})^T \cdot w, \nabla \times w) \neq 0$, the equality (4) means that the Leray-α model does not conserve helicity.*

The Leray-α model (3) does not treat helicity, and thus three dimensional rotational structures, accurately. It is evident from (8) that the $((\nabla \overline{w})^T \cdot w, \nabla \times w)$ term arising from the nonlinearity is responsible for non-conservation of helicity. Thus, to achieve helicity conservation in the model, $(\nabla \overline{w})^T \cdot w$ needs to be added to the left hand side of the momentum equation in (3). With the addition of $(\nabla \overline{w})^T \cdot w$ to the left hand side of (3) becomes

$$w_t + \overline{w} \cdot \nabla w + (\nabla \overline{w})^T \cdot w + \nabla q = \nu \Delta w + f, \quad \nabla \cdot w = 0, \tag{9}$$

Proposition 2.3. *A solution to the equations (9) resulting from the addition of* $(\nabla\overline{w})^T \cdot w$ *to the left hand side of (3) satisfies*

$$\frac{d}{dt}(w, \nabla \times w) = \frac{d}{dt}H(t) = 0 \tag{10}$$

Proof. This follows directly from the proof of Lemma 2.1. □

Remark 2.4. *The Leray-α model (3) is thus corrected to conserve helicity by adding the extra term* $(\nabla\overline{w})^T \cdot w$ *to its momentum equation. The model resulting from this correction (9) is precisely the NS-α model (1)-(2).*

Using (6), the NS-α model can be written in rotational form as

$$v_t - \overline{v} \times (\nabla \times v) + \nabla\tilde{p} = \nu\Delta v + f, \quad \nabla \cdot \overline{v} = 0, \tag{11}$$

where $\tilde{p}$ is a dynamic pressure. This form of the NS-α model could also be considered a Leray model for the rotational form of the nonlinearity. It is interesting because from here it is clear that the NS-α model can be discretized using methods similar to those used for the rotational form of the NSE. Recent work of Benzi and Liu in [13] has shown that solving the NSE in this form can lead to better efficiency, and so these advances may be able to be applied to the NS-α model. This form of the NS-α model also lends itself to the less efficient, but more physically accurate, energy and helicity conserving scheme [14].

Remark 2.5. *Although the focus of this report is 3d turbulence, in 2d the NS-α model also conserves enstrophy* ($Ens = \frac{1}{2}\|\nabla \times v\|^2$) *for periodic inviscid flow, which follows from the fact that* $(\overline{v}\times(\nabla \times v), \Delta v) = 0$. *The Leray-α model conserves a model enstrophy,* $Ens_L := \frac{1}{2}(\|\nabla \times \overline{v}\|^2 + \alpha^2\|\Delta v\|^2)$ *[7].*

We next define consistency error, and discuss the models' accuracies in terms of it.

Definition 2.6 (Consistency Error). *The consistency error of an LES model is the residual of the solution of the Navier-Stokes equations in the model.*

Even though the NS-α model has better physical fidelity than the Leray-α model, both of these models have the same, low order, asymptotic (as $\alpha \to 0$) consistency error.

Lemma 2.7. *The consistency errors in the NS-α and Leray-α models,* τ_{NS} *and* τ_L *satisfy for smooth* u,

$$\|\tau_{NS-\alpha}\| = C\alpha^2 \qquad \|\tau_L\| = C\alpha^2. \tag{12}$$

Proof. To show the stated consistency error in NS-α, we start with its rotational form (11). Substitute the (rotational form) NSE solutions u and p (i.e. p is Bernoulli pressure) for v and $\tilde{p}$ in (11), and add τ_{NS} to the right hand side of (11). This gives

$$\begin{aligned} \tau_{NS-\alpha} &= u \times (\nabla \times u) - \overline{u} \times (\nabla \times u) & (13)\\ &= (u - \overline{u}) \times (\nabla \times u) = (u - (-\alpha^2\Delta + I)^{-1}u) \times (\nabla \times u) & (14)\\ &= (-\alpha^2\Delta + I)^{-1}(\alpha^2\Delta u) \times (\nabla \times u) = \alpha^2\Delta\overline{u} \times (\nabla \times u), & (15) \end{aligned}$$

