On Lighthill's acoustic analogy for low Mach number flows

William Layton* Antonín Novotný†

Department of Mathematics, University of Pittsburgh,
301 Thackeray Hall, Pittsburgh PA 15 260, USA

Institut Mathématiques de Toulon,
Université du Sud Toulon-Var, BP 132, 839 57 La Garde, France

Dedicated to A. V. Kazhikov

Abstract

Most predictions of the noise generated by a turbulent flow are done using a model due to Lighthill from the 1950's (the Lighthill analogy). In a large region of a fluid at rest surrounding a small region containing a small Mach number, high Reynolds number turbulent flow, this is

- Step 1: Solve the *incompressible* Navier-Stokes equations with the constant density ϱ_∞ for the velocity $\mathbf{u}$.
- Step 2: Compute $\mathrm{div}(\mathrm{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u}))$ and solve the inhomogeneous acoustic equation in both regions for the acoustic density fluctuations R:

$$\partial_t^2 R - \omega \triangle R = \mathrm{div}(\mathrm{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u})),$$

  where $\sqrt{\omega}$ is the speed of sound.

Current understanding of the derivation of the Lighthill analogy seems to be a variation on Lighthill's origional reasoning and has resisted elaboration by the tools of both formal asymptotics and rigorous mathematics. In this report we give a rigorous derivation of Lighthill's acoustic analogy (including the sound source $\mathrm{div}(\mathrm{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u}))$ being derived from an *incompressible* flow simulation. from the compressible Navier-Stokes and energy equation as $\mathrm{Ma} \to 0$.

1 Introduction

1.1 The Lighthill equation and Lighthill's acoustic analogy

The mathematical simulation of aeroacoustic sound presents many technical problems related to modeling of its generation and propagation. Its importance for diverse

*wjl@pitt.edu, http://www.math.pitt.edu/˜wjl. Partially supported by NSF grants DMS 0508260 and 0810385

†novotny@univ-tln.fr. Work was completed during the stay of A.N. at the Department of Mathematics and the Department of Mechanical Engineering of the University of Pittsburgh under the financial support of the exchange visitor programm number P-1-00048. The authors thank to G.P. Galdi for inspiring discussions.

industrial applications is without any doubt in view of the demands of user comfort and environmental regulations. A few examples where aeroacoustic noise is a critical effect include the sounds produced by jet engines of an airliner, the noise produced in high speed trains and cars, wind noise around buildings, ventilator noise in various household appliances, ...

The departure point of most methods of acoustic simulations (at least those called hybrid methods) is the Lighthill theory [36], [37]. The starting point in the Lighthill approach in the simplest case is the system of Navier-Stokes-Poisson equations describing motion of a viscous compressible gas in isentropic regime, for unknown functions density ϱ and velocity $\mathbf{u}$. They read

$$\begin{aligned} &\partial_t\varrho + \mathrm{div}\varrho\mathbf{u} = 0, \\ &\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \varrho\mathbf{f} + \mathrm{div}\mathbb{S}, \end{aligned} \tag{1.1}$$

where $p = p(\varrho)$ is the pressure and $\mathbb{S} = \mathbb{S}(\nabla(\mathbf{u}))$ is the viscous stress tensor; p and $\mathbb{S}$ are given functions characterizing the gas and will be specified later. In this case we can rewrite the system as follows

$$\begin{aligned} &\partial_t R + \mathrm{div}\mathbf{Q} = \sigma \equiv \partial_t\Sigma, \\ &\partial_t\mathbf{Q} + \omega\nabla_x R = \mathbf{F} - \mathrm{div}\mathbb{T} \end{aligned} \tag{1.2}$$

where we have denoted

$$\mathbf{Q} = \varrho\mathbf{u}, \quad R = \varrho - \varrho_\infty \tag{1.3}$$

as the momentum and the density fluctuations from the basic density distribution ϱ_∞ of the background flow, and where we have set

$$\begin{aligned} &\Sigma = -\varrho_\infty, \quad \omega = p'(\varrho_\infty) > 0, \quad \mathbf{F} = \varrho\mathbf{f}, \\ &\mathbb{T} = \varrho\mathbf{u}\otimes\mathbf{u} + \Big(p - \omega(\varrho - \varrho_\infty)\Big)\mathbb{I} - \mathbb{S} \end{aligned} \tag{1.4}$$

Taking the time derivative of the first equation in (1.2) and the divergence of the second one, we obtain

$$\partial_t^2 R - \omega\Delta R = \partial_t\sigma - \mathrm{div}\mathbf{F} + \mathrm{div}(\mathrm{div}(\mathbb{T}). \tag{1.5}$$

Lighthill reasoned that, because of the large differences in energy, there is very little feedback from acoustics to the flow. Thus, according to the Lighthill's interpretation, equation (1.5) (or equivalently the system of equations (1.2)) is a non homogenous wave equation describing the acoustic waves (fluctuations of density), where the terms at the right hand side correspond to the mononopolar ($\partial_t\sigma$), bipolar ($-\mathrm{div}\mathbf{F}$) and quadrupolar ($\mathrm{div}(\mathrm{div}(\mathbb{T}))$) acoustic sources respectively, and are considered as known and calculable from the background fluid flow field. In the sequel, we shall deal rather with the formulation (1.2) and will refer to it as to the Lighthill equation or to the Lighthill acoustic analogy.

The physical sense of the terms at the right hand side of equation (1.5) is the following.

The first term $\partial_t\sigma$ represents the acoustic sources created by the changes of control volumes due to changes of pressure or displacements of a rigid surface: this source can be schematically described via a particle whose diameter changes (pulsates) creating acoustic waves (density perturbations). It may be interpreted as well as an instationary injection of a fluid mass σ per unit volume. The acoustic noise of a gun shot is a typical example.

The second term div$\mathbf{F}$ describes the acoustic sources due to external forces (usually resulting from the action of a solid surface on the fluid). This sources are responsible for the most of the acoustic noise in the machines and ventilators.

