Existence Theory of Abstract Approximate Deconvolution Models of Turbulence

Iuliana Stanculescu†

Abstract

This report studies an abstract approach to modeling the motion of large eddies in a turbulent flow. If the Navier-Stokes equations (NSE) are averaged with a local, spatial convolution type filter, $\overline{\boldsymbol{\phi}} = g_\delta * \boldsymbol{\phi}$, the resulting system is not closed due to the filtered nonlinear term $\overline{\mathbf{u}\mathbf{u}}$. An approximate deconvolution operator D is a bounded linear operator which is an approximate filter inverse

$$D(\overline{\mathbf{u}}) = \text{ approximation of } \mathbf{u}.$$

Using this general deconvolution operator yields the closure approximation to the filtered nonlinear term in the NSE

$$\overline{\mathbf{u}\mathbf{u}} \simeq \overline{D(\overline{\mathbf{u}})D(\overline{\mathbf{u}})}.$$

Averaging the Navier-Stokes equations using the above closure, possible including a time relaxation term to damp unresolved scales, yields the approximate deconvolution model (ADM)

$$\mathbf{w}_t + \nabla \cdot \overline{D(\mathbf{w})\ D(\mathbf{w})} - \nu \triangle \mathbf{w} + \nabla q + \chi \mathbf{w}^* = \overline{\mathbf{f}} \text{ and } \nabla \cdot \mathbf{w} = 0.$$

Here $\mathbf{w} \simeq \overline{\mathbf{u}}$, $\chi \geq 0$, and $\mathbf{w}^*$ is a generalized fluctuation. We derive conditions on the general deconvolution operator D that guarantee the existence and uniqueness of strong solutions of the model. We also derive the model's energy balance.

1 Introduction

Many approaches have been used to simulate turbulent flows. In large eddy simulation (LES) the evolution of averages is sought. These averages are defined through a local spatial averaging process associated with an averaging radius δ. Once an averaging radius and a filtering process is selected, an LES model can be developed and then solved numerically. One of the most interesting approaches to generate LES model is via approximate deconvolution or approximate/asymptotic inverse of the filtering operator. Approximate deconvolution models (ADM) are systematic (rather than ad hoc). They can achieve high theoretical accuracy and shine in practical tests; they contain few or no fitting/tuning parameters. The ADM approach has thus proven itself to be very promising. However, among the very many known approximate deconvolution operators from image processing, e.g. [2], so far only two have been studied for LES modeling, the van Cittert deconvolution operator and Geurts' approximate inverse filter. Their success suggest that it is time to develop a general theory of LES-ADM as a guide to development of models based on other, possibly better, deconvolution operators and refinement of existing ones.

Two basic requirements of an acceptable ADM are that a unique, strong solution exists and that the model's global energy balance be close in some sense to that of the NSE. Herein, we consider these two important questions. *We find, in Section 3 (see **P1**, **P2**, and **P3**), conditions on the approximate deconvolution operator D that guarantee that the ADM has a unique strong solution.*

†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA,15260, U.S.A.; email: ius1@pitt.edu, www.pitt.edu/~ius1, partially supported by NSF grant DMS 0508260

In Section 4 (see Theorem 4.2), we derive the model's energy balance and show that under these conditions on the deconvolution operator *the model correctly captures the global energy balance of the large scales.*

The underlying fluid velocity $\mathbf{u}(\mathbf{x},t)$ and pressure $p(\mathbf{x},t)$ are solutions of the Navier-Stokes equations (NSE):

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\mathbf{f}$ is the body force, and $\mathbb{R}^n$ ($n = 2$ or 3) is the flow domain. We consider L-periodic boundary conditions to separate the interior closure problem from other important problems associated with filtering through a boundary, [4] and finding boundary conditions for local averages (near wall laws), [11],

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t), \; j = 1, ..., n.$$

The Navier-Stokes equations are supplemented by the initial condition, the usual normalization condition in the periodic case of zero mean velocity and pressure, and the assumption that all data are square integrable with zero mean

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \text{ and } \int_Q \mathbf{u} \; d\mathbf{x} = \int_Q p \; d\mathbf{x} = 0,$$
$$\int_Q |\mathbf{u}_0(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \; \int_Q |\mathbf{f}(\mathbf{x}, t)|^2 d\mathbf{x} < \infty, \text{ and } \int_Q \mathbf{f}(\mathbf{x}, t) d\mathbf{x} = 0, \text{ for } \; 0 \leq t, \tag{1.2}$$

where $Q = (0, L)^n$. Let overbar denote a local, spatial averaging operator, such as averaging by convolution, that is linear and commutes with differentiation. Averaging the NSE, the (non-closed) equations for $\overline{\mathbf{u}}$ and $\overline{p}$, known as the Space Filtered Navier-Stokes equations (SFNSE), are

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u}\,\mathbf{u}}) - \nu \triangle \overline{\mathbf{u}} + \nabla \overline{p} = \overline{\mathbf{f}} \text{ and } \; \nabla \cdot \overline{\mathbf{u}} = 0. \tag{1.3}$$

Since in general $\overline{\mathbf{u}\mathbf{u}} \neq \overline{\mathbf{u}}\;\overline{\mathbf{u}}$, the closure problem is to replace $\overline{\mathbf{u}\mathbf{u}}$ by a tensor $S(\overline{\mathbf{u}}, \overline{\mathbf{u}})$ depending only on $\overline{\mathbf{u}}$, not on $\mathbf{u}$. If we denote filtering by $\overline{\mathbf{u}} = G\mathbf{u}$ and D is an approximate inverse of G (so $\mathbf{u} \cong D\overline{\mathbf{u}}$), then the variant of approximate deconvolution model we consider herein approximates

$$\overline{\mathbf{u}\mathbf{u}} \cong \overline{D(\overline{\mathbf{u}}) D(\overline{\mathbf{u}})} =: S(\overline{\mathbf{u}}, \overline{\mathbf{u}}).$$

The deconvolution problem is central in image processing, [2]. Thus many algorithms can be adapted to give a possible LES closure model. The goal of deconvolution in LES is to recover accurately the resolved scales *asymptotically* as $\delta \to 0$. The resulting LES model should have a lucid energy balance and favorable properties for its approximate solution. As an example, the N^{th} van Cittert approximate deconvolution operator D_N, defined precisely in Section 5, see also [1], [6], [14], is an easy to construct bounded linear operator on $L^2(Q)$ satisfying

$$\boldsymbol{\phi} = D_N(\overline{\boldsymbol{\phi}}) + O(\delta^{2N+2}), \text{ for smooth } \boldsymbol{\phi}.$$

In other words, D_N is an asymptotic (as $\delta \to 0$) approximate inverse of G. With the van Cittert approximate deconvolution operator, the closure problem in (1.3) can be solved approximately but systematically by:

$$\overline{\mathbf{u}\,\mathbf{u}} \simeq \overline{D_N(\overline{\mathbf{u}}) \; D_N(\overline{\mathbf{u}})} + O(\delta^{2N+2}), \text{ for smooth } \mathbf{u}.$$

More generally and beyond the van Cittert operator, there is little analytical guidance as to the properties needed of a deconvolution operator to produce a reliable LES model. Since inverting a filter is an ill-posed problem, a deconvolution operator can also be generated by any method for solving approximately ill-posed problems. Thus, there are many possible choices of D available. Once D is selected, we define the higher order fluctuation $\mathbf{w}^* = \mathbf{w} - D(\overline{\mathbf{w}})$, if $I - DG$ is symmetric positive semi-definite and $\mathbf{w}^* = (I - DG)^*(I - DG)\mathbf{w}$ otherwise.

In general, *any* approximate deconvolution operator D, used as a closure approximation, leads to the approximate deconvolution LES model

$$\begin{aligned} \mathbf{w}_t + \nabla \cdot \overline{D(\mathbf{w})\ D(\mathbf{w})} - \nu \triangle \mathbf{w} + \nabla q + \chi \mathbf{w}^* &= \overline{\mathbf{f}} \\ \nabla \cdot \mathbf{w} &= 0 \\ \mathbf{w}|_{t=0} &= \overline{\mathbf{u}}_0 \\ \int_Q q dx &= 0. \end{aligned} \tag{1.4}$$

The time relaxation term $\chi \mathbf{w}^*$, where $\chi \geq 0$ is a model parameter, is often included in the ADM to damp marginally resolved scales, see [19] and [16], so we include it in our analysis.

If D is a symmetric positive definite operator then it induces a deconvolution weighted L^2 inner product and norm defined by $(\mathbf{u}, \mathbf{v})_D := \int_Q \mathbf{u} \cdot D(\mathbf{v}) d\mathbf{x}$ and $||\mathbf{v}||_D^2 = (\mathbf{v}, \mathbf{v})_D$. For specificity, we select the filtering operation $\overline{\mathbf{v}} := (I - \delta^2 \triangle)^{-1} \mathbf{v}$.