and so

$$\|\tau_{NS-\alpha}\| = C(u)\alpha^2. \tag{16}$$

For Leray-α, insert the NSE solution u, p into the momentum equation in (3), and add τ_L to the right hand side of the equation to get

$$\tau_L = \overline{u} \cdot \nabla u - u \cdot \nabla u = (\overline{u} - u) \cdot \nabla u = C(u)\alpha^2. \tag{17}$$

□

Lemma 2.7 shows both of these models have only $O(\alpha^2)$ accuracy. In the next section, we show the same technique of helicity correction also works on a higher order accurate Leray-deconvolution family of models, of which the Leray-α is the zeroth order model.

3 High Order NS-α Models

A new family of Leray-type models has been proposed and studied in [1], the Leray-deconvolution models. For ease of notation, let $A = (-\alpha^2 + I)$, so $A^{-1}w := \overline{w}$. Define the N^{th} deconvolution operator, D_N, by

$$D_N w = \sum_{n=0}^{N} (I - A^{-1})^n w \tag{18}$$

In [15] it is shown that D_N is an approximate inverse to A^{-1}: for smooth u,

$$D_N \overline{u} = u + C(u)(\alpha^{2N+2}). \tag{19}$$

When using the D_N operator, it is sometimes convenient to use the following norm:

$$\|w\|_{D_N} := (w, D_N w)^{1/2} \tag{20}$$

It is shown in [16] that (20) defines a norm uniformly in α equivalent to the L^2 norm since the D_N operator is positive.

The Leray-deconvolution family of models are given by

$$w_t + D_N \overline{w} \cdot \nabla w + \nabla q = \nu \Delta w + f, \quad \nabla \cdot w = 0. \tag{21}$$

Lemma 3.1. *The consistency error in the Leray-deconvolution model is*

$$\tau_{LD} = D_N \overline{u} \cdot \nabla u - u \cdot \nabla u \tag{22}$$

and for smooth u satisfies

$$\|\tau_{LD}\| = C(u)\alpha^{2N+2} \tag{23}$$

Proof. This follows the same as for NS-α and Leray-α, except using (19). □

Lemma 3.1 shows that in smooth flow regions, Leray-deconvolution models have higher order consistency error that the zeroth order case, the Leray-α model. In addition, recent work in [7], [2], [1] has also shown that for $N \geq 1$, (21) has significantly greater accuracy than the Leray-α model (the $N = 0$ case). However, like the Leray-α model, the Leray-deconvolution family fails to conserve helicity in the case of periodic boundaries and no external force or viscosity [7].

Lemma 3.2. *Under periodic boundary conditions with no viscosity or external force, a solution w to a Leray-deconvolution model (21) satisfies*

$$\frac{d}{dt}(w, \nabla \times w) = \frac{d}{dt}H(t) = 2\int_0^T ((\nabla D_N \overline{w})^T \cdot w, \nabla \times w)\ dt \tag{24}$$

Proof. This proof is analogous to that of Lemma 2.1. □

Remark 3.3. *Since $((\nabla D_N \overline{w})^T \cdot w, \nabla \times w) \neq 0$ in general, helicity is not conserved in the Leray-deconvolution models.*

By the same techniques used for the Leray model in Section 2, the Leray-deconvolution models can be *corrected* to conserve helicity. Consider the model resulting from adding $(\nabla D_N \overline{w})^T \cdot w$ to the left hand side of (21):

$$w_t + D_N \overline{w} \cdot \nabla w + (\nabla D_N \overline{w})^T \cdot w + \nabla q = \nu \Delta w + f, \quad \nabla \cdot w = 0. \tag{25}$$

This model (25) no longer will conserve usual energy. However, we will show next it conserves a *model* energy, defined by

$$E_{model} := \frac{1}{2}\|\overline{w}\|_{D_N}^2 + \frac{\alpha^2}{2}\|\nabla \overline{w}\|_{D_N}^2. \tag{26}$$

