A COMPACT EMBEDDING OF A SOBOLEV SPACE IS EQUIVALENT TO AN EMBEDDING INTO A BETTER SPACE

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Abstract. We prove that in the presence of the Δ_2 condition the compact embedding of the Orlicz-Sobolev space is equivalent to the existence of a bounded embedding into a higher Orlicz space. We formulate results in an abstract setting of spaces of measurable functions with the property that every bounded sequence has a subsequence convergent a.e. We also provide an example showing that the theorem is not true without the Δ_2 condition.

1. Introduction

If $\Omega \subset \mathbb{R}^n$ is a bounded domain and the Sobolev space $W^{1,p}(\Omega)$ is emebdded into $L^q(\Omega)$, then for any $1 \leq s < q$, the embedding $W^{1,p}(\Omega) \Subset L^s(\Omega)$ is compact, see e.g. [4, Theorem 4]. This result generalizes to the setting of Orlicz-Sobolev spaces. Let A, Φ, Ψ be Young functions. If the embedding $W^{1,A}(\Omega) \subset L^{\Psi}(\Omega)$ is bounded and Ψ increases essentially faster than Φ, $\Psi \succ\succ \Phi$, then the embedding $W^{1,A}(\Omega) \Subset L^{\Phi}(\Omega)$ is compact, see [1, Theroem 8.24]. The last statement contains the previous one since the function t^q grows essentially faster than t^s for $q > s$. The proof given in [4, Theorem 4] is based on the following consequence of the Rellich-Kondrachov theorem: every bounded sequence in $W^{1,p}(\Omega)$ (or $W^{1,A}(\Omega)$) has a subsequence that is convergent a.e. and thus it is not surprising that the results can be generalized to the setting of abstract normed spaces W of measurable functions with the property that every bounded sequence has a subsequence convergent a.e., see Theorem 3.1 for a precise statement. This is nothing really new. What is new is that under the Δ_2 condition the converse implication is also true: If a Young function Φ satisfies the Δ_2 condition near infinity and the embedding $W \Subset L^{\Phi}$ is compact, then there is a Young function Ψ that grows essentially faster than Φ such that the embedding $W \subset L^{\Psi}$ is bounded. Hence in the presence of the Δ_2 condition compact embedding is equivalent with an embedding into a better space, see Theorem 3.2. This

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result has several natural consequences. In particular it shows that the optimal embedding is never compact, of course under the Δ_2 condition, see Corollary 3.3. In the last section we show an example (Theorem 4.1) that Theorem 3.2 is not true without the Δ_2 condition.

2. Notation and basic definitions

In this section we recall basic definitions and facts from the theory of Orlicz spaces. For more details, see [1], [6].

We say that $\Phi : [0,\infty) \to [0,\infty)$ is a *Young function* if it is convex, continuous, strictly increasing, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. If Φ and Ψ are two Young functions, we say that Ψ *grows essentially faster near infinity than* Φ if for every $k > 0$, $\Psi(t)/\Phi(kt) \to \infty$ as $t \to \infty$. We denote it by $\Psi \succ\succ \Phi$. Finally a Young function Φ is said to satisfy the Δ_2 *condition near infinity* if there are constants $K, t_0 > 0$ such that $\Phi(2t) \le K\Phi(t)$ for all $t > t_0$.

Observe that if Φ satisfies the Δ_2 condition near infinity, then $\Psi \succ\succ \Phi$ if and only if $\Psi(t)/\Phi(t) \to \infty$ as $t \to \infty$.

Let Φ be a Young function and (X,μ) be a measure space. For simplicity we will *always* assume that $\mu(X) < \infty$. The *Orlicz space* $L^\Phi(X)$ consists of all measurable functions u on X such that

$$\int_X \Phi(\lambda|u(x)|)\, d\mu < \infty \quad \text{for some } \lambda > 0.$$

It follows from the convexity of Φ that $L^\Phi(X)$ is a linear space and one can prove that this space equipped with the *Luxemburg norm*

$$\|u\|_\Phi = \inf\left\{ k > 0 : \int_X \Phi\left(\frac{|u(x)|}{k}\right) d\mu \le 1 \right\}$$

is a Banach space. Note that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_\Phi}\right) d\mu \le 1\,.$$

If Φ satisfies the Δ_2 condition near infinity, then

$$L^\Phi(X) = \left\{ u : \int_X \Phi(|u(x)|)\, d\mu < \infty \right\}$$

but this claim is not true without the Δ_2 condition.