In Sections 3 and 4 we show that if the deconvolution operator $D : L^2(Q) \to L^2(Q)$ is a bounded, self adjoint, and positive definite operator that commutes with differentiation, then a weak solution exists for the deconvolution model (1.4). In Section 5 we show that the weak solution is a unique strong solution and satisfies the energy equality:

$$\begin{aligned} &\frac{1}{2}||\mathbf{w}(t)||_D^2 + \frac{\delta^2}{2}||\nabla \mathbf{w}(t)||_D^2 + \int_0^t \left(\nu||\nabla \mathbf{w}||_D^2 + \nu\delta^2||\triangle \mathbf{w}||_D^2 + \chi(\mathbf{w}^*, (I - \delta^2 \triangle)\mathbf{w})_D\right) d\tau \\ &\quad = \int_0^t (\mathbf{f}, \mathbf{w})_D d\tau + \frac{1}{2}\left(||\mathbf{w}(0)||_D^2 + \delta^2||\nabla \mathbf{w}(0)||_D^2\right). \end{aligned} \tag{1.5}$$

Remark 1.1. [*An Important Difference Between Deconvolution Models*] *The SFNSE can be rewritten as*

$$\overline{\mathbf{u}}_t + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} - \nu \triangle \overline{\mathbf{u}} + \nabla \overline{p} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\ \overline{\mathbf{u}}) = \overline{\mathbf{f}}.$$

Another possible deconvolution LES model approximates

$$\overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\ \overline{\mathbf{u}} = \overline{D(\overline{\mathbf{u}})\ D(\overline{\mathbf{u}})} - \overline{D(\overline{\mathbf{u}})}\ \overline{D(\overline{\mathbf{u}})}.$$

In the simplest case ($D = I$) this is the Bardina model

$$\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} - \nu \triangle \mathbf{w} + \nabla q + \nabla \cdot (\overline{\mathbf{w}\mathbf{w}} - \overline{\mathbf{w}}\ \overline{\mathbf{w}}) = \overline{\mathbf{f}}. \tag{1.6}$$

This common approach is NOT covered by the theory herein. In fact, we believe (but cannot prove yet) that this approach can be unstable, unless sufficient ad hoc eddy viscosity is added (thereby destroying the high accuracy of the ADM approach). We therefore have a strong preference for the simpler, accurate, and unconditionally stable model (1.4) over the similarity type deconvolution model (1.6).

2 Preliminaries, Notations, and Function Spaces

We start by briefly reviewing the concept of averaging/filtering in LES and define the function spaces and the norms needed for the variational formulation of the scale similarity model.

2.1 Averaging operators used in LES

The idea of LES is to split the velocity into $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ a local, spatial average $\overline{\mathbf{u}}$ and a fluctuation about the mean $\mathbf{u}'$. It is widely believed that given the random and chaotic character of the fluctuations, their average effects on the mean motion can successfully be modeled and thus the mean can be predicted accurately. The mean is defined by filtering or mollification (convolution with an approximate identity). The goal is to predict the mean accurately. Let $g(\cdot)$ denote a filter, such as the Gaussian filter $g(\mathbf{x}) = \frac{6}{\pi} e^{-6|\mathbf{x}|^2}$, and let δ denote the selected averaging radius. Define $g_\delta(\mathbf{x}) := \delta^{-3} g(\mathbf{x}/\delta)$. Averages are defined by convolution with the kernel $g(\cdot)$. Given a velocity $\mathbf{u}$, its mean $\overline{\mathbf{u}}$ and fluctuation $\mathbf{u}'$ are defined by $\overline{\mathbf{u}} = G\mathbf{u}$ and $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$ when

$$\overline{\mathbf{u}}(\mathbf{x}) := \int_{R^3} g_\delta(\mathbf{x} - \mathbf{x}')\mathbf{u}(\mathbf{x}')d\mathbf{x}'. \tag{2.1}$$

Typical choices used in LES include the top-hat filter, sharp spectral cut-off, the Gaussian filter and differential filters, defined next, which also fit into the convolution formalism.

Definition 2.1. [*Differential Filter*] *Let $A := -\delta^2 \triangle + I$ and let $\overline{\boldsymbol{\varphi}}$ denote the unique L-periodic solution in Q of:*

$$A\overline{\boldsymbol{\varphi}} := -\delta^2 \triangle \overline{\boldsymbol{\varphi}} + \overline{\boldsymbol{\varphi}} = \boldsymbol{\varphi}. \tag{2.2}$$

Remark 2.1. *Under periodic boundary conditions and for constant δ both (2.1) and (2.2) commute with differentiation, preserve incompressibility, and can be written as convolution with approximate identity.*

Differential filters are well-established in LES, starting with the work of Germano [11] and continuing with [10], [18]. They have many connections to regularization processes such as the Yoshida regularization of semigroups and the very interesting work of Foias, Holm, Titi [7] (and others) on Lagrange averaging of the Navier-Stokes equations.

The mean $\overline{\mathbf{u}}(\mathbf{x})$ is the weighted average of $\mathbf{u}$ about the point $\mathbf{x}$. As $\delta \to 0$, the points near $\mathbf{x}$ are weighted more and more heavily, so intuitively we expect that $\overline{\mathbf{u}} \to \mathbf{u}$ as $\delta \to 0$. This is known for convolution filters. For differential filters, it is also not difficult to show.

Remark 2.2. *Expanding the velocity $\mathbf{u}$ in Fourier series we obtain*

$$\mathbf{u}(\mathbf{x}) = \sum_{\boldsymbol{k}} \widehat{\mathbf{u}}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\mathbf{x}} \text{ where } \boldsymbol{k} \text{ is the wave number vector and } \boldsymbol{k} = \frac{2\pi \boldsymbol{n}}{L}, \text{ for } \boldsymbol{n} \in \mathbb{Z}^3.$$

Let $k = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ be its magnitude. The Fourier coefficients are

$$\widehat{\mathbf{u}}(\boldsymbol{k}) = \frac{1}{L^3} \int_Q \mathbf{u}(\mathbf{x}) e^{-i\boldsymbol{k}\cdot\mathbf{x}} d\mathbf{x}.$$

Parseval's equality leads to

$$\frac{1}{L^3} \int_Q |\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} = \sum_{\boldsymbol{k}} |\widehat{\mathbf{u}}(\boldsymbol{k})|^2 = \sum_k \sum_{|\boldsymbol{k}|=k} |\widehat{\mathbf{u}}(\boldsymbol{k})|^2.$$

Lemma 2.1. *Let $\overline{\mathbf{u}} = (-\delta^2 \triangle + I)^{-1} \mathbf{u}$. Then for any $\mathbf{u} \in L^2(Q)$*

$$\overline{\mathbf{u}} \to \mathbf{u} \text{ in } L^2(Q) \text{ as } \delta \to 0.$$

Proof. Given $\varepsilon > 0$, we show that for δ small enough $||\overline{\mathbf{u}} - \mathbf{u}|| < \varepsilon$.
Indeed, using Parseval's equality, as in Remark 2.2, we obtain

$$||\overline{\mathbf{u}} - \mathbf{u}||^2 = \sum_k \sum_{|\mathbf{k}|=k} \left| 1 - \frac{1}{\delta^2 k^2 + 1} \right| |\widehat{\mathbf{u}}(\mathbf{k})|^2 = \sum_k \sum_{|\mathbf{k}|=k} \left(\frac{\delta^2 k^2}{1 + \delta^2 k^2} \right) |\widehat{\mathbf{u}}(\mathbf{k})|^2$$

and for any positive integer M we can write

$$||\overline{\mathbf{u}} - \mathbf{u}||^2 = \sum_{0<k\leq M} \sum_{|\mathbf{k}|=k} \left(\frac{\delta^2 k^2}{1 + \delta^2 k^2} \right) |\widehat{\mathbf{u}}(\mathbf{k})|^2 + \sum_{k>M} \sum_{|\mathbf{k}|=k} \left(\frac{\delta^2 k^2}{1 + \delta^2 k^2} \right) |\widehat{\mathbf{u}}(\mathbf{k})|^2.$$

Furthermore, we have

$$\sum_{|\mathbf{k}|=k}\left(\frac{\delta^2k^2}{1+\delta^2k^2}\right)|\widehat{\mathbf{u}}(\mathbf{k})|^2 \le \sum_{|\mathbf{k}|=k}|\widehat{\mathbf{u}}(\mathbf{k})|^2$$

and thus, for M large enough

$$\sum_{0<k\le M}\sum_{|\mathbf{k}|=k}\left(\frac{\delta^2k^2}{1+\delta^2k^2}\right)|\widehat{\mathbf{u}}(\mathbf{k})|^2 < \frac{\varepsilon}{2}. \tag{2.3}$$

For this M we have

$$\begin{aligned}\sum_{k>M}\sum_{|\mathbf{k}|=k}\left(\frac{\delta^2k^2}{1+\delta^2k^2}\right)|\widehat{\mathbf{u}}(\mathbf{k})|^2 &\le \frac{\delta^2M^2}{1+\delta^2M^2}||\mathbf{u}||^2 \\ &< \frac{\varepsilon}{2},\ \text{for some } \delta=\delta(M).\end{aligned} \tag{2.4}$$