Proposition 3.4. *Under periodic boundary conditions and no viscosity or external force, the model (25) conserves helicity a model energy,*

$$E_{model}(T) = E_{model}(0), \tag{27}$$

$$H(T) = H(0), \tag{28}$$

and in 2d, enstrophy.

$$Ens(T) = Ens(0) \tag{29}$$

Proof. We begin this proof by rewriting (25) in rotational form, by using (6) and grouping the resulting gradient term with the pressure to form a new dynamic pressure, yielding

$$w_t - (D_N\overline{w} \times (\nabla \times w)) + \nabla \tilde{q} = \nu \Delta w + f. \tag{30}$$

Helicity conservation follows exactly as in the $N = 0$ case proven in Section 2. For model energy conservation, multiply (30) by $D_N\overline{w}$, integrate over Ω and set $\nu = f = 0$. This vanishes the pressure term, leaving

$$0 = (w_t, D_N\overline{w}) = \frac{1}{2}(w, D_N\overline{w}) = \frac{1}{2}(A\overline{w}, D_N\overline{w}). \tag{31}$$

Expanding A and integrating by parts gives

$$0 = \frac{d}{dt}\left(\frac{1}{2}(\overline{w}, D_N\overline{w}) + \frac{\alpha^2}{2}(\nabla\overline{w}, \nabla D_N\overline{w})\right) = \frac{d}{dt}\left(\frac{1}{2}\|\overline{w}\|^2_{D_N} + \frac{\alpha^2}{2}\|\nabla\overline{w}\|^2_{D_N}\right) = \frac{d}{dt}E_{model}, \tag{32}$$

and thus the energy proof is complete. For enstrophy conservation we start with the rotational form of the momentum equation (30), set $\nu = f = 0$, multiply by Δw, and integrate over the domain. This gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla \times w\|^2 - (D_N\overline{w} \times (\nabla \times w), \Delta w) + (\nabla \tilde{q}, \Delta v) = 0. \tag{33}$$

The pressure term vanishes after integration by parts, and the nonlinearity also vanishes using integration by parts, the commutation of differential operators under periodic boundary condtitions, and the fact that the solution is divergence free:

$$\begin{aligned}(D_N\overline{w} \times (\nabla \times w), \Delta w) &= -\left(\nabla \times (D_N\overline{w} \times (\nabla \times w)), \nabla \times w\right) \\ &= -\left(D_N\overline{w} \cdot \nabla(\nabla \times w) - (\nabla \times w) \cdot \nabla D_N\overline{w}, \nabla \times w\right) = ((\nabla \times w) \cdot \nabla D_N\overline{w}, \nabla \times w)\end{aligned} \tag{34}$$

which is zero in two dimensions. This completes the proof. ☐

This new family of models (25) thus has conservation laws analogous to those of the NSE, in both two and three dimensions. Furthermore, we can show it has consistency error of arbitrary order of accuracy:

Proposition 3.5. *The consistency error in the N^{th} NS-α-deconvolution model satisfies*

$$\tau_{NS\alpha D} = u \times (\nabla \times u) - D_N\overline{u} \times (\nabla \times u), \tag{35}$$

and for smooth u,

$$\|\tau_{NS\alpha D}\| = C(u)\alpha^{2N+2} \tag{36}$$

Proof. After starting with the rotational form of the model (30), this result follows similar to the consistency error in Leray-deconvolution. ☐

Thus this new model (25) retains the high accuracy of Leray-deconvolution model (21), but treats important physical quantities more accurately.

4 Conclusions

The NS-α model has been re-derived as the Leray-α model with a helicity correction term. It was also shown that a new, more accurate generalization of the NS-α model, the NS-α-deconvolution family of models, can be derived as a helicity-corrected Leray-α-deconvolution models. We believe this new family of models, which is more physically relevant than Leray-deconvolution and more accurate than NS-α, deserves future study.

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