The third term div(div($\mathbb{T}$) is the acoustic source due to the turbulence and viscous effects in the background fluid flow which supports the density oscillations (acoustic waves). The noise of steady or non steady jets in aero-acoustics is the typical example.

The tensor $\mathbb{T}$ is called the Lighthill tensor. It is composed from three tensors whose physical interpretation is the following: the first term is the Reynolds tensor with components $\varrho u_i u_j$ describing the (nonlinear) turbulence effects, the term $(p - \omega(\varrho - \varrho_\infty))\mathbb{I}$ expresses the entropy fluctuations and the third one is the viscous stress tensor $\mathbb{S}$.

The method for predicting noise using Lighthill's equation is usually referred to as a hybrid method since noise generation and propagation are treated separately. The first step consists in using data provided by numerical simulations to form the sound sources. The second step then consists in solving the wave equation forced by these source terms to determine the sound radiation. The main advantage of this approach is that most of the conventional flow simulations can be used in the first step.

In its simplest form of a large region of a fluid at rest surrounding a small region containing a small Mach number, high Reynolds number turbulent flow, the, so called, hybrid method based upon Lighthill's theory is, e.g., Wagner, Huttl and Sagaut [46],

- Step 1: Solve the *incompressible* Navier-Stokes equations for the velocity $\mathbf{u}$.
- Step 2: Compute $\mathrm{div}(\mathrm{div}(\rho_\infty \mathbf{u} \otimes \mathbf{u}))$ (possibly plus more terms that are often dropped) and solve the inhomogeneous acoustic equation (1.5) in both regions for the acoustic density fluctuations R.

In practical numerical simulations, the Lighthill tensor is calculated from the velocity and density fields obtained by using various direct numerical methods and solvers for compressible Navier-Stokes equations. Then the acoustic effects are evaluated from the Lighthill equation by using diverse direct numerical methods for solving the non homogenous wave equations (see e.g. Colonius [8], Mitchell at al [35], Freud at al [23], among others). For flows in the low Mach number regimes the direct simulations are often costly, unstable, inefficient and unreliable, essentially due to the presence of rapidly oscillating acoustic waves (with periods proportional to the Mach number) in the equations themselves. In the low Mach number regimes the acoustic analogies as the Lighthill equation, in combination with the incompressible flow solvers, give more reliable results, see [23].

Indeed, if the Mach number is small, the background flow can be considered as incompressible implying negligible entropy fluctuations for non heated or isentropic flows [36], Bogey at al [4], Freud at al [23]; thus the Lighthill tensor reduces to

$$\mathbb{T} = \varrho \mathbf{u} \otimes \mathbf{u} - \mathbb{S}, \quad \text{where} \quad \varrho_\infty = \text{const.}, \quad \mathrm{div}\mathbf{u} = 0. \tag{1.6}$$

Moreover, due to the latter condition, for newtonian fluids, $\mathrm{div}(\mathrm{div}\mathbb{S}) = 0$, and the only relevant part of the Lighthill tensor in the Lighthill equation is the Reynolds tensor $\varrho \mathbf{u} \otimes \mathbf{u}$, cf. Lighthill [36].

The comparison of numerical simulations using compressible solvers on one hand and incompressible solvers on other hand at low Mach number regimes show no noticeable difference in the evaluated acoustic fields and a good agreement with experiments up to Ma $= 0.6$, see Boersma [3] and references quoted there.

For a complete review of numerical methods, evaluation and approximation of various sound sources in the Lighthill equation from the point of view of mathematical modeling and acoustic simulations see Freud at al [23].

The Lighthill acoustic analogy as described above involves the interaction of two motions of different time scales: the slow variables describe the background fluid flow governed by the Navier-Stokes equations; the fast variables describe the sound propagation and are governed by a non homogenous wave equation.

The main goal of the present paper is to establish a link between the recent theory of low Mach number limits in various models describing viscous compressible fluids (which started with the pioneering paper of Lions, Masmoudi [30]) on one hand, and Lighthill's acoustic analogy ([36], [37]) as well as underlying hybrid methods used by numerical analysts in acoustics (see e.g. Boersma [3] or Feud at al. [23]). The point of view presented in this paper should be compared and combined with another interpretations and results as [22] or [18].

We shall prove rigorously, that the Lighthill equation (1.2) with the right hand side calculated from incompressible Navier-Stokes equations can be obtained as a particular low Mach number limit of the Navier-Stokes-Poisson system describing viscous compressible gas in isentropic regime, more precisely, as a superposition of slow variables being governed by the incompressible Navier-Stokes equations and fast time variables solving a homogenous wave equation.

This result is obtained in the context of weak solutions, on an arbitrary large time interval and for the ill prepared initial data. It is formulated in Theorem 2.1.

We also prove, under certain assumptions on initial data, that the right hand side of the Lighthill equation is independent of time and can be calculated from the steady incompressible Navier-Stokes equations. This result is formulated in Theorem 2.2.

All these results can be reformulated and proved for the complete Navier-Stokes-Fourier system describing the motion of viscous heat conducting gasses modulo overcoming additional technical difficulties in the underlying mathematical analysis.

One may anticipate that the future development of mathematical fluid mechanics as well as capabilities of the related numerical simulations will depend on understanding not only the asymptotic models (the Lighthill equation being an example) but the way they can be rigorously derived. We therefore believe that the theorems itself are as important as the methods leading to their proofs. These methods have their mathematical background in physically motivated scaling analysis of the Navier-Stokes-Poisson system; the main challenge and difficulty consists in the fact the we deal with interaction of fluid motions characterized by two different time scales.

2 Original and target problems, main results

2.1 Weak formulation of the Lighthill equations

We shall investigate the Lighthill equation (1.2) with $\sigma = \partial_t \Sigma$, $\mathbf{F}$ and $\mathbb{T}$ a given scalar-, resp. vector- resp. tensor-valued functions on a sufficiently smooth (at least Lipschitz) bounded domain Ω on an arbitrary large time interval $(0,T)$, $T>0$.