Convexity of Φ implies that for $0 < \varepsilon \le 1$, $\Phi(x) \le \varepsilon\Phi(x/\varepsilon)$ and hence it is easy to see that convergence $u_n \to u$ in L^Φ implies

$$\int_X \Phi(|u_n - u|)\, d\mu \to 0\,. \tag{2.1}$$

Convergence (2.1) is called *convergence in mean* and we note here that convergence in mean implies convergence in the Luxemburg norm only if Φ satisfies the Δ_2 condition near infinity.

Given an open set $\Omega \subset \mathbb{R}^n$ and a Young function A we can define in a natural way the *Orlicz-Sobolev* space $W^{1,A}(\Omega)$. If $A(t) = t^p$, then $W^{1,A}(\Omega) = W^{1,p}(\Omega)$. Convexity of A implies that $A(t) \geq at$ for $t \geq t_0$ and hence $W^{1,A}(\Omega) \subset W^{1,1}_{\rm loc}(\Omega)$. Thus it follows from the Rellich-Kondrachov theorem and the standard diagonal argument that every bounded sequence in $W^{1,A}(\Omega)$ has a subsequence that is convergent a.e.

We say that a family of functions $\mathcal{F} \subset L^1(X)$, is *equi-integrable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sup_{f\in\mathcal{F}} \int_E |f| d\mu < \varepsilon \quad \text{whenever } \mu(E) < \delta.$$

Note that equi-integrability does not imply in general that the family $\mathcal{F}$ is bounded in $L^1(X)$ even if $\mu(X) < \infty$ (which is our standing assumption), because the measure may have atoms.

We will need the following result of de la Vallée Poussin which we state as a lemma. For a proof, see [3], [6].

Lemma 2.1 (de la Vallée Poussin)**.** *Let (X, μ) be a measure space with $\mu(X) < \infty$ and let $\mathcal{F} \subset L^1(\mu)$ be bounded. Then $\mathcal{F}$ is equi-integrable if and only if there is a Young function Φ, $\lim_{t\to\infty} \Phi(t)/t = \infty$ such that*

$$\sup_{f\in\mathcal{F}} \int_X \Phi(|f|) d\mu \leq 1 \,. \tag{2.2}$$

In most of the statements found in the literature the condition is that the integral (2.2) is finite. Dividing Φ by an appropriate constant we may further require that the integral is less than or equal to 1, as we do in (2.2).

3. Main Theorems

The following result is a common generalization of [1, Theorem 8.24], [4, Theorem 4].

Theorem 3.1. *Let $W(X)$ be a normed space of measurable functions on (X, μ), $\mu(X) < \infty$, with the property that every bounded sequence in $W(X)$ has a subsequence that is convergent a.e. If Ψ is a Young function such that the embedding $W(X) \subset L^\Psi(X)$ is bounded, then for every Young function Φ such that $\Psi \succ\succ \Phi$, the embedding $W(X) \Subset L^\Phi(X)$ is compact.*

Proof. Since the embedding $W \subset L^\Psi$ is bounded, there is a constant $C > 0$ such that $\|f\|_\Psi \leq C\|f\|_W$ for all $f \in W$. Let $\{f_i\} \subset W$ be a bounded sequence, $\|f_i\|_W \leq M$. It suffices to prove that a subsequence of f_i is a Cauchy sequence in L^Φ. Let $u_i = f_i/\varepsilon$. Then

$$\|u_i - u_j\|_\Psi \leq C\|u_i - u_j\|_W \leq 2CM\varepsilon^{-1}$$

and hence

$$\int_X \Psi\left(\frac{|u_i - u_j|}{2CM\varepsilon^{-1}}\right) d\mu \leq 1 \quad \text{for all } i, j.$$

