Sharp Global Bounds for the Hessian on Pseudo-Hermitian Manifolds

Sagun Chanillo[1] and Juan J. Manfredi[2]

[1] Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854, chanillo@math.rutgers.edu
[2] Department of Mathematics, University of Pittsburgh, Pittsburgh PA 15260, manfredi@pitt.edu

Dedicated to the memory of our friend and colleague Carlos Segovia.

1 Introduction

In PDE theory, Harmonic Analysis enters in a fundamental way through the basic estimate valid for $f \in C_0^\infty(\mathbb{R}^n)$, which states,

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^n)} \leq c(n,p) \left\| \Delta f \right\|_{L^p(\mathbb{R}^n)}, \text{ for } 1 < p < \infty. \tag{1}$$

This estimate is really a statement of the L^p boundedness of the Riesz transforms, and thus (1) is a consequence of the multiplier theorems of Marcinkiewicz and Hörmander-Mikhlin, [15]. More sophisticated variants of (1) can be proved by relying on the square function [15] and [14]. In particular (1) leads to a-priori $W^{2,p}$ estimates for solutions of

$$\Delta u = f, \text{ for } f \in L^p. \tag{2}$$

Knowledge of $c(p,n)$ allows one to perform a perturbation of (2) and study

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f \tag{3}$$

as was done by Cordes [4], where $A = (a^{ij})$ is bounded, measurable, elliptic and close to the identity in a sense made precise by Cordes. The availability of the estimates of Alexandrov-Bakelman-Pucci and the Krylov-Safonov theory

[7] allows one to obtain estimates for (3) in full generality without relying on a perturbation argument. See also [12].

Our focus here will be to study the CR analog of (3). Since at this moment in time there is no suitable Alexandrov-Bakelman-Pucci estimate for the CR analog of (3) we will be seeking a perturbation approach based on an analog of (1) on a CR manifold. Our main interest is the case $p = 2$ in (1). In this case a simple integration by parts suffices to prove (1) in $\mathbb{R}^n$. We easily see that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|^2_{L^2(\mathbb{R}^n)} = \|\Delta f\|^2_{L^2(\mathbb{R}^n)} . \tag{4}$$

In the case of (1) on a CR manifold a result has been recently obtained by Domokos-Manfredi [6] in the Heisenberg group. The proof in [6] makes uses of the harmonic analysis techniques in the Heisenberg group developed by Strichartz [16] that will not apply to studying such inequalities for the Hessian on a general CR manifold, although other nilpotent groups of step 2 can be treated similarly [5].

Instead we shall proceed by integration by parts and use of the Bochner technique. A Bochner identity on a CR manifold was obtained by Greenleaf [8] and will play an important role in our computations.

We now turn to our setup. We consider a smooth orientable manifold M^{2n+1}. Let $\mathcal{V}$ be a vector sub-bundle of the complexified tangent bundle $\mathbb{C}TM$. We say that $\mathcal{V}$ is a CR bundle if

$$\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}, \ [\mathcal{V}, \mathcal{V}] \subset \mathcal{V}, \text{ and } \dim_{\mathbb{C}} \mathcal{V} = n. \tag{5}$$

A manifold equipped with a sub-bundle satisfying (5) will be called a CR manifold. See the book by Trèves [18]. Consider the sub-bundle

$$H = \operatorname{Re}\left(\mathcal{V} \oplus \overline{\mathcal{V}}\right). \tag{6}$$

H is a $2n$-dimensional vector sub-bundle of the tangent bundle TM. We assume that the real line bundle $H^\perp \subset T^*M$, where T^*M is the cotangent bundle, has a smooth non-vanishing global section. This is a choice of a non-vanishing 1-form θ on M and (M, θ) is said to define a pseudo-hermitian structure. M is then called a pseudo-hermitian manifold. Associated to θ we have the Levi form L_θ given by

$$L_\theta(V, \overline{W}) = -i \, d\theta(V \wedge \overline{W}), \text{ for } V, W \in \mathcal{V}. \tag{7}$$

We shall assume that L_θ is definite and orient θ by requiring that L_θ is positive definite. In this case, we say that M is strongly pseudo-convex. We shall always assume that M is strongly pseudo-convex.