Hereafter, we explain what we mean under the weak solution of system (1.2):

Definition 1.1
Let $\mathbb{T}$, $\mathbf{F}$ and Σ belong to $L^1(0,T;L^1(\Omega))$. We say that a couple $(R,\mathbf{Q}) \in L^1(0,T;L^1(\Omega))\times L^1(0,T;L^1(\Omega;R^3))$ represents a weak solution of the Lighthill equation (1.2) on $(0,T)\times\Omega$ if there exists a couple $(R_0^{(1)},\mathbf{Q}_0) \in L^1(\Omega)\times L^1(\Omega;R^3)$ (of initial conditions) such that

$$\int_0^T \int_\Omega \Big((R-\Sigma)\partial_t\varphi + \mathbf{Q}\cdot\nabla_x\varphi \Big) \mathrm{d}x\mathrm{d}t$$
$$= -\int_\Omega R_0^{(1)}\varphi(0)\mathrm{d}x, \quad \varphi \in C_c^\infty([0,T)\times\overline{\Omega})), \tag{2.1}$$

$$\int_0^T \int_\Omega \Big(\mathbf{Q}\partial_t\varphi + \omega R \mathrm{div}\varphi \Big) \mathrm{d}x\mathrm{d}t$$
$$= -\int_0^T \int_\Omega \Big(\mathbf{F}\cdot\varphi + \mathbb{T} : \nabla_x\varphi \Big)\mathrm{d}x\mathrm{d}t - \int_\Omega \mathbf{Q}_0\cdot\varphi(0,\cdot)\mathrm{d}x, \tag{2.2}$$
$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega}); R^3), \quad \varphi\cdot\mathbf{n} = 0 \quad \textit{on } [0,T)\times\partial\Omega.$$

If $(R,\mathbf{Q})$ is a sufficiently smooth weak solution to the Lighthill equation in a sufficiently smooth domain Ω with sufficiently smooth external data $\mathbf{F}$, Σ and $\mathbb{T}$, $(\mathbb{T}\mathbf{n})\times\mathbf{n} = 0$ at $\partial\Omega$, corresponding to initial conditions $(R_0^{(1)},\mathbf{Q}_0)$ then it verifies

$$\begin{gathered} \partial_t\mathbf{R} + \mathrm{div}\mathbf{Q} = \partial_t\Sigma \quad \text{in } (0,T)\times\Omega, \\ \partial_t\mathbf{Q} + \omega\nabla_x R = \mathbf{F} - \mathrm{div}\mathbb{T} \quad \text{in } (0,T)\times\Omega, \\ \mathbf{Q}\cdot\mathbf{n} = 0 \quad \text{at } (0,T)\times\partial\Omega, \end{gathered} \tag{2.3}$$

and

$$R(0,x) = R_0^{(1)}(x) - \Sigma(0,x), \quad x\in\Omega \tag{2.4}$$

in the classical sense.

2.2 The Navier-Stokes-Poisson system with small Mach number

The time evolution of the density ϱ, the velocity $\mathbf{u}$ of a viscous, compressible fluid in isentropic and low Mach number regime characterized by the Mach number, $\mathrm{Ma} = \varepsilon$, is governed by the *Navier-Stokes-Poisson system* :

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0, \tag{2.5}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}\,\mathbb{S} + \varrho\mathbf{f}, \tag{2.6}$$

The other nondimensional parameters in this system, as Strouhal number, Reynolds number and eventually Froude number, have been normalized to 1 (see, for instance, Klein [25] or the survey paper by Klein at al. [26] for more details about the dimensional analysiys of fluid dynamics equations).

Once completed with the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0, \ (\mathbb{S}\mathbf{n})\times\mathbf{n} = 0 \text{ on the boundary } \partial\Omega, \tag{2.7}$$

the conservation of total energy in Ω

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big)\mathrm{d}x + \varepsilon^2\int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\mathbf{u}\mathrm{d}x\mathrm{d}t = \varepsilon^2\int_\Omega \varrho\mathbf{f}\mathrm{d}x \tag{2.8}$$

follows provided both $\varrho > 0$ and $\mathbf{u}$ are "sufficiently" smooth.

In (2.5-2.8),

$$\mathbb{S} = \mathbb{S}(\nabla_x\mathbf{u}) = \mu\Big(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}\mathbf{u}\ \mathbb{I}\Big) + \zeta\mathrm{div}\mathbf{u}\mathbb{I}, \tag{2.9}$$

is the viscous stress tensor with shear (μ) and bulk (ζ) constant viscosities which satisfy

$$\mu > 0, \quad \zeta \geq 0 \tag{2.10}$$

and the so called potential energy H is given by

$$H(\varrho) = \varrho P(\varrho) \quad \text{where} \quad P(\varrho) = P(1) + \int_1^\varrho \frac{p(s)}{s^2}\mathrm{d}s. \tag{2.11}$$

Here and hereafter, the symbol $\frac{1}{2}(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u})$ denotes the symmetrized gradient. Since the entropy is supposed to be constant through the flow, the pressure takes the form

$$p(\varrho) = \varrho^\gamma, \quad \gamma > 1, \quad \text{yielding} \quad H(\varrho) = \frac{1}{\gamma - 1}\varrho^\gamma. \tag{2.12}$$

The physical values of the adiabatic constant γ are given by formula $\gamma = \frac{R+c_v}{c_v}$, where c_v is the specific heat at constant volume and R is the universal gas constant; physically reasonable values of γ's are in the range $(1, \frac{5}{3}]$, $\gamma = \frac{5}{3}$ being the adiabatic constant of the monoatomic gases. We notice, that $p(\varrho)$ satisfies standard themodynamics stability condition asserting its strict monotonicity (cf. Bechtel et al. [2]) and that function H in (2.11) is strictly convex on $(0, \infty)$. Later on, we shall suppose

$$\gamma > \frac{3}{2} \tag{2.13}$$

which is the condition required both by the existence theory and low Mach number limit performed in this paper. Notice that at least adiabatic constants of monoatomic gases do enter into this range.