Since Ψ grows essentially faster than Φ, there is $t_0 > 0$ such that

$$\Phi(t) \leq \frac{1}{4}\Psi\left(\frac{t}{2CM\varepsilon^{-1}}\right) \quad \text{for } t > t_0.$$

On the set $\{|u_i - u_j| < t_0\}$ we have $\Phi(|u_i - u_j|) < \Phi(t_0)$. Let $\delta = (4\Phi(t_0))^{-1}$. If $E \subset X$ is such that $\mu(X \setminus E) < \delta$, then

$$\begin{aligned}\int_{X\setminus E} \Phi(|u_i - u_j|)\, d\mu &\leq \int_{\{|u_i - u_j| > t_0\}} \Phi(|u_i - u_j|)\, d\mu + \int_{X\setminus E} \Phi(t_0)\, d\mu \\ &\leq \frac{1}{4}\int_X \Psi\left(\frac{|u_i - u_j|}{2CM\varepsilon^{-1}}\right) d\mu + \frac{\Phi(t_0)}{4\Phi(t_0)} \\ &\leq \frac{1}{2} \quad \text{for all } i, j.\end{aligned}$$

By our assumptions u_i has a subsequence u_{i_j} that is convergent a.e. According to the Egorov theorem there is a measurable set $E \subset X$ such that $\mu(X \setminus E) < \delta$ and u_{i_j} converges uniformly on E. Hence there is N such that

$$|u_{i_j}(x) - u_{i_k}(x)| \leq \Phi^{-1}\left(\frac{1}{2\mu(X)}\right) \quad \text{for all } x \in E \text{ and } j, k \geq N.$$

Then for $j, k \geq N$ we have

$$\begin{aligned}\int_X \Phi\left(\frac{|f_{i_j} - f_{i_k}|}{\varepsilon}\right) d\mu &= \int_E \Phi(|u_{i_j} - u_{i_k}|)\, d\mu + \int_{X\setminus E} \Phi(|u_{i_j} - u_{i_k}|)\, d\mu \\ &\leq \frac{\mu(X)}{2\mu(X)} + \frac{1}{2} = 1\end{aligned}$$

and hence

$$\|f_{i_j} - f_{i_k}\|_\Phi \leq \varepsilon \quad \text{for all } j, k \geq N.$$

The proof is complete. □

The following theorem is the main result of the paper.

Theorem 3.2. *Let $W(X)$ be a normed space of measurable functions on (X, μ), $\mu(X) < \infty$, with the property that every bounded sequence in $W(X)$ has a subsequence that is convergent a.e. Let Φ be a Young function that satisfies the Δ_2 condition near infinity. Then the following conditions are equivalent.*

(a) *$W(X)$ is compactly embedded into $L^\Phi(X)$, $W(X) \Subset L^\Phi(X)$.*

(b) *There is a Young function $\Psi \succ\succ \Phi$ such that $W(X)$ is continuously embedded into $L^\Psi(X)$, $W(X) \subset L^\Psi(X)$.*

Proof. The implication from (b) to (a) is contained in Theorem 3.1, so we are left with the proof of the implication from (a) to (b). Suppose that $W \Subset L^\Phi$ and Φ satisfies the Δ_2 condition near infinity. Let $C > 0$ be such that $\|f\|_\Phi \leq C\|f\|_W$ for $f \in W$. Consider the unit sphere in W

$$S = \{f \in W : \|f\|_W = 1\}.$$

We claim that the family

$$\mathcal{F} = \{\Phi(|f|/C) : f \in S\}$$

is bounded and equi-integrable in $L^1(X)$. Boundedness follows from the definition of the Luxemburg norm. Indeed, $\|f\|_\Phi \leq C$ for $f \in S$ and hence

$$\int_X \Phi(|f|/C)\, d\mu \leq 1\,.$$

Thus $\mathcal{F}$ is contained in the unit ball in $L^1(X)$. By contrary suppose that $\mathcal{F}$ is not equi-integrable. Then there is $\varepsilon > 0$ and two sequences $E_n \subset X$, $f_n \in S$ such that $\mu(E_n) < 1/n$, while

$$\int_{E_n} \Phi\left(\frac{|f_n|}{C}\right) d\mu \geq \varepsilon\,. \tag{3.1}$$