From (2.3) and (2.4) the conclusion follows. □

2.2 Function Spaces, Weak and Strong Solution

The notation for the function spaces we are using follows Temam, [20] and Layton and Lewandowski [15]. Let $||\cdot||$ represent the L^2 norm. For $k\ge 1$, let $V^k_{\#}(Q)$ denote the space of all $[0,L]^3$-periodic functions with restriction on the cell $Q=(0,L)^3$ in the Sobolev space $H^k(Q)$. Thus

$$V^k_{\#}(Q)=\left\{\ \mathbf{u}\in H^k_{loc}(Q)|\ \mathbf{u} \text{ is L-periodic }\right\}.$$

We denote by $||\cdot||_{V^k_{\#}(Q)}$ the associated norm:

$$||\boldsymbol{\varphi}||_{V^k_{\#}}(Q)=\sum_{j=0}^{k}\left(\int_Q|\nabla^j\boldsymbol{\varphi}(x)|^2dx\right)^{1/2},\ \text{for all } \boldsymbol{\varphi}\in V^k_{\#}(Q).$$

For the variational formulation of the approximate deconvolution model, we consider the spaces of periodic, divergence-free functions:

$$\begin{aligned}&\overline{V}^k_{\#}(Q)=\left\{\ \mathbf{u}\in V^k_{\#}(Q)|\ \nabla\cdot\mathbf{u}=0\ \right\}\\ &H=\left\{\ \mathbf{u}\in L^2(Q)|\ \mathbf{u} \text{ is L-periodic and } \nabla\cdot\mathbf{u}=0\ \right\}\end{aligned}$$

and

$$V=\left\{\ \mathbf{u}\in H^1(Q)|\ \nabla\cdot\mathbf{u}=0\ \right\}.$$

Further, we define

$$D(Q)=\left\{\boldsymbol{\psi}\in C^\infty(Q)|\ \boldsymbol{\psi} \text{ is L-periodic, has compact support and } \int_Q\boldsymbol{\psi}d\mathbf{x}=0\right\}$$

$$D_T(Q)=\left\{\boldsymbol{\psi}\in C^\infty\left([0,T)\times Q\right)|\ \boldsymbol{\psi}(\cdot,t) \text{ is L-periodic, has compact support and } \int_Q\boldsymbol{\psi}d\mathbf{x}=0\ \right\}.$$

Definition 2.2. *Let D be a symmetric, positive definite and bounded operator on $L^2(Q)$. The D inner product and norm on $L^2(Q)$ are $(\boldsymbol{\phi},\boldsymbol{\psi})_D:=\int_Q\boldsymbol{\phi}\cdot D(\boldsymbol{\psi})d\mathbf{x}$ and*

$$||\boldsymbol{\phi}||^2_D=(\boldsymbol{\phi},\boldsymbol{\phi})_D,\ \textit{for every}\ \boldsymbol{\phi}\in L^2(Q).$$

Definition 2.3. *Let D and G be bounded operators on $L^2(Q)$. We define*

$$\phi^* := \begin{cases} (I - DG)\ \phi, & \text{if } I - DG \text{ is symmetric positive semi-definite} \\ (I - DG)^*(I - DG)\ \phi, & \text{otherwise.} \end{cases}$$

The $*$ semi-inner product and semi-norm on $L^2(Q)$ are $(\phi, \psi)_* := (\phi^*, \psi)$ and

$$||\phi||_*^2 = (\phi, \phi)_*, \text{ for every } \phi \in L^2(Q).$$

To make progress in the mathematical understanding of an LES model the key idea is the notion of weak solution. For the NSE the notion of weak solution was introduced by Leray, [13]. We now introduce the notion of weak solution for the ADM (1.4).

Definition 2.4. *Let $\mathbf{f} \in L^2(0, T; V')$ and $\mathbf{w}_0 \in \overline{V}^1_{\#}(Q)$. A function $\mathbf{w} : Q \times [0, T] \to \mathbb{R}^n$ is a weak solution of* (1.4) *if*

$$\mathbf{w} \in L^2\left(0, T; \overline{V}^2_{\#}(Q)\right) \cap L^\infty(0, T; H) \tag{2.5}$$

and

$$\begin{aligned}(\mathbf{w}(T), \varphi(T)) \quad &- \quad \int_0^T \left[\left(\mathbf{w}, \frac{\partial \varphi}{\partial t}\right) - \nu(\nabla\mathbf{w}, \nabla\varphi) - \left(\nabla \cdot \overline{D(\mathbf{w})D(\mathbf{w})}, \varphi\right) - \chi(\mathbf{w}^*, \varphi)\right] dt \\ &= \quad \int_0^T (\overline{\mathbf{f}}, \varphi) dt + (\mathbf{w}_0, \varphi(0)) \end{aligned} \tag{2.6}$$

for all $\varphi \in D(Q_T)$.

Definition 2.5. *The pair $(\mathbf{w}, q)$ is a strong solution of the deconvolution model (1.4) if $\mathbf{w}$ is a weak solution and*

$$\begin{aligned} \mathbf{w} \in \overline{V}^2_{\#}(\mathrm{Q}) \cap H, &\qquad \text{for a.e. } t \in [0, T] \\ \mathbf{w} \in \overline{V}^1_{\#}([0, T]), &\qquad \text{for a.e. } \mathbf{x} \in Q \\ q \in \overline{V}^1_{\#}(\mathrm{Q}) \cap L^2_0(Q), &\qquad \text{for a.e. } t \in (0, T]. \end{aligned} \tag{2.7}$$

Remark 2.3. *Because the inclusion $V^2_{\#}(Q) \to H$ is compact, the inverse of the Stokes operator is bounded, self-adjoint, and compact. This implies that there exists an orthonormal basis of H consisting of eigenfunctions of the Stokes operator.*

3 Existence of Weak Solutions of Approximate Deconvolution Models

Due to the nonlinearity in (1.4) small changes in the deconvolution operator can yield significant (positive or negative) changes in the solution of the induced model. The averaging/convolution operator G is bounded, self-adjoint, positive definite and commutes with differentiation. We thus postulate that deconvolution operator D, its approximate inverse, is also bounded, self-adjoint, positive definite and commutes with differentiation. We postulate the following properties of the deconvolution operator D:

P1. *D is a bounded linear operator on $L^2(Q)$,*
P2. *D is self-adjoint and positive definite,*
P3. *D commutes with differentiation.*

Note that **P**1, **P**2, and **P**3 imply in particular that the D norm and the L^2 norm are equivalent on $L^2(Q)$, i.e. there exist two positive constants C_1 and C_2 such that

$$C_1||\varphi|| \le ||\varphi||_D \le C_2||\varphi||, \text{ for all } \varphi \in L^2(Q).$$

Remark 3.1. *Let $\{\boldsymbol{\psi}_j\}$ be eigenfunctions of the Stokes operator in H. Then, due to the periodic boundary conditions, span$\{\boldsymbol{\psi}_j\}_j$ is invariant under the Stokes operator and any other constant coefficient differential operators. By **P**3, it is also invariant under D.*

Under postulates **P**1, **P**2, and **P**3 on the deconvolution operator D, we prove that the ADM model (1.4) admits weak solutions, in the sense of the Definition 2.4. We follow the exposition on the existence of the weak solutions of the NSE, in Galdi [8]. The analysis includes the case $\chi = 0$ of no time relaxation.

Theorem 3.1. *Let $T > 0$ and D be a deconvolution operator satisfying the properties **P1**, **P2**, and **P3**. For $\mathbf{w}_0 \in \overline{V}^1_{\#}(Q) \cap H$ and $\mathbf{f} \in L^2(0,T;V')$, there exists a weak solution $\mathbf{w} \in L^2\left(0,T;\overline{V}^2_{\#}(Q)\right) \cap L^\infty(0,T;H)$ of (1.4) in the sense of the Definition 2.4. Moreover, for any $t \in (0,T]$, the following stability bound holds:*

$$\frac{1}{2}||\mathbf{w}(t)||^2 + \frac{1}{2}\delta^2||\nabla \mathbf{w}(t)||^2 + \nu \int_0^t \left(||\nabla \mathbf{w}(\tau)||^2 + \delta^2||\triangle \mathbf{w}(\tau)||^2\right) d\tau$$
$$\leq C\left(\int_0^t ||\mathbf{f}(\tau)||^2 d\tau + ||\mathbf{w}(0)||^2 + \delta^2||\nabla \mathbf{w}(0)||^2\right). \tag{3.1}$$

Proof. Let $\{\boldsymbol{\psi}_j\} \in D(Q)$ be an orthonormal basis of H consisting of eigenfunctions of the Stokes operator, as in Remark 2.3. We are looking for Galerkin approximate solutions of the form:

$$\mathbf{w}_k(\mathbf{x},t) = \sum_{r=1}^{k} \eta_{kr}(t)\boldsymbol{\psi}_r(\mathbf{x}), \text{ for } k \in \mathbb{N}. \tag{3.2}$$