We shall investigate the density fluctuations around a positive constant density $\overline{\varrho}$. Conformably to this fact, the Navier-Stokes-Poisson system (2.5-2.6) will be supplemented with initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0, \tag{2.14}$$

where the initial data

$$\varrho_0 = \varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon\varrho_{\varepsilon,0}^{(1)}, \ \mathbf{u}_0 = \mathbf{u}_{\varepsilon,0}, \tag{2.15}$$

are chosen so that

$$\overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_{\varepsilon,0}\ \mathrm{d}x = \text{const.} > 0, \tag{2.16}$$

with the quantities $\varrho_{\varepsilon,0}^{(1)}$, $\mathbf{u}_{\varepsilon,0}$, bounded uniformly with respect to $\varepsilon \to 0$.

Here and in what follows, we should always keep in mind that the absence of the density in the dependence of the transport coefficients μ and ζ is required by the *existence theory* and does not play any significant role in the present paper. We also should keep in mind that more general pressure law then (2.12) could be taken into account, provided $p(\varrho) \sim \varrho^\gamma$ for large ϱ's and provided the function H in (2.11) remains strictly convex on $(0,\infty)$.

Definition 1.2
We shall say that a couple $\{\varrho, \mathbf{u}\}$ *is a* **bounded energy weak solution** *of the Navier-Stokes-Poisson system (2.5 - 2.12) on a time interval* $(0,T)$ *if the following conditions are satisfied:*

- *the density* ϱ *is a non-negative function,* $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$*, the velocity field* $\mathbf{u}$ *belongs to the space* $L^2(0,T;W^{1,2}(\Omega;R^3))$*,*

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } (0,T)\times\partial\Omega$$

and the integral identity

$$\int_0^T\int_\Omega \Big(\varrho B(\varrho)\partial_t\varphi + \varrho B(\varrho)\mathbf{u}\cdot\nabla_x\varphi - b(\varrho)\ \mathrm{div}\mathbf{u}\ \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = \tag{2.17}$$

$$-\int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x$$

holds for any test function $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$*), and any* b *such that*

$$b \in L^\infty \cap C[0,\infty),\ \ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\ \mathrm{d}z; \tag{2.18}$$

- *the momentum* $\varrho\mathbf{u}$ *belongs to* $L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3))$*, and the integral identity*

$$\int_0^T\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + \frac{1}{\varepsilon^2}p(\varrho,\vartheta)\ \mathrm{div}\varphi\Big)\ \mathrm{d}x\mathrm{d}t = \tag{2.19}$$

$$\int_0^T\int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

is satisfied for any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega};R^3)$*,* $\varphi\cdot\mathbf{n} = 0$ *on* $(0,T)\times\partial\Omega$*;*

- *the total energy balance*

$$\Big[\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big)\ \mathrm{d}x\Big](\tau) + \varepsilon^2\int_0^\tau\int_\Omega \mathbb{S} : \nabla_x\mathbf{u}\mathrm{d}x\mathrm{d}t \tag{2.20}$$

$$\le \varepsilon^2\int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t + \int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2 + H(\varrho_0)\Big)\ \mathrm{d}x$$

holds for a.a. $\tau \in (0,T)$*.*

It follows from (2.17) and (2.19) that $(\varrho, \varrho\mathbf{u})$ admit pointwise time values, namely $\varrho \in C_{\rm weak}([0,T];L^\gamma(\Omega)$ and $\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, meaning among other things the satisfaction of initial conditions ϱ_0, $\varrho_0\mathbf{u}_0$ in the weak sense.

Note that (2.17) is the so-called renormalized formulation of continuity equation introduced by DiPerna and Lions [13].

The existence of variational solutions in the sense of Definition 1.2 was established in Lions [29] for $\gamma \geq 9/5$ and in [19] for $\gamma > 3/2$ for $\Omega \subset R^3$ a bounded spatial domain, where the velocity field was supposed to vanish on the boundary. The details about necessary modifications to accommodate the slip boundary conditions (2.7) can be found in [18]. More recent information about the existence results for the Navier-Stokes-Poisson system or for the Navier-Stokes-Fourier system can be found in the monographs [14], [39], [18].

2.3 Incompressible Navier-Stokes equations, nonsteady case

In order to conclude this part, we introduce a standard concept of weak solutions to the system of Navier-Stokes equations describing incompressible newtoninan fluid introduced more than 70 years ago by Leray [28].

In the classical framework, one is searching for a couple $(\Pi, \mathbf{U})$ representing pressure and velocity fields, Π a scalar-valued and $\mathbf{U}$ a vector-valued functions of time $t \in [0,T)$ and space $x \in \Omega$, which satisfies

$$\begin{aligned} \overline{\varrho}\partial_t \mathbf{U} + \overline{\varrho}\mathrm{div}(\mathbf{U}\otimes\mathbf{U}) + \nabla\Pi = \mathrm{div}\mathbb{S} + \overline{\varrho}\mathbf{f}, \\ \mathrm{div}\mathbf{U} = 0 \end{aligned} \tag{2.21}$$

endowed with the initial conditions

$$\mathbf{U}(0,x) = \mathbf{U}_0(x) \tag{2.22}$$

and boundary conditions (2.7).