The sequence $2f_n/C$ is bounded in W and since the embedding $W \Subset L^\Phi$ is compact, the sequence has a subsequence (still denoted by $2f_n/C$) convergent in L^Φ to some function $g \in L^\Phi$. The convergence in mean (2.1) gives

$$\int_X \Phi\left(\left|\frac{2f_n}{C} - g\right|\right) d\mu < \varepsilon \quad \text{for } n \geq n_1.$$

Since $g \in L^\Phi$ and Φ satisfies the Δ_2 condition near infinity, $\int_X \Phi(|g|)\, d\mu < \infty$ and hence there is n_2 such that

$$\int_{E_n} \Phi(|g|)\, d\mu < \varepsilon \quad \text{for } n \geq n_2$$

by absolute continuity of the integral. For $n > \max\{n_1, n_2\}$ convexity of Φ gives

$$\begin{aligned}
\int_{E_n} \Phi\left(\frac{|f_n|}{C}\right) d\mu \;&\leq\; \int_{E_n} \Phi\left(\frac{1}{2}\left|\frac{2f_n}{C} - g\right| + \frac{1}{2}|g|\right) d\mu \\
&\leq\; \frac{1}{2}\int_{E_n} \Phi\left(\left|\frac{2f_n}{C} - g\right|\right) d\mu + \frac{1}{2}\int_{E_n} \Phi(|g|)\, d\mu < \varepsilon
\end{aligned}$$

which contradicts (3.1). We proved that the family $\mathcal{F}$ satisfies assumptions of the de la Vallée Poussin theorem and hence there is a Young function η

such that $\eta(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{f\in S} \int_X \eta\left(\Phi\left(\frac{|f|}{C}\right)\right)\, d\mu \leq 1\,.$$

Hence for all $0 \neq f \in W$ and $\Psi = \eta \circ \Phi$

$$\int_X \Psi\left(\frac{|f|}{C\|f\|_W}\right)\, d\mu \leq 1$$

which proves boundedness of the embedding $W \subset L^\Psi$ with the same constant $\|f\|_\Psi \leq C\|f\|_W$. It remains to observe that $\Psi \succ\succ \Phi$. Indeed, for any $k > 0$

$$\lim_{t\to\infty} \frac{\Psi(t)}{\Phi(kt)} = \lim_{t\to\infty} \frac{\eta(\Phi(t))}{\Phi(t)}\,\frac{\Phi(t)}{\Phi(kt)} = \infty$$

since $\Phi(t)/\Phi(kt)$ is bounded away from 0 by the Δ_2 condition. □

Corollary 3.3. *If Φ satisfies the Δ_2 condition near infinity and a bounded embedding $W(X) \subset L^\Phi(X)$ is optimal in the category of Orlicz spaces, then it is not compact.*

All of the above theorems apply to Sobolev and Orlicz-Sobolev spaces since in the statements we can take $W = W^{1,A}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set of finite measure.

Nečas [5, Théorème 1.4] proved that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with continuous boundary (i.e. the boundary is locally a graph of a continuous function), then the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact. As an immediate consequence of this result and Theorem 3.2 we obtain

Corollary 3.4. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with continuous boundary, then there is a Young function Φ that grows essentially faster at infinity than t^2, such that the embedding $W^{1,2}(\Omega) \subset L^\Phi(\Omega)$ is bounded.*

A more precise description of the function Φ can be obtained from the information about the modulus of continuity of the functions used to represent the boundary as a graph, but it is interesting to observe that our argument implies the existence of Φ without any careful investigation of the structure of the boundary.

4. Example

The following example shows that we cannot avoid the Δ_2 condition in Theorem 3.2. In particular the example shows that if we do not assume the Δ_2 condition, the optimal embedding can be compact, differently than in the case of Corollary 3.3. We do not know if there is a similar example in the setting of Sobolev spaces.