Since $(\nabla p, \boldsymbol{\psi}_j) = -(p, \nabla \cdot \boldsymbol{\psi}_j) = 0$ for $j = 1, ..., k$, the functions $\{\mathbf{w}_k\}_k$ satisfy the ODE system:

$$\left(\frac{\partial \mathbf{w}_k}{\partial t}, \boldsymbol{\psi}_j\right) + \nu\left(\nabla \mathbf{w}_k, \nabla \boldsymbol{\psi}_j\right) + \left(\nabla \cdot \overline{D(\mathbf{w}_k)D(\mathbf{w}_k)}, \boldsymbol{\psi}_j\right) = \left(\overline{\mathbf{f}}, \boldsymbol{\psi}_j\right) \tag{3.3}$$
$$\left(\mathbf{w}_k(0), \boldsymbol{\psi}_j\right) = (\mathbf{w}_0, \boldsymbol{\psi}_j), \tag{3.4}$$

for all $j = 1, ..., k$. Using (3.2) in (3.3) and (3.4) it follows that

$$\sum_{r=1}^{k} \frac{\partial \eta_{kr}}{\partial t}(\boldsymbol{\psi}_r, \boldsymbol{\psi}_j) + \nu \sum_{r=1}^{k} \eta_{kr}(\nabla \boldsymbol{\psi}_r, \nabla \boldsymbol{\psi}_j) + \sum_{r,i=1}^{k} \eta_{kr}\eta_{ki}\left(\nabla \cdot \overline{D(\boldsymbol{\psi}_r)D(\boldsymbol{\psi}_i)}, \boldsymbol{\psi}_j\right) = \left(\overline{\mathbf{f}}, \boldsymbol{\psi}_j\right) \tag{3.5}$$

$$\sum_{r=1}^{k} \eta_{kr}(0)\left(\boldsymbol{\psi}_r, \boldsymbol{\psi}_j\right) = \left(w_0, \boldsymbol{\psi}_j\right) \tag{3.6}$$

for all $j = 1, ..., k$. For simplification, let $a_{rj} = (\nabla \boldsymbol{\psi}_r, \nabla \boldsymbol{\psi}_j)$, $a_{rij} = \left(\nabla \cdot \overline{D(\boldsymbol{\psi}_r)D(\boldsymbol{\psi}_i)}, \boldsymbol{\psi}_j\right)$, $f_j = \left(\overline{\mathbf{f}}, \boldsymbol{\psi}_j\right)$ and $c_{0j} = (\mathbf{w}_0, \boldsymbol{\psi}_j)$. With these and since $(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j) = \delta_{ij}$, equations (3.5) and (3.6) become

$$\frac{\partial \eta_{kj}}{\partial t} + \nu \sum_{r=1}^{k} a_{rj}\eta_{kr} + \sum_{r,i=1}^{k} a_{rij}\eta_{kr}\eta_{ki} = f_j \tag{3.7}$$
$$\eta_{kj}(0) = c_{0j}. \tag{3.8}$$

for $j = 1, ..., k$. Since $f_j \in L^2(0,T)$, for $j = 1, ..., k$ from the theory of ODEs we know that the problem (3.7)-(3.8) has a unique solution $\eta_{kj} \in W^{1,2}(0,T_k)$, for a small enough time interval $T_k \leq T$.

Because $\mathbf{w}_0 \in \overline{V}^1_{\#}(Q) \cap H$, there exists $\mathbf{u}_0 \in H$ such that $\overline{\mathbf{u}}_0 = \mathbf{w}_0$ and:

$$(\mathbf{w}_0, \boldsymbol{\psi}_j) = (\overline{\mathbf{u}}_0, \boldsymbol{\psi}_j), \; j = 1, ..., k. \tag{3.9}$$

From Remark 3.1, span$\{\boldsymbol{\psi}_j\}_j$ is invariant under the Stokes and differential operators. In particular span$\{\boldsymbol{\psi}_j\}_j$ is invariant under the deconvolution operator D and $(-\delta^2\triangle+I)$. Since $D(\mathbf{w}_k)$ is a linear combination of $\{\boldsymbol{\psi}_j\}$ we deduce that $D(\mathbf{w}_k)\in$ span $\{\boldsymbol{\psi}_j\}_j$ and $(-\delta^2\triangle+I)\mathbf{w}_k(0)\in$ span$\{\boldsymbol{\psi}_j\}_j$. So, we can use $(-\delta^2\triangle+I)$ $\mathbf{w}_k(0)$ as test function in (3.9).

$$\big(\mathbf{w}_k(0),(-\delta^2\triangle+I)\mathbf{w}_k(0)\big)=\big(\overline{\mathbf{u}}_0,(-\delta^2\triangle+I)\mathbf{w}_k(0)\big)=(\mathbf{u}_0,\mathbf{w}_k(0)). \tag{3.10}$$

Thus, $||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\leq\frac{1}{2}\left(||\mathbf{u}_0||^2+||\mathbf{w}_k(0)||^2\right)$ and we have the following à priori estimate

$$\frac{1}{2}||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\ \leq\ \frac{1}{2}||\mathbf{u}_0||^2. \tag{3.11}$$

Further, in (3.3) we can replace $\boldsymbol{\psi}_j$ by $(I-\delta^2\triangle)D(\mathbf{w}_k)\in$ span $\{\boldsymbol{\psi}_j\}_j$. Since $(I-\delta^2\triangle)$ is symmetric, the nonlinear term vanishes:

$$\left(\nabla\cdot\overline{D(\mathbf{w}_k)D(\mathbf{w}_k)},\left(I-\delta^2\triangle\right)D(\mathbf{w}_k)\right)=(\nabla\cdot D(\mathbf{w}_k)D(\mathbf{w}_k),\ D(\mathbf{w}_k))=0$$

and (3.3) becomes

$$\left(\frac{\partial\mathbf{w}_k}{\partial t},(I-\delta^2\triangle)D(\mathbf{w}_k)\right)+\nu\left(\nabla\mathbf{w}_k,\nabla(I-\delta^2\triangle)D(\mathbf{w}_k)\right)=\left(\mathbf{f},D(\mathbf{w}_k)\right).$$

Applying properties **P**1, **P**2, and **P**3 of D, and integrating between 0 and t we are led to

$$\begin{aligned}\frac{1}{2}||\mathbf{w}_k(t)||_D^2+\frac{1}{2}\delta^2||\nabla\mathbf{w}_k(t)||_D^2+\nu\int_0^t\left(||\nabla\mathbf{w}_k(\tau)||_D^2+\delta^2||\triangle\mathbf{w}_k(\tau)||_D^2\right)d\tau\\ =\int_0^t(\mathbf{f},D\mathbf{w}_k(\tau))d\tau+\frac{1}{2}||\mathbf{w}_k(0)||_D^2+\frac{1}{2}\delta^2||\nabla\mathbf{w}_k(0)||_D^2.\end{aligned} \tag{3.12}$$

Moreover, since the D-norm and the L^2-norm are equivalent on $L^2(Q)$ there exists a constant C such that

$$\begin{aligned}\frac{1}{2}||\mathbf{w}_k(t)||^2+\frac{1}{2}\delta^2||\nabla\mathbf{w}_k(t)||^2+\nu\int_0^t\left(||\nabla\mathbf{w}_k(\tau)||^2+\delta^2||\triangle\mathbf{w}_k(\tau)||^2\right)d\tau\\ \leq C\left(\int_0^t||\mathbf{f}||^2d\tau+\int_0^t||\mathbf{w}_k(\tau)||^2d\tau+||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\right).\end{aligned} \tag{3.13}$$

Gronwall's inequality in (3.13) implies that for all $t\in(0,T]$

$$\begin{aligned}\frac{1}{2}||\mathbf{w}_k(t)||^2+\frac{1}{2}\delta^2||\nabla\mathbf{w}_k(t)||^2+\nu\int_0^t\left(||\nabla\mathbf{w}_k(\tau)||^2+\delta^2||\triangle\mathbf{w}_k(\tau)||^2\right)d\tau\\ \leq Ce^t\left(\int_0^t||\mathbf{f}||^2d\tau+||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\right).\end{aligned} \tag{3.14}$$

Also, for all $t\in(0,T]$ we have that

$$e^t\left(\int_0^t||\mathbf{f}||^2d\tau+||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\right)\leq Ce^T\left(\int_0^T||\mathbf{f}||^2d\tau+||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\right).$$

Let $M:=e^T\left(\int_0^T||\mathbf{f}||^2d\tau+||\mathbf{w}_k(0)||^2+\delta^2||\nabla\mathbf{w}_k(0)||^2\right)$, so that M is a constant independent of t and k. In particular, from (3.14) we deduce

$$||\mathbf{w}_k(t)||\leq M^{1/2}. \tag{3.15}$$

Using the definition of the L^2-norm and (3.2) we get $||\mathbf{w}_k(t)||^2=(\mathbf{w}_k(t),\mathbf{w}_k(t))=\sum_{r=1}^k\eta_{kr}^2(t)$. Thus

$$|\eta_{kr}(t)|\leq\left(\sum_{r=1}^k\eta_{kr}^2(t)\right)^{1/2}=||\mathbf{w}_k(t)||\leq M^{1/2}.$$

We already know that $T_k \leq T$. The à priori bound (3.15) shows that we cannot have $T_k < T$, since it would imply $|\eta_{kr}(t)|$ is unbounded as $t \to T_k$, for some $r \in \{1, ..., k\}$. This shows $T_k = T$, for all $k \in \mathbb{N}$.