Multiplying the first (momentum) equation in (2.21) scalarly by $\mathbf{U}$, and integrating over Ω, yields, for a classical solution $(\Pi, \mathbf{U})$, the energy identity, which reads

$$\frac{1}{2}\overline{\varrho}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{U}|^2 \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{U} \mathrm{d}x = \overline{\varrho}\int_\Omega \mathbf{f}\cdot\mathbf{U}\mathrm{d}x. \tag{2.23}$$

In (2.22) we have denoted by $\overline{\varrho} > 0$ the constant density of the fluid and by $\mathbb{S}$ the same viscous stress tensor as that one defined in (2.9-2.10); obviously, by virtue of the second (continuity) equation in (2.21), it simplifies to

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}\Big). \tag{2.24}$$

Definition 1.3

(i) *We shall say that function* $\mathbf{U}$ *is a weak solution of Navier-Stokes system (2.21), supplemented with the boundary conditions (2.7) and the initial conditions (2.22) if the following conditions are satisfied:*

- $$\mathbf{U} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W^{1,2}(\Omega;R^3))$$
  $$\mathrm{div}\mathbf{U} = 0 \ \ a.a. \ on \ (0,T)\times\Omega, \ \ \mathbf{U}\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0;$$

- *the integral identity*

$$\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho}(\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi \Big) \,\mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi \mathrm{d}x\mathrm{d}t + \tag{2.25}$$

$$\int_0^T \int_\Omega \mu(\nabla_x\mathbf{U} + \nabla_x^\perp\mathbf{U}) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{U}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

holds for any test function

$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega};R^3),\ \mathrm{div}\ \varphi = 0,\ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0. \tag{2.26}$$

(ii) *We say that* $\mathbf{U}$ *is a weak solution with bounded energy if* $\mathbf{U}$ *is a weak solution which satisfies the energy inequality*

$$\Big[\tfrac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{U}|^2\mathrm{d}x\Big](\tau) + \int_0^\tau\int_\Omega \mathbb{S} : \nabla_x\mathbf{U}\mathrm{d}x\mathrm{d}t \tag{2.27}$$
$$\leq \overline{\varrho}\int_0^\tau\int_\Omega \mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t + A$$

for a.a. $\tau\in(0,T)$*, where* A *is a positive constant depending only on initial data.*

(iii) *We shall say that function* $\mathbf{U}$ *is a Leray-Hopf weak solution of Navier-Stokes system (2.21), supplemented with the boundary conditions (2.7) and the initial conditions (2.22) if it is a weak solution with bounded energy and the constant* A *in (2.27) has the form*

$$A = \frac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{U}_0|^2\mathrm{d}x. \tag{2.28}$$

2.4 Incompressible Navier-Stokes equations, steady case

With the notation of the previous section the steady Navier-Stokes equations read

$$\overline{\varrho}\mathrm{div}(\mathbf{U}\otimes\mathbf{U}) + \nabla_x\Pi = \mathrm{div}\mathbb{S} + \overline{\varrho}\mathbf{f}, \tag{2.29}$$
$$\mathrm{div}\mathbf{U} = 0.$$

They are completed with slip boundary condition (2.7). Weak solutions to this system are defined as follows.

Definition 1.4
We say that a vector field $\mathbf{U}$ *is a weak solution of the problem (2.29), (2.7) if*

$$\mathbf{U}\in W^{1,2}(\Omega;R^3), \quad \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathrm{div}\mathbf{U} = 0 \tag{2.30}$$

and

$$\forall\varphi\in C_c^\infty(\overline{\Omega};R^3) \quad \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.31}$$
$$\int_\Omega \Big(\overline{\varrho}\mathbf{U}\otimes\mathbf{U}) - \mu(\nabla\mathbf{U} + \nabla^\perp\mathbf{U})\Big) : \nabla_x\varphi\mathrm{d}x = -\int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi\mathrm{d}x$$

In the context of steady Navier-Stokes equations, we shall investigate the asymptotic limits to the Navier-Stokes-Poisson system (2.5-2.7) emanating from initial data (2.15), (2.16) which are, in addition, well prepared, meaning

$$\begin{gathered} \varrho_{\varepsilon,0}^{(1)} \to 0 \quad \text{a.e. in } L^\infty(\Omega), \\ \varrho_{\varepsilon,0}|\mathbf{u}_{\varepsilon,0}|^2 \to \overline{\varrho}|\mathbf{u}_0|^2 \quad \text{weakly in } L^1(\Omega). \end{gathered} \tag{2.32}$$

2.5 The main results - asymptotic limits

For the formulation of the main theorem and also throughout the proofs we shall need the Helmholtz projection on the divergence free vector fields, $\mathbf{H}$, and its orthogonal complement, projection on gradients, $\mathbf{H}^\perp$.

For $\mathbf{v} \in L^p(\Omega; R^3)$, $1 < p < \infty$, where Ω is a Lipschitz domain,

$$\mathbf{H}^\perp(\mathbf{v}) = \nabla\psi, \quad \mathbf{H}(\mathbf{v}) = \mathbf{v} - \nabla\psi, \tag{2.33}$$

where $\psi \in \overline{W^{1,p}}(\Omega) := \{z \in W^{1,p}(\Omega) \,|\, \int_\Omega z \mathrm{d}x = 0\}$ is a (unique) solution of the weak Neumann problem

$$\forall \eta \in C_c^\infty(\overline{\Omega}), \quad \int_\Omega \nabla\psi \cdot \nabla\eta \mathrm{d}x = \int_\Omega \mathbf{v} \cdot \nabla\eta \mathrm{d}x. \tag{2.34}$$

Thus

$$\begin{gathered} \mathbf{H}^\perp : L^p(\Omega; R^3) \to \mathbf{G}^p(\Omega) := \{\nabla z \,|\, z \in \overline{W^{1,p}}(\Omega)\}, \\ \mathbf{H} : L^p(\Omega; R^3) \to \dot{\mathbf{L}}^p(\Omega) := \{\mathbf{z} \in L^p(\Omega; R^3) \,|\mathrm{div}\mathbf{z} = 0\} \end{gathered} \tag{2.35}$$

are continuous linear operators from $L^p(\Omega; R^3)$ to $L^p(\Omega; R^3)$, where the spaces at the right hand side are closed subspaces of $L^p(\Omega; R^3)$. In particular,

$$L^2(\Omega; R^3) = \mathbf{H}(L^2(\Omega)) \oplus \mathbf{H}^\perp(L^2(\Omega)) \tag{2.36}$$

where the direct sum is orthogonal.