On a manifold M that carries a pseudo-hermitian structure, or a pseudo-hermitian manifold, there is a unique vector field T, transverse to H defined in (6) with the properties

$$\theta(T) = 1 \quad \text{and} \quad d\theta(T, \cdot) = 0. \tag{8}$$

T is also called the Reeb vector field. The volume element on M is given by

$$dV = \theta \wedge (d\theta)^n. \tag{9}$$

A complex valued 1-form η is said to be of type $(1,0)$ if $\eta(\overline{W}) = 0$ for all $W \in \mathcal{V}$, and of type $(0,1)$ if $\eta(W) = 0$ for all $W \in \mathcal{V}$.

An admissible co-frame on an open subset of M is a collection of $(1,0)$ forms $\{\theta^1, \ldots, \theta^\alpha, \ldots, \theta^n\}$ that locally form a basis for $\mathcal{V}^*$ and such that $\theta^\alpha(T) = 0$ for $1 \le \alpha \le n$. We set $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. We then have that $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ locally form a basis of the complex co-vectors, and the dual basis are the complex vector fields $\{T, Z_\alpha, \overline{Z_\alpha}\}$. For $f \in C^2(M)$ we set

$$Tf = f_0, \qquad Z_\alpha f = f_\alpha, \qquad \overline{Z_\alpha} f = f_{\overline{\alpha}}. \tag{10}$$

We note that in the sequel all our functions f will be real valued.

If follows from (5), (7), and (8) that we can express

$$d\theta = i\, h_{\alpha\overline{\beta}}\, \theta^\alpha \wedge \theta^{\overline{\beta}}. \tag{11}$$

The hermitian matrix $(h_{\alpha\overline{\beta}})$ is called the Levi matrix.

On pseudo-hermitian manifolds Webster [19] has defined a connection, with connection forms ω_α^β and torsion forms $\tau_\beta = A_{\beta\alpha}\theta^\alpha$, with structure relations

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau_\beta, \qquad \omega_{\alpha\overline{\beta}} + \omega_{\bar{\beta}\alpha} = dh_{\alpha\overline{\beta}} \tag{12}$$

and

$$A_{\alpha\beta} = A_{\beta\alpha}. \tag{13}$$

Webster defines a curvature form

$$\prod\nolimits_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta,$$

where we have used the Einstein summation convention. Furthermore in [19] it is shown that

$$\prod\nolimits_\alpha^\beta = R_{\alpha\bar{\beta}\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + \text{ other terms.}$$

Contracting two indices using the Levi matrix $(h_{\alpha\bar{\beta}})$ we get

$$R_{\alpha\bar{\beta}} = h^{\rho\bar{\sigma}}\ R_{\alpha\bar{\beta}\rho\bar{\sigma}}. \tag{14}$$

The Webster-Ricci tensor $\mathrm{Ric}(V,V)$ for $V \in \mathcal{V}$ is then defined as

$$\mathrm{Ric}(V,V) = R_{\alpha\bar{\beta}}x^\alpha\overline{x^\beta}, \text{ for } V = \sigma_\alpha x^\alpha Z_\alpha. \tag{15}$$

The torsion tensor is defined for $V \in \mathcal{V}$ as follows

$$\mathrm{Tor}(V,V) = i\left(A_{\bar{\alpha}\bar{\beta}}\overline{x^{\alpha}}\bar{x}_{\beta} - A_{\alpha\beta}x^{\alpha}x^{\beta}\right). \tag{16}$$

In [19], Prop. (2.2), Webster proves that the torsion vanishes if $\mathcal{L}_T$ preserves H, where $\mathcal{L}_T$ is the Lie derivative. In particular if M is a hypersurface in $\mathbb{C}^{n+1}$ given by the defining function ρ

$$\mathrm{Im} z_{n+1} = \rho(z,\overline{z}), \qquad z = (z_1, z_2, \ldots, z_n) \tag{17}$$

then Webster's hypothesis is fulfilled and the torsion tensor vanishes on M. Thus for the standard CR structure on the sphere S^{2n+1} and on the Heisenberg group the torsion vanishes.