Next, we investigate the properties of convergence of $\{\mathbf{w}_k\}$ as $k \to \infty$. To begin, let j be fixed but arbitrary. Define the sequence $\{N_k(t)\}_k$, where

$$N_k(t) = (\mathbf{w}_k(t), \boldsymbol{\psi}_j), \text{ for } t \in (0, T].$$

We shall first show that the sequence so defined satisfies the properties

1. $\{N_k(t)\}_k$ is uniformly bounded,

2. $\{N_k(t)\}_k$ is equicontinuous.

The first property follows from $||\boldsymbol{\psi}_j|| = 1$, the Cauchy- Schwarz inequality, and (3.15).

$$|N_k(t)| \leq ||\mathbf{w}_k(t)||\,||\boldsymbol{\psi}_j|| \leq M^{1/2}.$$

To prove the second, let us note that for every t and s in $(0, T)$

$$|N_k(t) - N_k(s)| = \left|\big(\mathbf{w}_k(t) - \mathbf{w}_k(s), \boldsymbol{\psi}_j\big)\right| = \left|\sum_{r=1}^{k} (\eta_{kr}(t) - \eta_{kr}(s))\,(\boldsymbol{\psi}_r, \boldsymbol{\psi}_j)\right|.$$

Integrating between 0 and t in (3.7) we first obtain that the coefficients η_{kr} satisfy the equation

$$\eta_{kj}(t) = \eta_{kj}(0) + \int_0^t (f_j - \nu \sum_{r=1}^{k} a_{rj}\eta_{kr} - \sum_{n,i=1}^{k} a_{rij}\eta_{kr}\eta_{ki}) d\tau \tag{3.16}$$

and thus

$$N_k(t) - N_k(s) = \int_s^t (\overline{\mathbf{f}}, \boldsymbol{\psi}_j) d\tau - \nu \int_s^t (\nabla \mathbf{w}_k, \nabla \boldsymbol{\psi}_j) d\tau - \int_s^t (\nabla \cdot \overline{D(\mathbf{w}_k)D(\mathbf{w}_k)}, \boldsymbol{\psi}_j) d\tau. \tag{3.17}$$

Using the Cauchy- Schwarz inequality and the orthogonality of $\{\boldsymbol{\psi}_j\}$, we can bound each term

$$\begin{aligned}
(\mathbf{f}, \boldsymbol{\psi}_j) \leq ||\mathbf{f}||\,||\boldsymbol{\psi}_j|| &= ||\mathbf{f}||, \qquad (\nabla \mathbf{w}_k, \nabla \boldsymbol{\psi}_j) \leq ||\nabla\, \mathbf{w}_k||\,||\nabla \boldsymbol{\psi}_j|| \\
(\nabla \cdot D(\mathbf{w}_k)D(\mathbf{w}_k), \boldsymbol{\psi}_j) &= \big(D(\mathbf{w}_k)\nabla \cdot D(\mathbf{w}_k), \boldsymbol{\psi}_j\big) \leq C(||D||) \max_j |\boldsymbol{\psi}_j|\; ||\mathbf{w}_k||\; ||\nabla \mathbf{w}_k|| \\
&\leq C(||D||)\; M^{1/2} \max_j |\boldsymbol{\psi}_j|\; ||\nabla \mathbf{w}_k||.
\end{aligned}$$

Putting everything together and letting $K_1 = \nu||\nabla \boldsymbol{\psi}_j||$ and $K_2 = \max_j |\boldsymbol{\psi}_j|$ we have

$$|N_k(t) - N_k(s)| \leq \int_s^t \Big(||\overline{\mathbf{f}}|| + K_1||\nabla \mathbf{w}_k|| + K_2 M^{1/2}||\nabla \mathbf{w}_k||\Big)\, d\tau. \tag{3.18}$$

Further more, Cauchy-Schwarz inequality in time gives:

$$|N_k(t) - N_k(s)| \leq \sqrt{|t-s|}\; \left\{||\overline{\mathbf{f}}||^2_{L^2(s,t;L^2(Q))} + \Big(K_1 + K_2 M^{1/2}\Big)\, ||\nabla \mathbf{w}_k||^2_{L^2(s,t;L^2(Q))}\right\}. \tag{3.19}$$

Finally, since $\mathbf{f} \in L^2(0, T; V')$ and since $||\nabla \mathbf{w}_k||^2_{L^2(s,t;L^2(Q))} \leq ||\nabla \mathbf{w}_k||^2_{L^2(0,T;L^2(Q))}$ for any $T > 0$, our argument is over and $\{N_k(t)\}_k$ is equicontinuous.

By the Arzela-Ascoli Theorem, from $\{N_k(t)\}$ we may select a subsequence, which we redenote by $\{N_k(t)\}$ uniformly convergent to a continuous function $N(t)$, i.e:

$$(\mathbf{w}_k(t), \boldsymbol{\psi}_j) \to N(t) \text{ uniformly in } t. \tag{3.20}$$

Since the estimate (3.15) is independent of t, we obtain that $\{\mathbf{w}_k\}$ is a bounded sequence in $L^\infty(0,T;H)$. By the weak compactness of $L^\infty(0,T;H)$ it follows that there exists a convergent subsequence of $\{\mathbf{w}_k\}$ in $L^\infty(0,T;H)$. If we redenote the convergent subsequence by $\{\mathbf{w}_k\}$ we obtain that there exists $\mathbf{w}(t)$ in H such that

$$(\mathbf{w}_k(t),\boldsymbol{\psi}_j) \to (\mathbf{w}(t),\boldsymbol{\psi}_j) \text{ uniformly in } t. \tag{3.21}$$

From (3.20) and (3.21) follows that

$$V(t) = (\mathbf{w}(t),\boldsymbol{\psi}_j), \text{ for all } t \in (0,T].$$

Using (3.13), we deduce that $\{\mathbf{w}_k\}$ is a bounded sequence in $L^2(0,T;\overline{V}^2_\#(Q))$. By the weak compactness of $L^2(0,T;\overline{V}^2_\#(Q))$, there exists a convergent subsequence of $\{\mathbf{w}_k\}$ in $L^2(0,T;\overline{V}^2_\#(Q))$. Again, if we redenote the convergent subsequence by $\{\mathbf{w}_k\}$, then there exists $\mathbf{w}'(t) \in \overline{V}^2_\#(Q)$ such that

$$(\mathbf{w}_k(t),\mathbf{v}(t)) \to (\mathbf{w}'(t),\mathbf{v}(t)) \text{ uniformily in } (0,T], \text{ for all } j \in \mathbb{N} \tag{3.22}$$

and for all $\mathbf{v} \in \overline{V}^2_\#(Q)$. However, $\overline{V}^2_\#(Q) \subset H$. Uniqueness of the limit of a sequence together with (3.21) and (3.22) yield that $\mathbf{w}(t) = \mathbf{w}'(t)$.

It means that $\mathbf{w} \in L^2(0,T;\overline{V}^2_\#(Q)) \cap L^\infty(0,T;H)$. This implies that along a subsequence

$$\begin{aligned} \mathbf{w}_k \to \mathbf{w} \quad &\text{in} \quad L^2(0,T,L^2(Q)) \\ \nabla\mathbf{w}_k \to \nabla\mathbf{w} \quad &\text{in} \quad L^2(0,T,L^2(Q)). \end{aligned} \tag{3.23}$$

To prove (3.1) we replace $\boldsymbol{\psi}_j$ by $(I-\delta^2\triangle)\boldsymbol{\psi}_j$ in (3.13) and let $k\to\infty$. The nonlinear term gives

$$\begin{aligned} &\int_0^T \left(\nabla\cdot\overline{D(\mathbf{w}_k)D(\mathbf{w}_k)},(I-\delta^2\triangle)\boldsymbol{\psi}_j\right)d\tau - \int_0^T \left(\nabla\cdot\overline{D(\mathbf{w})D(\mathbf{w})},(I-\delta^2\triangle)\boldsymbol{\psi}_j\right)d\tau \\ &\quad= \int_0^T \left(\nabla\cdot D(\mathbf{w}_k)D(\mathbf{w}_k),\boldsymbol{\psi}_j\right)d\tau - \int_0^T \left(\nabla\cdot D(\mathbf{w})D(\mathbf{w}),\boldsymbol{\psi}_j\right)d\tau \\ &\quad= \int_0^T \left(\nabla\cdot D(\mathbf{w}_k-\mathbf{w})D(\mathbf{w}_k),\boldsymbol{\psi}_j\right)d\tau - \int_0^T \left(\nabla\cdot D(\mathbf{w})D(\mathbf{w}-\mathbf{w}_k),\boldsymbol{\psi}_j\right)d\tau \\ &\leq C(||D||)\ \max_j|\boldsymbol{\psi}_j|\ \ ||\mathbf{w}_k-\mathbf{w}||_{L^2(0,T,L^2(Q))}\ \ ||\nabla\mathbf{w}_k||_{L^2(0,T,L^2(Q))} \\ &\qquad + C(||D||)\ \max_j|\boldsymbol{\psi}_j|\ \ ||\nabla\mathbf{w}_k-\nabla\mathbf{w}||_{L^2(0,T,L^2(Q))}\ \ ||\mathbf{w}||_{L^2(0,T,L^2(Q))}, \end{aligned}$$

which tends to 0 as $k\to\infty$ by (3.23). This concludes our proof. □

Proposition 3.1. *If $\mathbf{w}$ is a weak solution of the model (1.4), then*

$$\mathbf{w}_t \in L^2(0,T;L^2(Q)). \tag{3.24}$$

Proof. By the Riesz Representation Theorem and density

$$||\mathbf{w}_t|| = \sup_{\phi\in D(Q)} \frac{\int_0^T(\mathbf{w}_t,\boldsymbol{\phi})dt}{||\boldsymbol{\phi}||}.$$