Due to the elliptic regularity applied to (2.34), $\mathbf{H}$ and $\mathbf{H}^\perp$ are continuous linear operators from $W^{k,p}(\Omega; R^3)$ to $W^{k,p}(\Omega; R^3)$, $k \in N$.

Having introduced all the necessary material we are ready to state the main result concerning the asymptotic limit of solutions to the Navier-Poison-Stokes system for low values of the Mach number.

Theorem 2.1. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$. Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of bounded energy weak solutions to the Navier-Stokes-Poisson system in the sense of Definition 1.2, with p, H determined in terms of ϱ_ε by (2.12-2.13) and $\mathbb{S}$ defined in (2.9-2.10). Furthermore, assume the solutions emanate from the initial state*

$$\varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon\varrho_{\varepsilon,0}^{(1)}, \quad \mathbf{u}_{\varepsilon,0}, \quad \textit{with } \textstyle\int_\Omega \varrho_{\varepsilon,0}^{(1)} \mathrm{d}x = 0,\ \overline{\varrho} = \text{const.} > 0, \tag{2.37}$$

where

$$\varrho_{\varepsilon,0}^{(1)} \to \varrho_0^{(1)},\ \mathbf{u}_{\varepsilon,0} \to \mathbf{u}_0, \quad \textit{weakly-(*) in } L^\infty(\Omega) \tag{2.38}$$

and from the right hand side

$$\mathbf{f} \in L^\infty(0,T; L^2(\Omega; R^3)). \tag{2.39}$$

Let us introduce fast time variables

$$
\begin{aligned}
r_\varepsilon(t,x) &= \varrho(\varepsilon t,x), \quad \mathbf{v}_\varepsilon(t,x) = \mathbf{u}(\varepsilon t,x),\\
r_\varepsilon^{(1)}(t,x) &= \tfrac{r_\varepsilon(t,x)-\overline{\varrho}}{\varepsilon}, \quad \mathbf{q}_\varepsilon(t,x) = (r_\varepsilon \mathbf{v}_\varepsilon)(t,x).
\end{aligned} \tag{2.40}
$$

Then the following holds true:

- $$\varrho_\varepsilon \to \overline{\varrho} \ \textit{in} \ C([0,T];L^r(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega)), \ r\in[1,\gamma) \tag{2.41}$$

  and, passing to a subsequence if necessary,

  $$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in} \ L^2(0,T;W^{1,2}(\Omega;R^3)), \tag{2.42}$$

  $$\forall \mathcal{T}\in(0,\infty), \quad r_\varepsilon^{(1)} \to r^{(1)} \ \ \textit{in} \ C_{weak}([0,\mathcal{T}];L^{s_\gamma}(\Omega)), \ s_\gamma = \min\{\gamma,2\}, \tag{2.43}$$

  $$\forall \mathcal{T}\in(0,\infty), \quad \mathbf{q}_\varepsilon \to \mathbf{q} \ \textit{weakly-}* \ \textit{in} \ L^\infty(0,\mathcal{T};L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)). \tag{2.44}$$

- *Vector filed* $\mathbf{U}$ *is a weak solution with bounded energy (in the sense of Definition 1.3) to the Navier-Stokes equations (2.21), (2.7) with initial conditions*

  $$\mathbf{U}_0 = \mathbf{H}(\mathbf{u}_0), \tag{2.45}$$

  where $\mathbf{H}$ *is the Helmholtz projection defined in (2.33-2.35).*

- *If moreover*

  $$\mathbf{H}(\mathbf{u}_0) \in W^{\frac{2}{5},\frac{5}{4}}(\Omega;R^3) \quad \textit{(Sobolev-Slobodeckii space)}, \tag{2.46}$$

  then there exists a unique function

  $$\Pi \in L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\Omega)) \tag{2.47}$$

  such that the couple $(\Pi,\mathbf{U})$ *satisfies the integral identity*

  $$
  \begin{aligned}
  &\int_0^T\int_\Omega \overline{\varrho}\mathbf{U}\cdot\partial_t\varphi \ \mathrm{d}x \ \mathrm{d}t + \int_0^T\int_\Omega \Big(\overline{\varrho}(\mathbf{U}\otimes\mathbf{U}) - \mu(\nabla_x\mathbf{U}+\nabla_x^T\mathbf{U})\Big) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t \\
  &+ \int_0^T\int_\Omega \Pi \mathrm{div}\varphi \ \mathrm{d}x \ \mathrm{d}t = -\int_0^T\int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{U}_0\cdot\varphi(0,\cdot) \ \mathrm{d}x
  \end{aligned} \tag{2.48}
  $$

  with test functions

  $$\varphi \in C_c^\infty([0,T)\times\overline{\Omega};R^3), \ \varphi\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0. \tag{2.49}$$

- *The couple* $(R,\mathbf{Q})$*, where*

  $$R = r^{(1)} - \Big(1/[p'(\overline{\varrho})]\Big)\Pi, \ \mathbf{Q} = \mathbf{q} - \overline{\varrho}\mathbf{U} \tag{2.50}$$

  satisfies equations (2.1-2.2) with

  $$\omega = p'(\overline{\varrho}), \ \mathbf{F} = \overline{\varrho}\mathbf{f}, \ \mathbb{T} = \overline{\varrho}\mathbf{U}\otimes\mathbf{U} - \mu(\nabla_x\mathbf{U} + \nabla_x^\perp\mathbf{U}), \ \Sigma = -\tfrac{1}{\omega}\Pi \tag{2.51}$$

  and with initial conditions

  $$\mathbf{Q}_0 = \overline{\varrho}\mathbf{u}_0 - \overline{\varrho}\mathbf{H}(\mathbf{u}_0), \quad R_0^{(1)} = \varrho_0^{(1)}. \tag{2.52}$$

Therorem 2.1 will be proved in several steps in Sections 3–5.