Our main focus will be the sub-Laplacian Δ_b. We define the complex horizontal gradient ∇_b and Δ_b as follows:

$$\nabla_b f = \sum_{\alpha} f_{\overline{\alpha}} Z_{\alpha}, \tag{18}$$

$$\Delta_b f = \sum_{\alpha} f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}. \tag{19}$$

When $n = 1$ we will need to frame our results in terms of the CR Paneitz operator. Define the Kohn Laplacian $\Box_b$ by

$$\Box_b = \Delta_b + i\,T. \tag{20}$$

Then the CR Paneitz operator P_0 is defined by

$$P_0 f = \left(\overline{\Box}_b \Box_b + \Box_b \overline{\Box}_b\right) f - 2\left(Q + \overline{Q}\right) f, \tag{21}$$

where

$$Qf = 2i\,(A^{11} f_1)_1.$$

See [10] and [9] for further details.

2 The Main Theorem

Theorem 1. *Let M^{2n+1} be a strictly pseudo-convex pseudo-hermitian manifold. When M is non compact assume that $f \in C_0^\infty(M)$. When M is compact with $\partial M = \emptyset$ we may assume $f \in C^\infty(M)$. When f is real valued and $n \geq 2$ we have*

$$\sum_{\alpha,\beta} \int_M |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \int_M \Big(Ric + \frac{n}{2}\,Tor\Big)(\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{2n} \int_M |\Delta_b f|^2. \quad (22)$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$\int_M |f_{11}|^2 + |f_{1\bar{1}}|^2 + \int_M \left(Ric - \frac{3}{2}\,Tor \right)(\nabla_b f, \nabla_b f) \leq \frac{3}{2} \int_M |\Delta_b f|^2. \quad (23)$$

Proof. We begin by noting the Bochner identity established by Greenleaf, Lemma 3 in [8]:

$$\frac{1}{2}\Delta_b\left(|\nabla_b f|^2\right) = \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \mathrm{Re}\left(\nabla_b f, \nabla_b(\Delta_b f)\right) \quad (24)$$
$$+ \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b, \nabla_b) + i \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}\right).$$

where for $V, W \in \mathcal{V}$ we use the notation $(V, W) = L_\theta(V, \overline{W})$ and $|V| = (V, V)^{1/2}$. Using the fact that $f \in C_0^\infty(M)$ or if $\partial M = \emptyset$, M is compact, integrate (24) over M using the volume (9) to get

$$\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) \quad (25)$$
$$+ i \int_M \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}\right) = -\int_M \mathrm{Re}\left(\nabla_b f, \nabla_b(\Delta_b f)\right).$$

Integration by parts in the term on the right yields (see (5.4) in [8])

$$-\int_M \mathrm{Re}(\nabla_b f, \nabla_b(\Delta_b f)) = \frac{1}{2}\int_M |\Delta_b f|^2. \quad (26)$$

Combining (25) and (26) we get

$$\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \int_M \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) \quad (27)$$
$$+ i \int_M \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}\right) = \frac{1}{2}\int_M |\Delta_b f|^2.$$

To handle the third integral in the left-hand side, we use Lemmas 4 and 5 of [8] (valid for real functions) according to which we have

$$i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) = \frac{2}{n}\int_M \Big(\sum_{\alpha,\beta} \big(|f_{\alpha\bar{\beta}}|^2 - |f_{\alpha\beta}|^2\big) - \mathrm{Ric}(\nabla_b f, \nabla_b f)\Big), \tag{28}$$

and

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) = &- \frac{4}{n}\int_M \big|\sum_\alpha f_{\alpha\bar{\alpha}}\big|^2 \\ &+ \frac{1}{n}\int_M |\Delta_b f|^2 \\ &+ \int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \tag{29}$$

Applying the Cauchy-Schwarz inequality to the the first term in the right-hand side of (29) we get