Theorem 4.1. *There is a Banach space W of measurable functions on $[0,1]$ with the following properties:*

(a) *Every bounded sequence in W has a subsequence convergent a.e.*
(b) $W \Subset L^\Phi([0,1])$ *for* $\Phi(t) = \frac{2}{\pi}(e^t - 1)$.
(c) *There is no Young function* $\Psi \succ\succ \Phi$ *such that* $W \subset L^\Psi([0,1])$.

Remark 4.2. A we do not require in (c) that the embedding $W \subset L^\Psi([0,1])$ has to be bounded. We only assume that every function in W belongs to $L^\Psi([0,1])$.

Proof. First we will define auxiliary functions that will be used to construct the space W. Let

$$f(x) = -\log(x + x\log^2 x), \quad x \in (0,1]\,.$$

Note that f is strictly decreasing from ∞ to 0. We have

$$\int_0^1 \Phi(|f(x)|)\,dx = \frac{2}{\pi}\int_0^1 \left(e^{f(x)} - 1\right)\,dx = \frac{2}{\pi}\left(\frac{\pi}{2} - 1\right) < 1$$

since the antiderivative of $e^{f(x)} = (x + x\log^2 x)^{-1}$ is $\arctan\log x$. It is easy to see that for any $0 < k < 1$

$$\int_0^1 \Phi\left(\frac{|f(x)|}{k}\right)\,dx = \infty$$

and hence $\|f\|_\Phi = 1$. For $n \geq 2$ we define

$$g_n = c_n\chi_{[0,\frac{1}{n}]}, \quad \text{where } c_n = -\log\left(\frac{\pi}{2} + \arctan\log\frac{1}{n}\right)\,.$$

Observe that $c_n > 0$ for $n \geq 2$. Finally let $f_n = f + g_n$. We have

$$\begin{aligned}\int_0^1 \Phi(|f_n(x)|)\,dx &= \frac{2}{\pi}\left(\int_0^{1/n}\left(e^{f(x)}e^{c_n} - 1\right)\,dx + \int_{1/n}^1\left(e^{f(x)} - 1\right)\,dx\right)\\ &\leq \frac{2}{\pi}\left(1 - \frac{1}{n} + \frac{\pi}{2} - 1\right) < 1\end{aligned}$$

and for $0 < k < 1$

$$\int_0^1 \Phi\left(\frac{|f_n(x)|}{k}\right)\,dx > \int_0^1 \Phi\left(\frac{|f(x)|}{k}\right)\,dx = \infty\,,$$

so $\|f_n\|_\Phi = 1$. Note also that $f_n \to f$ in L^Φ as $n \to \infty$. Indeed, for every $\varepsilon > 0$

$$\lim_{n\to\infty}\int_0^1 \Phi\left(\frac{|f_n - f|}{\varepsilon}\right)\,dx = \lim_{n\to\infty}\frac{2}{\pi}\frac{1}{n}\left(e^{c_n/\varepsilon} - 1\right) = 0$$

by a simple application of the l'Hospital rule.

Now we define a Banach space W of measurable functions on $[0,1]$ as

$$W = \left\{ h = \sum_{i=1}^{\infty} a_i f_i \, : \, (a_i)_{i=1}^{\infty} \in \ell^1 \right\}$$

with the norm

$$\|h\|_W = \Big\| \sum_{i=2}^{\infty} a_i f_i \Big\|_W := \sum_{i=2}^{\infty} |a_i| \, .$$

Since for every $x \in (0,1]$, $f_i(x) = f(x)$ for all sufficiently large i, the series $\sum_{i=2}^{\infty} a_i f_i(x)$ converges at every $x \in (0,1]$ and hence it defines a measurable function. Considering intervals $(1/(n+1), 1/n]$, $n = 1,2,3,\ldots$ one can easily check by induction that if $\sum_{i=2}^{\infty} a_i f_i = 0$ a.e., then $a_i = 0$ for all i, so the coefficients a_i are uniquely determined and hence $\|\cdot\|_W$ is a well defined norm. Now it is obvious that W is isometric to ℓ^1 and hence W is a Banach space.