If the initial data are well prepared and close to a steady solution, the system in the limit is again the Lighthill equation with the more regular right hand side which emanates from the steady Navier-Stokes equations. This result is subject of the following theorem.

Theorem 2.2. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$. Let $\{(\varrho_\varepsilon, \mathbf{u}_\varepsilon)\}$ be a family of the bounded energy weak solutions to the Navier-Stokes-Poisson system investigated in Theorem 2.1, which emanates from the same initial conditions and right hand side, meaning that $(\varrho_{0,\varepsilon},\ \mathbf{u}_{0,\varepsilon})$ and $\mathbf{f}$ satisfy (2.37-2.39). Suppose, in addition, that the right hand side $\mathbf{f}$ is time independent, i.e.*

$$\mathbf{f} \in L^2(\Omega; R^3), \tag{2.53}$$

that the initial data are well prepared, i.e. (2.32) holds, and close to the steady state corresponding to $\mathbf{f}$, meaning that

$$\mathbf{u}_0 \textit{ satisfies (2.30-2.31).} \tag{2.54}$$

Then the sequences ϱ_ε, $\mathbf{u}_\varepsilon$, $r_\varepsilon^{(1)}$, $\mathbf{q}_\varepsilon$, where $r_\varepsilon^{(1)}$, $\mathbf{q}_\varepsilon$ are defined in (2.40), admit limits $\overline{\varrho}$, $\mathbf{u}$, $r^{(1)}$, $\mathbf{q}$ specified in (2.41–2.44), and these limits have the following properties:

- $$\mathbf{U} = \mathbf{u}_0; \tag{2.55}$$

- *there exists a function $\Pi \in L^2(\Omega)$ such that*

  $$\int_\Omega \Big(\overline{\varrho}\mathbf{u}_0 \otimes \mathbf{u}_0 - \mu(\nabla_x \mathbf{u}_0 + \nabla_x^\perp \mathbf{u}_0)\Big) : \nabla_x \varphi \mathrm{d}x + \int_\Omega \Pi \mathrm{div}\varphi \mathrm{d}x = -\int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi \mathrm{d}x \tag{2.56}$$

  for all

  $$\varphi \in C_c^\infty(\overline{\Omega}; R^3), \quad \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0;$$

- *the couple*

  $$(R, \mathbf{Q}) = \Big(r^{(1)} - (p(\overline{\varrho}))^{-1}\Pi\,,\, \mathbf{q}\Big) \tag{2.57}$$

  belongs to

  $$\Big(C_{\rm weak}([0,T]; L^{s_\gamma}(\Omega)) \cap L^\infty(0,T; L^{s_\gamma}(\Omega))\Big) \times L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)) \tag{2.58}$$

  and satisfies equations (2.5), (2.6) with

  $$\omega = p'(\overline{\varrho}),\ \mathbf{F} = \overline{\varrho}\mathbf{f},\ \mathbb{T} = \overline{\varrho}\mathbf{u}_0 \otimes \mathbf{u}_0 - \mu(\nabla_x \mathbf{U} + \nabla_x^\perp \mathbf{U}), \quad \Sigma = 0 \tag{2.59}$$

  and with initial conditions

  $$\mathbf{Q}_0 = \overline{\varrho}\mathbf{u}_0, \quad R_0^{(1)} = \varrho_0^{(1)}. \tag{2.60}$$

Theorem 2.2 will be proved in Section 7.

From the point of view of mathematical modeling of acoustic waves, where the standard procedure consists in solving directly the non homogenous wave equation (2.3) for unknowns $(R, \mathbf{Q})$ with zero initial data $R(0)$ and $\mathbf{Q}(0)$, Theorems 2.1 and

2.2 suggest an alternative approach: To construct the solution $(R, \mathbf{Q})$ of the Lighthill equation via formulas (2.50) resp. (2.57) by using the solution $(\Pi, \mathbf{U})$ of the incompressible Navier-Stokes equations and the solution $(r^{(1)}, \mathbf{q})$ of the homogenous wave equation with the initial data $r^{(1)}(0) = \frac{1}{\omega}\Pi(0)$, $\mathbf{q}(0) = 0$.

In what follows we shortly describe the organization of proofs.

Section 3 is devoted to the a priori estimates. Lemma 3.1 shows the energy inequality in the form which takes into account the way the initial data are bounded. It is then used in Lemma 3.2 to deduce estimates for the real-time variables $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ and in Lemma 3.3 to derive estimates for the fast-time variables $(r_\varepsilon, \mathbf{v}_\varepsilon)$.

Section 4 concerns the passage to the limit in the Navier–Stokes–Poisson equations rescaled to the fast time; we show that the limiting density fluctuations and gradient part of the momentum satisfy a conveniently weakly formulated homogenous wave equation. These results are concisely announced in Lemmas 4.1.