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) \geq &- 4\int_M \sum_{\alpha,\beta} |f_{\alpha\bar{\beta}}|^2 \\ &+ \frac{1}{n}\int_M |\Delta_b f|^2 \\ &+ \int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \tag{30}$$

Multiply (28) by $1-c$ and (30) by c , $0 < c < 1$, and where c will eventually be chosen to be $1/(n+1)$, and add to get

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) \geq \ & 2\ \frac{(1-c)}{n}\int_M \sum_{\alpha,\beta} \big(|f_{\alpha\bar{\beta}}|^2 - |f_{\alpha\beta}|^2\big) \\ &- 2\frac{(1-c)}{n}\int_M \mathrm{Ric}(\nabla_b f, \nabla_b f) \\ &- 4c\int_M \sum_{\alpha,\beta} |f_{\alpha\overline{\beta}}|^2 \\ &+ \frac{c}{n}\int_M |\Delta_b f|^2 + c\int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \tag{31}$$

We now insert (31) into (27) and simplify. We have

$$\begin{aligned}
&\left(1-\frac{2(1-c)}{n}\right)\int_M \mathrm{Ric}(\nabla_b f,\nabla_b f)+\\
&\left(\frac{(n-2)}{2}+c\right)\int_M \mathrm{Tor}(\nabla_b f,\nabla_b f)+\\
&\left(1+\frac{2(1-c)}{n}-4c\right)\int_M \sum_{\alpha,\beta}|f_{\alpha\bar\beta}|^2+ \qquad (32)\\
&\left(1-\frac{2(1-c)}{n}\right)\int_M \sum_{\alpha,\beta}|f_{\alpha\beta}|^2 \le \left(\frac{1}{2}-\frac{c}{n}\right)\int_M |\Delta_b f|^2.
\end{aligned}$$

Let $c=1/(n+1)$. Then (32) becomes

$$\left(\frac{n-1}{n+1}\right)\left[\int_M \sum_{\alpha,\beta}\left(|f_{\alpha\beta}|^2+|f_{\alpha\bar\beta}|^2\right)+\int_M\left(\mathrm{Ric}+\frac{n}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)\right] \qquad (33)$$
$$\le \left(\frac{n-1}{n+1}\right)\left(\frac{n+2}{2n}\right)\int_M |\Delta_b f|^2.$$

Since $n\ge 2$, $n-1>0$ and we can cancel the factor $\frac{n-1}{n+1}$ from both sides to get (22).

We now establish (23) using some results by Li-Luk [11] and [9]. When $n=1$, identity (27) becomes

$$\int_M |f_{1\bar1}|^2+|f_{11}|^2+\int_M\left(\mathrm{Ric}-\frac{1}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f) \qquad (34)$$
$$+\,i\int_M (f_{10}f_{\bar1}-f_{\bar10}f_1)=\frac{1}{2}\int_M |\Delta_b f|^2.$$

By (3.8) in [11] we have

$$i\int_M (f_{01}f_{\bar1}-f_{0\bar1}f_1)=-\int_M f_0^2.$$

Moreover, by (3.6) in [11] we also have

$$i\,(f_{10}f_{\bar1}-f_{\bar10}f_1)=i\,(f_{01}f_{\bar1}-f_{0\bar1}f_1)+\mathrm{Tor}(\nabla_b f,\nabla_b f)$$

and combining the last two identities we get

$$i\int_M (f_{10}f_{\bar1}-f_{\bar10}f_1)=-\int_M f_0^2+\int_M \mathrm{Tor}(\nabla_b f,\nabla_b f). \qquad (35)$$

Substituting (35) into (34) we obtain

$$\int_M |f_{1\bar1}|^2+|f_{11}|^2+\int_M\left(\mathrm{Ric}+\frac{1}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)-\int_M f_0^2 \qquad (36)$$
$$=\frac{1}{2}\int_M |\Delta_b f|^2.$$

Next, we use (3.4) in [9],

$$\int_M f_0^2 = \int_M |\Delta_b f|^2 + 2\int_M \mathrm{Tor}(\nabla_b f, \nabla_b f) - \frac{1}{2}\int_M P_0 f \cdot f. \tag{37}$$