Let $\boldsymbol{\phi}\in D(Q)$. Multiplying the first equation of (1.4) by $\boldsymbol{\phi}$, applying the symmetric positive definite property of the filtering operation and the incompressibility condition we obtain

$$(\mathbf{w}_t,\boldsymbol{\phi}) = (\mathbf{f},\overline{\boldsymbol{\phi}}) - (\nabla\cdot D(\mathbf{w})\ D(\mathbf{w}),\overline{\boldsymbol{\phi}}) + \nu(\triangle\mathbf{w},\boldsymbol{\phi}). \tag{3.25}$$

We can bound each term in the right hand side. First, we note that Definition 2.1 leads to $||\overline{\boldsymbol{\phi}}|| \leq ||\boldsymbol{\phi}||$. Using the Cauchy-Schwarz inequality it follows that

$$(\mathbf{f},\overline{\boldsymbol{\phi}}) \ \leq\ ||\mathbf{f}||\ ||\overline{\boldsymbol{\phi}}|| \ \leq\ ||\mathbf{f}||\ ||\boldsymbol{\phi}||, \tag{3.26}$$

$$(\triangle\mathbf{w},\boldsymbol{\phi}) \ \leq\ ||\triangle\mathbf{w}||\ ||\boldsymbol{\phi}||. \tag{3.27}$$

If we also use Ladyshenskaya's and Poicaré's inequalities we obtain

$$\begin{aligned}(\nabla \cdot D(\mathbf{w})\ D(\mathbf{w}), \overline{\boldsymbol{\phi}}) &\leq C||D(\mathbf{w})||^{1/2}||\nabla D(\mathbf{w})||^{1/2}||\nabla D(\mathbf{w})||\ ||\nabla \overline{\boldsymbol{\phi}}|| \\ &\leq C(\delta, ||D||)\ ||\mathbf{w}||^{1/2}||\nabla \mathbf{w}||^{3/2}||\boldsymbol{\phi}||. \end{aligned} \tag{3.28}$$

With inequalities (3.26)-(3.28) and the regularity properties of $\mathbf{w}$, from (3.25) we deduce

$$(\mathbf{w}_t, \boldsymbol{\phi}) \leq C\,(\delta, ||D||, ||\mathbf{w}||, ||\nabla \mathbf{w}||, ||\triangle \mathbf{w}||)\ \ ||\boldsymbol{\phi}||.$$

Let $K := C\,(\delta, ||D||, ||\mathbf{w}||, ||\nabla \mathbf{w}||, ||\triangle \mathbf{w}||)$. Then

$$(\mathbf{w}_t, \boldsymbol{\phi}) \leq K||\boldsymbol{\phi}||.$$

Integration in time and Cauchy-Schwarz inequality in the right hand side lead to

$$\frac{\int_0^T (\mathbf{w}_t, \boldsymbol{\phi}) dt}{||\boldsymbol{\phi}||} \leq C(K, T).$$

Now take $\sup_{\boldsymbol{\phi}}$ and the conclusion follows. □

Remark 3.2. *An important consequence of Proposition 3.1 is that*

$$(\mathbf{w}_t, \mathbf{v}) + (\nabla \cdot D(\mathbf{w})\ D(\mathbf{w}), \overline{\mathbf{v}}) + \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) = (\mathbf{f}, \overline{\mathbf{v}}), \tag{3.29}$$

for any $\mathbf{v} \in L^2(0,T;L^2(Q))$. Indeed, since $D(Q)$ is dense in $L^2(0,T;L^2(Q))$, there exist a sequence of functions $\{\boldsymbol{\phi}_n\}_n$ in $D(Q)$ such that $\boldsymbol{\phi}_n \to \mathbf{w}$. If we multiply the first equation of (1.4) by $\{\boldsymbol{\phi}_n\}_n$ and let $n \to \infty$ we obtain (3.29), using the Proposition 3.1 to pull the limit into the first term and the regularity proven in the Theorem 3.1 for the remaining terms.

4 ADM Energy Balance and Uniqueness

In this section we prove that the weak solution of the ADM (1.4) is a unique strong solution. We also show that the model satisfies an energy equality rather than inequality. In our proofs we include the case when $I - DG$ is symmetric positive semi-definite and thus $\boldsymbol{\phi}^* := (I - DG)\boldsymbol{\phi}$. First we prove uniqueness.

Theorem 4.1. *Assume that $\mathbf{w}_0 \in \overline{H}^1_{\#}(Q) \cap H$ and $\mathbf{f} \in L^2(0,T;V')$. The weak solution of (1.4) is unique.*

Proof. By contradiction, assume that there exist two solutions $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$ of (1.4). Let $\boldsymbol{\phi} := \mathbf{w}_2 - \mathbf{w}_1$ (thus $\boldsymbol{\phi}^* := \mathbf{w}_2^* - \mathbf{w}_1^*$) and $r := q_2 - q_1$. Then, subtracting the weak formulations of $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$, it follows that $(\boldsymbol{\phi}, r)$ is a weak solution of

$$\begin{aligned}\boldsymbol{\phi}_t + \nabla \cdot (\overline{D(\mathbf{w}_2)D(\mathbf{w}_2) - D(\mathbf{w}_1)D(\mathbf{w}_1)}) - \nu \triangle \boldsymbol{\phi} + \nabla r + \chi \boldsymbol{\phi}^* &= 0 \\ \nabla \cdot \boldsymbol{\phi} &= 0 \\ \boldsymbol{\phi}(0) &= 0. \end{aligned} \tag{4.1}$$

subject to periodic boundary condition and zero mean. From Remark 3.2, we can multiply the first equation of (4.1) by $(I - \delta^2 \triangle)D(\boldsymbol{\phi}) \in L^2(0,T;L^2(Q))$. After algebraic manipulation and integration by parts, the nonlinear term becomes:

$$(\nabla \cdot (\overline{D(\mathbf{w}_2)D(\mathbf{w}_2) - D(\mathbf{w}_1)D(\mathbf{w}_1)}), (I - \delta^2 \triangle)D(\boldsymbol{\phi}))d\mathbf{x}\ = -\int_Q D(\boldsymbol{\phi}) \cdot \nabla D(\boldsymbol{\phi}) \cdot D(\mathbf{w}_1) d\mathbf{x}.$$

Since both $I - DG$ and D are symmetric, positive semi-definite and positive definite respectively, and all operators commute, the higher order fluctuation term is non-negative

$$\chi\ (\boldsymbol{\phi}^*, (I - \delta^2 \triangle)D(\boldsymbol{\phi}))\ =\ \chi\ (\boldsymbol{\psi}^*, \boldsymbol{\psi}) + \chi\ \ ((\nabla \boldsymbol{\psi})^*, (\nabla \boldsymbol{\psi}))\ \geq\ 0, \text{ where } \boldsymbol{\psi} = D^{1/2}(\boldsymbol{\phi}). \tag{4.2}$$

The other terms are

$$
\begin{aligned}
(\boldsymbol{\phi}_t, (I-\delta^2\triangle)D(\boldsymbol{\phi})) &= \frac{1}{2}\frac{d}{dt}(||\boldsymbol{\phi}||_D^2 + \delta^2||\nabla\boldsymbol{\phi}||_D^2) \\
-\nu\ (\triangle\boldsymbol{\phi}, (I-\delta^2\triangle)D(\boldsymbol{\phi})) &= \nu\ (||\nabla\boldsymbol{\phi}||_D^2 + \delta^2||\triangle\boldsymbol{\phi}||_D^2) \\
(\nabla r, (I-\delta^2\triangle)D(\boldsymbol{\phi})) &= -(r, (I-\delta^2\triangle)D(\nabla\cdot\boldsymbol{\phi})) = 0.
\end{aligned}
$$