The partial sums of the series series $\sum_{i=2}^{\infty} a_i f_i$ form a Cauchy sequence in L^Φ because

$$\Big\| \sum_{i=k}^{\ell} a_i f_i \Big\|_\Phi \leq \sum_{i=k}^{\ell} |a_i| \|f_i\|_\Phi = \sum_{i=k}^{\ell} |a_i|$$

and hence the series converges in the Banach space L^Φ. This also shows that W is continuously embedded into L^Φ

$$\|h\|_\Phi = \Big\| \sum_{i=2}^{\infty} a_i f_i \Big\|_\Phi \leq \sum_{i=2}^{\infty} |a_i| \|f_i\|_\Phi = \sum_{i=2}^{\infty} |a_i| = \|h\|_W \, ,$$

but what is more interesting, the embedding is compact, $W \Subset L^\Phi([0,1])$. Before we prove this fact observe that compactness of the embedding implies that every bounded sequence in W has a subsequence that is convergent a.e. which is the property (a).

Recall that $f_n \to f$ in L^Φ as $n \to \infty$ and hence the set

$$F = \{f_i\}_{i=1}^{\infty} \subset L^\Phi, \quad \text{where } f_1 = f$$

is compact. Then also family of functions

$$K = \{x \mapsto t f_i(x) \, : \, t \in [-M, M], \ i \geq 1\} \subset L^\Phi$$

is compact. Indeed, K is the image of a continuous mapping defined on a compact set

$$\lambda : [-M,M] \times F \to L^\Phi, \quad \lambda(t, f_i) = t f_i, \quad \lambda([-M,M] \times F) = K \, .$$

According to Mazur's theorem [2, Theorem 4.8], the convex hull $co(K)$ is relatively compact in L^Φ. With this introduction we can complete the proof of (b) as follows.

Let $h_n \in W$ be a bounded sequence and let $\tilde{h}_n = \sum_{i=2}^{k(n)} a_i^n f_i$ be such that $\|h_n - \tilde{h}_n\|_W < 1/n$. The sequence $\tilde{h}_n$ is bounded, say

$$\|\tilde{h}_n\|_W = \sum_{i=2}^{k(n)} |a_i^n| \leq M\,.$$

Then

$$\tilde{h}_n = \sum_{i=2}^{k(n)} \frac{|a_i^n|}{\|\tilde{h}_n\|_W} \Big(\mathrm{sgn}\,(a_i^n)\|\tilde{h}_n\|_W f_i\Big) \in co(K)$$

and hence $\tilde{h}_n$ has a subsequence convergent in L^Φ. This also implies that h_n has a subsequence convergent in L^Φ to the same limit.

We are left with the proof of (c). Suppose that there is $\Psi \succ\succ \Phi$ such that $W \subset L^\Psi$. It follows from the closed graph theorem that the embedding is bounded. Indeed, if $h_n \to h$ in W and $h_n \to g$ in L^Ψ, then from boundedness of the embedding into L^Φ, $h_n \to h$ in L^Φ and hence $g = h$.

Since $\|f_n\|_W = 1$, the sequence f_n is bounded in L^Ψ, say $\|f\|_\Psi \leq C$, so

$$\int_0^1 \Psi\left(\frac{|f_n(x)|}{C}\right) dx \leq 1\,. \tag{4.1}$$

Note that

$$\inf_{x\in[0,1/n]} f_n(x) \geq f(1/n) \to \infty \quad \text{as } n \to \infty$$

and therefore the condition $\Psi \succ\succ \Phi$ implies

$$A_n = \inf_{x\in[0,1/n]} \frac{\Psi\left(\frac{|f_n(x)|}{C}\right)}{\Phi(|f_n(x)|)} \to \infty \quad \text{as } n \to \infty.$$

Thus

$$\begin{aligned}\int_0^1 \Psi\left(\frac{|f_n(x)|}{C}\right) dx &\geq A_n \int_0^{1/n} \Phi(|f_n(x)|)\, dx \\ &= \frac{2}{\pi} A_n \int_0^{1/n} \left(e^{f_n(x)} - 1\right) dx \\ &= \frac{2}{\pi} A_n \left(1 - \frac{1}{n}\right) \to \infty\end{aligned}$$

which contradicts (4.1). The proof is complete. □

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