Finally, substitute (37) into (36) and simplify to get

$$\int_M |f_{1\bar{1}}|^2 + |f_{11}|^2 + \int_M \left(\mathrm{Ric} - \frac{3}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) + \frac{1}{2}\int_M P_0 f \cdot f$$
$$= \frac{3}{2}\int_M |\Delta_b f|^2.$$

Assuming $P_0 \geq 0$ we obtain (23). □

We now wish to make some remarks about our theorem:

(a) It is shown in [6] that on the Heisenberg group the constant $(n+2)/2n$ is sharp. Since the Heisenberg group is a pseudo-hermitian manifold with Ric $\equiv 0$ and Tor $\equiv 0$, we easily conclude our theorem is sharp and contains the result proved in [6].

(b) We notice that when we consider manifolds such that Ric$+(n/2)$Tor $>$ 0, then for $n \geq 2$, in general we have the strict inequality

$$\sum_{\alpha,\beta}\int_M |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 < \frac{n+2}{2n}\int_M |\Delta_b f|^2.$$

On the Heisenberg group Ric $\equiv 0$, Tor $\equiv 0$ and the constant $(n+2)/2n$ is achieved by a function with fast decay [6]. Thus, the Heisenberg group is, in a sense, extremal for inequality (22) in Theorem 1. A similar remark holds for inequality (23).

(c) The hypothesis on the Paneitz operator in the case $n = 1$ in our theorem is satisfied on manifolds with zero torsion. A result from [2] shows that if the torsion vanishes the Paneitz operator is non-negative.

(d) We note that Chiu [9] shows how to perturb the standard pseudo-hermitian structure in $\mathbb{S}^3$ to get a structure with non-zero torsion, for which $P_0 > 0$ and Ric $- (3/2)$Tor > 1. To get such a structure, let θ be the contact form associated to the standard structure on $\mathbb{S}^3$. Fix g a smooth function on $\mathbb{S}^3$. For $\epsilon > 0$ consider

$$\tilde{\theta} = e^{2f}\theta, \text{ where } f = \epsilon^3 \sin(\frac{g}{\epsilon}). \tag{38}$$

Since the sign of the Paneitz operator is a CR invariant and θ has zero torsion we conclude by [2] that the CR Paneitz operator $\tilde{P}_0$ associated to $\tilde{\theta}$ satisfies $\tilde{P}_0 > 0$. Furthermore following the computation in Lemma (4.7) of [9], we easily have for small ϵ that

$$\text{Ric} - \frac{3}{2}\text{Tor} \geq (2 + O(\epsilon))\, e^{-2f} \geq 1 \geq 0.$$

Thus, the hypothesis of the case $n = 1$ in our theorem are met, and for such $(M, \tilde{\theta})$ we have, for $f \in C^\infty(M)$ the estimate

$$\int_M |f_{11}|^2 + |f_{1\bar{1}}|^2 \, dV \leq \frac{3}{2} \int_M |\Delta_b f|^2 \, dV.$$

(e) Compact pseudo-hermitian 3-manifolds with negative Webster curvature may be constructed by considering the co-sphere bundle of a compact Riemann surface of genus g, $g \geq 2$. Such a construction is given in [3].

3 Applications to PDE

For applications to subelliptic PDE it is helpful to re-state our main result Theorem 1 in its real version. We set

$$X_i = \text{Re}(Z_i) \text{ and } X_{i+n} = \text{Im}(Z_i)$$

for $i = 1, 2 \ldots, n$. The real horizontal gradient of a function is the vector field

$$\mathfrak{X}(f) = \sum_{i=1}^{2n} X_i(f) X_i.$$

Its sublaplacian is given by

$$\Delta_{\mathfrak{X}} f = \sum_{i=1}^{2n} X_i X_i(f),$$

and the horizontal second derivatives are the $2n \times 2n$ matrix

$$\mathfrak{X}^2 f = (X_i X_j(f)).$$

For f real we have the following relationships

$$\nabla_b f = \mathfrak{X}(f) + i \left(\sum_{i=1}^{n} X_i(f) X_{i+n} - X_{i+n}(f) X_i \right),$$

$$\Delta_b f = 2\, \Delta_{\mathfrak{X}} f,$$

and

$$\sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 = 2 \sum_{i,j} |X_i X_j(f)|^2 = 2|\mathfrak{X}^2 f|^2.$$