Thus, we obtain

$$
\frac{1}{2}\frac{d}{dt}(||\boldsymbol{\phi}||_D^2 + \delta^2||\nabla\boldsymbol{\phi}||_D^2) + \nu(||\nabla\boldsymbol{\phi}||_D^2 + \delta^2||\triangle\boldsymbol{\phi}||_D^2) \leq \int_Q D(\boldsymbol{\phi})\cdot\nabla(D\boldsymbol{\phi})\cdot D(\mathbf{w}_1)d\mathbf{x}. \qquad (4.3)
$$

Next, we apply the Cauchy-Schwarz inequality in the right hand side

$$
\left|\int_Q D(\boldsymbol{\phi})\cdot\nabla D(\boldsymbol{\phi})\cdot D(\mathbf{w}_1)d\mathbf{x}\right| \leq ||D(\mathbf{w}_1)||_{L^4(Q)}||D(\boldsymbol{\phi})||_{L^4(Q)}||\nabla D(\boldsymbol{\phi})||.
$$

The Sobolev embedding theorem gives that there exists $C = C(Q, \mathbf{f}, \mathbf{w}_0, ||D||)$ such that

$$
\left|\int_Q D(\boldsymbol{\phi})\cdot\nabla D(\boldsymbol{\phi})\cdot D(\mathbf{w}_1)d\mathbf{x}\right| \leq C||\nabla\boldsymbol{\phi}||.
$$

Furthermore, using the equivalence of the D and the L^2 norm, there is a constant $C' = C'(\delta)$ such that

$$
||\nabla\boldsymbol{\phi}|| \leq C'(||\boldsymbol{\phi}||^2 + \delta^2||\nabla\boldsymbol{\phi}||^2).
$$

Putting everything together, (4.3) implies

$$
\frac{1}{2}\frac{d}{dt}(||\boldsymbol{\phi}||_D^2 + \delta^2||\nabla\boldsymbol{\phi}||_D^2) \leq C'(||\boldsymbol{\phi}||^2 + \delta^2||\nabla\boldsymbol{\phi}||^2) \qquad (4.4)
$$

But, $\boldsymbol{\phi}(0) = 0$. Gronwall's Lemma implies that $\boldsymbol{\phi}$ vanishes everywhere for all t. Hence, uniqueness follows. □

Lemma 4.1. *The LES ADM model (1.4) has a unique strong solution.*

Proof. From the Definition 2.5, any strong solution is, in particular, a weak solution as well. From Theorems 3.1 and 4.1 a unique weak solution exists. Remark 3.2 shows that this weak solution is also a strong solution. □

Theorem 4.2. *Let* $\mathbf{w}$ *be the unique strong solution of (1.4). Then* $\mathbf{w}$ *satisfies the energy equality:*

$$
\begin{aligned}
&\frac{1}{2}\left(||\mathbf{w}(t)||_D^2 + \delta^2||\nabla\mathbf{w}(t)||_D^2\right) + \nu\int_0^t \left(||\nabla\mathbf{w}(\tau)||_D^2 + \delta^2||\triangle\mathbf{w}(\tau)||_D^2\right)d\tau \\
&+\chi\int_0^t \left(\mathbf{w}^*(\tau), (-\delta^2\triangle + I)D\mathbf{w}(\tau)\right)d\tau = \int_0^t (\mathbf{f}, \mathbf{w}(\tau))_D d\tau + \frac{1}{2}\left(||\mathbf{w}(0)||_D^2 + \delta^2||\nabla\mathbf{w}(0)||_D^2\right), \qquad (4.5)
\end{aligned}
$$

for all $t \in (0, T]$.

Proof. Multiply the first equation of (1.4) by the test function $(-\delta^2\triangle + I)D(\mathbf{w})$. The nonlinear term vanishes because:

$$
\left(\nabla\cdot\overline{D(\mathbf{w})D(\mathbf{w})}, (I-\delta^2\triangle)D(\mathbf{w})\right) = (\nabla\cdot D(\mathbf{w})D(\mathbf{w}), D(\mathbf{w})) = 0
$$

Rewriting, we obtain:

$$
\left(\mathbf{w}_t, (I-\delta^2\triangle)D(\mathbf{w})\right) - \nu\left(\triangle\mathbf{w}, (I-\delta^2\triangle)D(\mathbf{w})\right) + \chi\left(\mathbf{w}^*, (-\delta^2\triangle + I)D(\mathbf{w})\right) = \left(\overline{\mathbf{f}}, (I-\delta^2\triangle)D(\mathbf{w})\right). \qquad (4.6)
$$

We have:

$$
\begin{aligned}
\left(\mathbf{w}_t, (I-\delta^2\triangle)D(\mathbf{w}_k)\right) &= \frac{1}{2}\frac{d}{dt}\left(||\mathbf{w}||_D^2 + \delta^2||\nabla\mathbf{w}||_D^2\right)\\
-\nu(\triangle\mathbf{w}, (I-\delta^2\triangle)D(\mathbf{w}_k)) &= \nu\left(||\nabla\mathbf{w}||_D^2 + \delta^2||\triangle\mathbf{w}||_D^2\right)\\
(\overline{\mathbf{f}}, (I-\delta^2\triangle)D(\mathbf{w})) &= (\mathbf{f}, D(\mathbf{w})).
\end{aligned}
\tag{4.7}
$$

Replacing (4.7) in (4.6) and integrating between 0 and t we obtain (4.5), for all $t \in (0, T]$. □

Remark 4.1. *In Theorem 4.2,*

$$
\begin{aligned}
&\textit{the kinetic energy of the model } = \frac{1}{2}\left(||\mathbf{w}(t)||_D^2 + \delta^2||\nabla\mathbf{w}(t)||_D^2\right),\\
&\textit{total energy dissipation } = \int_0^t \left[\nu\left(||\nabla\mathbf{w}(\tau)||_D^2 + \delta^2||\triangle\mathbf{w}(\tau)||_D^2\right) + \chi\left(\mathbf{w}^*(\tau), (-\delta^2\triangle + I)\mathbf{w}(\tau)\right)_D\right] d\tau,\\
&\textit{the initial kinetic energy of the model } = \frac{1}{2}\left(||\mathbf{w}(0)||_D^2 + \delta^2||\nabla\mathbf{w}(0)||_D^2\right),\\
&\textit{total energy input by the body force } \mathbf{f} = \int_0^t (\mathbf{f}, \mathbf{w}(\tau))_D d\tau.
\end{aligned}
$$

Thus Theorem 4.2 means that the kinetic energy of the model + total energy dissipation = the initial kinetic energy of the model + total energy input by the body force **f**.

5 Examples of Deconvolution Operators and their properties

The basic problem in deconvolution is: given $\overline{\mathbf{u}} + noise$ find $\mathbf{u}$ approximately. In other words

$$
\text{given } \overline{\mathbf{u}} \text{ solve } G\mathbf{u} = \overline{\mathbf{u}} \text{ for } \mathbf{u}. \tag{5.1}
$$

Many averaging operators G are symmetric and positive semi-definite. If the averaging operator is smoothing, the deconvolution problem will be not stably invertible due to small divisor problems. With these constraints in mind we review a few examples of deconvolution operators and their properties. In this section we consider the filtering operation be given by (2.1).

1. **The van Cittert deconvolution operator**.

   The van Cittert method of approximate deconvolution, see [2], constructs a family D_N of inverses to G using N steps of fixed point iterations.

   Algorithm 5.1. [*van Cittert Algorithm*]*: Choose*

   $$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

   For $n = 0, 1, 2, ..., N-1$ *perform*

   $$\mathbf{u}_{n+1} = \mathbf{u}_n + \{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

   Set $D_N\mathbf{u} := \mathbf{u}_N$.

   A very detailed mathematical theory of the van Cittert deconvolution operator and resulting approximate deconvolution models are already known, see [1], [3], [15], [5] and [6]. We point out the following lemma, proved in [6], concerned with properties of the approximate deconvolution operator D_N.

   Lemma 5.1. *The operator* $D_N : \mathrm{L}^2(Q) \to \mathrm{L}^2(Q)$ *is bounded, symmetric and positive definite.*

   Proof. For the proof see [[6], Lemma 2.1]. □

2. **The Accelerated van Cittert deconvolution operator.**

In our second example, approximations to $\mathbf{u}$ are obtained via the algorithm defined below.

Algorithm 5.2. [*Accelerated van Cittert Algorithm*]: *Given relaxation parameters* ω_n, *choose*

$$\mathbf{u}_0 = \overline{\mathbf{u}}.$$

For $n = 0, 1, 2, ..., N-1$ *perform*

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega_n\{\overline{\mathbf{u}} - G\mathbf{u}_n\}.$$

Set $D_N^{\omega}\mathbf{u} := \mathbf{u}_N$.