Theorem 2. *Let M^{2n+1} be a strictly pseudo-convex pseudo-hermitian manifold. When M is non compact assume that $f \in C_0^\infty(M)$. When M is compact with $\partial M = \emptyset$ we may assume $f \in C^\infty(M)$. When f is real valued and $n \geq 2$ we have*

$$\int_M |\mathfrak{X}^2 f|^2 + \int_M \frac{1}{2}\left(Ric + \frac{n}{2} Tor\right)(\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{n}\int_M |\Delta_{\mathfrak{X}} f|^2. \tag{39}$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$\int_M |\mathfrak{X}^2 f|^2 + \int_M \frac{1}{2}\left(Ric - \frac{3}{2} Tor\right)(\nabla_b f, \nabla_b f) \leq 3\int_M |\Delta_{\mathfrak{X}} f|^2. \tag{40}$$

Let $A(x) = (a_{ij}(x))$ a $2n \times 2n$ matrix. Consider the second order linear operator in non-divergence form

$$\mathcal{A}u(x) = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u(x), \tag{41}$$

where coefficients $a_{ij}(x)$ are bounded measurable functions in a domain $\Omega \subset M^{2n+1}$. Cordes [4] and Talenti [17] identified the optimal condition expressing how far $\mathcal{A}$ can be from the identity and still be able to understand (41) as a perturbation of the case $A(x) = I_{2n}$, when the operator is just the sublaplacian. This is the so called Cordes condition that roughly says that all eigenvalues of A must cluster around a single value.

Definition 1. *([4],[17], [6]) We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that*

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2n} a_{ij}^2(x) \leq \frac{1}{2n-1+\varepsilon}\left(\sum_{i=1}^{2n} a_{ii}(x)\right)^2 \tag{42}$$

for a. e. $x \in \Omega$.

Let $c_n = \frac{(n+2)}{n}$ for $n \geq 2$ and $c_1 = 3$ the constants in the right-hand sides of Theorem 2. We can now adapt the proof of Theorem 2.1 in [6] to get

Theorem 3. *Let M^{2n+1} be a strictly pseudo-convex pseudo-hermitian manifold such that $Ric + \frac{n}{2} Tor \geq 0$ if $n \geq 2$ and $Ric - \frac{3}{2} Tor \geq 0$ if $n = 1$. Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{(1-\varepsilon)c_n} < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in C_0^\infty(\Omega)$ we have the a-priori estimate*

$$\|\mathfrak{X}^2 u\|_{L^2} \leq \sqrt{1 + \frac{2}{n}}\ \frac{1}{1-\gamma}\ \|\alpha\|_{L^\infty} \|\mathcal{A}u\|_{L^2}\,, \tag{43}$$

where

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} = \frac{\sum_{i=1}^{2n} a_{ii}(x)}{\sum_{i,j=1}^{2n} a_{ij}^2(x)}\,.$$

Proof. We start from formula (2.7) in [6] which gives

$$\int_{\Omega} |\Delta_{\mathfrak{X}} u(x) - \alpha(x)\mathcal{A}u(x)|^2\, dx \leq (1-\varepsilon) \int_{\Omega} |\mathfrak{X}u|^2 dx.$$

We now apply Theorem 2 to get

$$\int_{\Omega} |\Delta_{\mathfrak{X}} u(x) - \alpha(x)\mathcal{A}u(x)|^2\, dx \leq (1-\varepsilon) c_n \int_{\Omega} |\Delta_{\mathfrak{X}} f|^2.$$

The theorem then follows as in [6]. □

Remark: The hypothesis of Theorem 2, $n \geq 2$, can be weakened to assume only a bound from below

$$\mathrm{Ric} + \frac{n}{2}\mathrm{Tor} \geq -K, \text{ with } K > 0$$

to obtain estimates of the type

$$\int_M |\mathfrak{X}^2 f|^2 \leq \frac{(n+2)}{n} \int_M |\Delta_{\mathfrak{X}} f|^2 + 2K \int_M |\mathfrak{X} f|^2. \tag{44}$$

A similar remark applies to the case $n = 1$.

We finish this paper by indicating how the *a priori* estimate of Theorem 3 can be used to prove regularity for p-harmonic functions in the Heisenberg group $\mathcal{H}^n$ when p is close to 2. We follow [6], where full details can be found. Recall that, for $1 < p < \infty$, a p-harmonic function u in a domain $\Omega \subset \mathcal{H}^n$ is a function in the horizontal Sobolev space

$$W^{1,p}_{\mathfrak{X},\mathrm{loc}}(\Omega) = \{u \colon \Omega \mapsto \mathbb{R} \text{ such that } u, \mathfrak{X}u \in L^p_{\mathrm{loc}}(\Omega)\}$$

such that

$$\sum_{i=1}^{2n} X_i \left(|\mathfrak{X}u|^{p-2}\, X_i u \right) = 0\,, \text{ in } \Omega \tag{45}$$

in the weak sense. That is, for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} |\mathfrak{X}u(x)|^{p-2} (\mathfrak{X}u(x), \mathfrak{X}\phi(x)\, dx = 0. \tag{46}$$

Assume for the moment that u is a smooth solution of (45). We can then differentiate to obtain

$$\sum_{i,j=1}^{2n} a_{ij}\, X_i X_j u = 0\,, \text{ in } \Omega \tag{47}$$

where

$$a_{ij}(x) = \delta_{ij} + (p-2)\frac{X_i u(x)\, X_j u(x)}{|\mathfrak{X} u(x)|^2}\,.$$

A calculation shows that this matrix satisfies the Cordes condition (42) precisely when

$$p-2 \in \left(\frac{n - n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\,,\ \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\right). \tag{48}$$

In the case $n = 1$ this simplifies to

$$p-2 \in \left(\frac{1-\sqrt{5}}{2}\,,\ \frac{1+\sqrt{5}}{2}\right).$$

We then deduce *a priori* estimates for $\mathfrak{X}^2 u$ from Theorem 3. To apply the Cordes machinery to functions that are only in $W^{1,p}_{\mathfrak{X}}$ we need to know that the second derivatives $\mathfrak{X}^2 u$ exist. This is done in the Euclidean case by a standard difference quotient argument applied to a regularized p-Laplacian. In the Heisenberg case this would correspond to proving that solutions to

$$\sum_{i=1}^{2n} X_i \left(\left(\frac{1}{m} + |\mathfrak{X} u|^2 \right)^{\frac{p-2}{2}} X_i u \right) = 0 \tag{49}$$

are smooth. Contrary to the Euclidean case (where solutions to the regularized p-Laplacian are C^∞-smooth) in the subelliptic case this is known only for $p \in [2, c(n))$ where $c(n) = 4$ for $n = 1, 2$, and $\lim_{n\to\infty} c(n) = 2$ (see [13].) The final result will combine the limitations given by (48) and $c(n)$.

Theorem 4. *(Theorem 3.1 in [6]) For*

$$2 \le p < 2 + \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}$$

we have that p-harmonic functions in the Heisenberg group $\mathcal{H}^n$ are in $W^{2,2}_{\mathfrak{X},loc}(\Omega)$.

At least in the one-dimensional case $\mathcal{H}^1$ one can also go below $p = 2$. See Theorem 3.2 in [6]. We also note that when p is away from 2, for example $p > 4$ nothing is known regarding the regularity of solutions to (45) or its regularized version (49) unless we assume a priori that the length of the gradient is bounded below and above

$$0 < \frac{1}{M} \le |\mathfrak{X} u| \le M < \infty.$$

See [1] and [13].

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