Proposition 5.1. *Let the averaging operator be the differential filter* $G\boldsymbol{\varphi} := (-\delta^2\triangle + I)^{-1}\boldsymbol{\varphi}$. *If the relaxation parameters* ω_i *are positive, for* $i = 0, 1, ..., N$, *then the Accelerated van Cittert deconvolution operator* $D_N^{\omega} : \mathrm{L}^2(Q) \to \mathrm{L}^2(Q)$ *is symmetric positive definite.*

Proof. The proof follows from [[17], Lemma 3.2]. □

3. **Tichonov regularization deconvolution operator.**

Our third example is the Tichonov regularization deconvolution operator. More precisely, since $G := (-\delta^2\triangle + I)^{-1}$ is symmetric positive-definite, given $\overline{\mathbf{u}}$ and $\mu > 0$ small, an approximate solution to the deconvolution problem (5.1) can be calculated as the unique minimizer in $L^2(Q)$ of the functional

$$F_\mu(\mathbf{v}) = \frac{1}{2}(G\mathbf{v}, \mathbf{v}) - (\overline{\mathbf{u}}, \mathbf{v}) + \frac{\mu}{2}(\mathbf{v}, \mathbf{v}).$$

The resulting family of Tichonov regularization deconvolution operators is

$$D_\mu = \lim_{\mu \to 0}(G + \mu I)^{-1} \tag{5.2}$$

and the approximate solution of (5.1) is $\mathbf{u}_\mu = \lim_{\mu\to 0}(G + \mu I)^{-1}\mathbf{u}$. The family of operators D_μ has the following properties, see [2]

1. for any $\mu > 0$, D_μ is a bounded linear operator,
2. $\lim_{\mu\to 0} D_\mu\boldsymbol{\varphi} = \boldsymbol{\varphi}$ for all $\boldsymbol{\varphi} \in L^2(Q)$.

Proposition 5.2. *Let the averaging operator be the differential filter* $G\boldsymbol{\varphi} := (-\delta^2\triangle + I)^{-1}\boldsymbol{\varphi}$. *Let* $\mu > 0$ *be fixed. The operator* $D_\mu : \mathrm{L}^2(Q) \to \mathrm{L}^2(Q)$ *is a bounded self-adjoint and positive definite.*

Proof. Remark that G is a linear, self-adjoint positive definite operator. Since D_μ is a function of G and the identity operator I, it is also linear and self-adjoint . The Spectral Mapping Theorem implies that the eigenvalues of D_μ are given by $\lambda(D_\mu) = (\lambda(G) + \mu I)^{-1}$.

Since $||G|| \leq 1$, see [6], we have that the eigenvalues of D_μ are strictly positive and

$$0 < \frac{1}{\mu} \leq \ \lambda(D_\mu) \ \leq \frac{1}{1+\mu} < \infty, \text{ for } \mu > 0. \tag{5.3}$$

Thus D_μ is also bounded and positive definite. □

4. **Geurts' approximate filter inverse.** One of the first studies of deconvolution as a basis for LES models was done by Geurts in [9]. Let $\overline{\boldsymbol{\phi}}$ be the top hat filter, $\overline{\boldsymbol{\phi}}(\mathbf{x}) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\boldsymbol{\phi}(x')dx'$. Briefly, he developed approximate inverse of the top-hat convolution filter,

$$D\boldsymbol{\phi} := \int_{x-\delta}^{x+\delta} d\,(x - x')\,\boldsymbol{\phi}(x')dx',$$

based on exactness of polynomials of degree $\leq \mu$. If $\Pi_\mu = \{p(x) : p(x) = a_0 + a_1 x + ... + a_\mu x^\mu\}$ the criteria that determined the deconvolution kernel d was

$$DG\phi = \phi, \text{ for all } \phi \in \Pi_\mu. \tag{5.4}$$

In $1D$ (in multiple dimension extension is by tensor product), the top-hat filter, $g(\cdot)$ has the special property that

$$g \star \{ \text{ polynomial of degree } \leq \mu\} = \{ \text{ polynomial of degree } \leq \mu\}.$$

Using this property, for $L \geq 0$, let

$$d(x) = \begin{cases} d_0 + d_1 x + ... + d_{2L} x^{2L}, & \text{if } |x| \leq 2\pi \\ 0, & \text{if } |x| > 2\pi. \end{cases}$$

On $[-\pi, \pi]$ the coefficients $d_0, d_1, ..., d_{2L}$ are uniquely determined by exactness of polynomials of degree $\leq 2L + 1$.

$$d \star g \star x^l = x^l, \text{ l = 0, 1, ..., 2L+1}. \tag{5.5}$$

The deconvolution operator $D\phi = d \star \phi$ is self adjoint, commutes with G. The theoretical development in [9] and associated test suggests that D satisfies:

$$\begin{aligned} ||DG\phi - \phi|| &\leq C(\phi)\delta^{2L+1} \text{ for smooth } \phi \\ ||D||_{\mathcal{L}(L^2 \to L^2)} &\leq C(\delta, L) < \infty. \end{aligned}$$

The deconvolution operator D_L are tabulated for $L = 0, 1, 2, 3$ in Geurts [[9], Table 1]. The top-hat filter, and thus the associated deconvolution operator, is important in many applications, but it is not clear if or how the theory developed herein could be extended to it. This is because $\widehat{g}(k) = sin(k\delta/2)/(k\delta/2)$, both changes sign and has zeros. At this point it appears to be an interesting and important open question to extend Geurts construction to the other filters, such as the Gaussian. (Extension to dynamic inverse models has been done in [12].)

5. **A variation of the Geurts' Approximate Filter Inverse**, [9]. The construction of Geurts [9], can be modified so as to fit in the theory herein. Indeed, first we shall interpret the construction in wave number space. With the differential filter $\widehat{g}(\delta k) = \frac{1}{\delta^2|k|^2+1}$ we have $|\widehat{g}(k)| \to 0$ as $|k| \to \infty$, since the filtering is smoothing. High order accuracy on the large scales means exactness on the high degree polynomials on x, (i.e. Guerts condition (5.4)) and, equivalently, high order contact of $\widehat{d}(k)$ to $\frac{1}{\widehat{g}(k)}$ at $k = 0$. Since the convolution should be a bounded operator and $\widehat{d}(0) = 1$ we can pose the problem seeking a rational deconvolution kernel

$$\widehat{d}(k) = \frac{1 + n_1 k + ... + n_l k^l}{1 + b_1 k + ... + b_l k^l}, \; b_l \neq 0.$$

Then the accuracy conditions are

$$\begin{aligned} &\widehat{d}(0) = 1 \text{ satisfied by the choice of the } 0^{th} \text{ coefficients } n_0 \text{ and } b_0 \\ &\frac{d^m}{dk^m}\widehat{d}(k)\bigg|_{k=0} = \frac{d^m}{dk^m}\frac{1}{\widehat{a}(k)}\bigg|_{k=0}, \; m = 1, 2, ..., \mu. \end{aligned}$$

If, additionally,

$$\begin{aligned} 1 + b_1 k + ... + b_l k^l \neq 0, \text{ for all } k \in \mathbb{R} \\ \widehat{d}(k) > 0, \text{ for all } k \in \mathbb{R}, \end{aligned}$$

then the deconvolution operator satisfies the conditions of the theory.

6. Not all deconvolution operators used in image processing satisfy $\mathbf{P}1, \mathbf{P}2$, and $\mathbf{P}3$ above. **Direct deconvolution**, $D = -\delta^2 \triangle + I$ is not bounded and thus not satisfying $\mathbf{P}1$. **The Accelerated van Cittert with negative relaxation parameters** is not positive definite, violating $\mathbf{P}2$. The resulting deconvolution operator is not positive definite. Furthermore, there are very many iterative methods, such as *steepest descent* and *the conjugate gradient method*, that can be truncated to give deconvolution operators. However these often produce approximations to $\mathbf{u}$ which depend nonlinearly on $\overline{\mathbf{u}}$. Thus, the resulting approximate deconvolution operators are nonlinear, violating $\mathbf{P}1$.

6 Extension to other filters and Conclusions

When developing a mathematical foundation of an LES model, the first analytical problem that arises is existence of solutions of the model. We developed a general theory about existence of solutions of deconvolution models. The averaging operator chosen was a specific differential filter. More generally, if the filter G satisfies $\widehat{g}(k) \neq 0$ for all k, then the exact filter inverse A can be defined as an unbounded operator with dense domain and close range. If additionally, $|\widehat{g}(k)| \to 0$ as $k \to \infty$ with $O(\frac{1}{|k|^2})$ (or faster) then the existence theory developed herein can be extended to the filter G. This includes the Gaussian filter, for example, but excludes the top filter and sharp spectral cutoff.

One main result of this work is finding near minimal conditions on the deconvolution operator that guarantee existence and uniqueness of the strong solution of a deconvolution model. We also proved that under $\mathbf{P}1, \mathbf{P}2$, and $\mathbf{P}3$ the models satisfy an energy equality, which describes the evolution of the kinetic energy in a fluid's flow. It is important to remark that there are many possible deconvolution operators that don't satisfy the conditions $\mathbf{P}1, \mathbf{P}2$, and $\mathbf{P}3$.

Acknowledgement

I thank Professor William Layton for suggesting the problem, all the support and help. I thank Professor Geurts for a helpful communication on inverse modeling in LES.

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