Section 5 deals with the real-time limit of the bounded energy weak solutions of the Navier-Stokes-Poisson equations to a weak solution with bounded energy of the Navier-Stokes equations in the sense of Definition 1.3. The principal result is formulated in Lemma 5.1 and proved through Sections 6.1–6.5. In Section 6.1, we start to investigate basic limits which can be deduced from à priori estimates listed in Lemma 4.1 via classical compactness tools of functional analysis and we show, among others, the strong convergence of divergenceless part of the sequence of velocity fields. The projection to the gradient part suffers from the lack of estimates of the time derivative; in fact, due to the presence of the singular term $(1/\varepsilon^2)\nabla p$ in the momentum equation, they may rapidly oscillate. Consequently, as usual in these type of problems, we will not be able to pass to the limit in the part $\operatorname{div}(\mathbf{H}^\perp(\varrho_\varepsilon \mathbf{u}_\varepsilon) \otimes \mathbf{H}^\perp(\mathbf{u}_\varepsilon))$ of the convective term. We shall rather prove that the latter expression tends to a gradient, and therefore is irrelevant from the point of view of definition of weak solutions. This property, discovered by Schochet [41] in the context of strong solutions (see also related papers by Kleinerman, Majda [24] da Veiga [9], Métivier, Schochet [34], Alazard [1] as well as the survey papers [42], [10], [11] plus references quoted there) , and by Lions, Masmoudi [30] (see also the the survey paper [32], [33]) in the context of weak solutions, can be proved nowadays by various methods see [12],[31], [6], [33], [15], [17], [16]. They are mostly based on the observation that $\mathbf{H}^\perp(\varrho_\varepsilon \mathbf{u}_\varepsilon)$ satisfy a non-homogenous wave equation with vanishing right hand side and exploit in various ways its structure. For the sake of completeness, we show in Sections 6.2-6.3 a "short" proof based on the spectral analysis of the underlying wave operator following Lions, Masmoudi [30]. Then we show in Section 6.4 that the limit velocity field $\mathbf{U}$ satisfies the energy inequality.

The definition of weak solutions with bounded energy is apparently silent about the pressure field, whose knowledge is, however, necessary to discover the "low Mach number" Lighthill acoustic analogy. For the non stationary Navier-Stokes equations this question is not an elementary problem (see e.g; the survey paper of Galdi [21]). To complete the proof of Lemma 5.1, we investigate this problem in Section 6.5 following Ladyzhenskaya [27].

Finally, the weak solutions of the Lighthill acoustic analogy in the sense of Definition 1.1 are obtained combining the weak solutions of the homogenous wave equation in the fast time limit constructed in Lemma 4.1 with the weak solutions of the non stationary Navier Stokes equation obtained in the real-time limit in Lemma 5.1. The time dependent pressure is responsible for the singular source term $\partial_t\Sigma$ which is equal to the distribution $-\frac{1}{\omega}\partial_t\Pi$.

In the steady case, when the initial data are well prepared in the sense of Theorem 2.2, the source term $\partial_t\Sigma = 0$ and we discover the weak formulation of the Lighthill equations with the right hand side emanating from a weak solution of the steady incompressible Navier-Stokes problem (2.29), (2.7). The precise formulation of this result is subject of Theorem 2.2.

The proof is performed in Section 7. Its first part consists of the material of Section 4 and the main auxiliary result serving for the construction of the Lighthill equation is formulated in Lemma 4.1. As far as the real-time limit, we start by Lemma 5.1, where we have showed existence of a weak limit in the case of ill-prepared data. We observe, that the weak solutions with bounded energy constructed in this lemma are Leray-Hopf solutions, provided the initial data are well prepared. If, in addition, the initial data are close to a steady state corresponding to the same specific external force $\mathbf{f}$, the limit appears to be a weak solution of steady Navier-Stokes problem with the same external force $\mathbf{f}$. Since this solution (as any steady weak solution) satisfies the Prodi-Serrin conditions, we can identify it with the Leray-Hopf weak solution constructed in Lemma 5.1. This observations are formulated in Lemmas 7.1–7.2. Theorem 2.2 is then obtained as a combination of the fast-time limit from Lemma 4.1 and real-time limit obtained in Lemma 7.2.

3 Estimates for real-time and fast-time variables

An immediate consequence of the energy inequality (2.20) is the following lemma:

Lemma 3.1. *Under assumptions of Theorem 2.1 the following estimate holds*

$$\begin{aligned}\Big[\int_\Omega \Big(\tfrac{1}{2}\varrho_\varepsilon\mathbf{u}_\varepsilon^2 + \tfrac{1}{\varepsilon^2}\mathcal{H}(\varrho_\varepsilon)\Big)\mathrm{d}x\Big](\tau) + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : \nabla\mathbf{u}_\varepsilon \mathrm{d}x\mathrm{d}t \\ \le \int_\Omega \Big(\tfrac{1}{2}\varrho_\varepsilon\mathbf{u}_{\varepsilon,0}^2 + \tfrac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x + \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.1}$$

where

$$\mathcal{H}(\varrho) = H(\varrho) + \partial_\varrho H(\overline{\varrho})(\varrho - \overline{\varrho}) - H(\overline{\varrho}) = \varrho^\gamma - \gamma\overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho}) - \overline{\varrho}^\gamma \tag{3.2}$$

is a strictly convex nonnegative function with minimum at $\varrho = \overline{\varrho}$.

Lemma 3.1 implies several estimates; in order to write them in a concise way we shall introduce, inspired by [17], the essential and residual sets as follows

$$\begin{aligned}\mathcal{M}_{\rm ess}(t) = \{x \in \Omega \,|\, \tfrac{\overline{\varrho}}{2} \le \varrho_\varepsilon(t,x) \le 2\overline{\varrho}\}, \quad \mathcal{M}_{\rm res}(t) = R^3 \setminus \mathcal{M}_{\rm ess}(t). \\ \tilde{\mathcal{M}}_{\rm ess}(t) = \mathcal{M}_{\rm ess}(\varepsilon t), \quad \tilde{\mathcal{M}}_{\rm res}(t) = \mathcal{M}_{\rm res}(\varepsilon t)\end{aligned} \tag{3.3}$$

For a function $h : \Omega \to R$, there holds

$$\begin{aligned}h = [h]_{\rm ess} + [h]_{\rm res}, \quad \text{where} \quad [h]_{\rm ess} = h1_{\mathcal{M}_{\rm ess}}, \quad [h]_{\rm res} = h1_{\mathcal{M}_{\rm res}}, \\ h = [h]_{\widetilde{\rm ess}} + [h]_{\widetilde{\rm res}}, \quad \text{where} \quad [h]_{\widetilde{\rm ess}} = h1_{\tilde{\mathcal{M}}_{\rm ess}}, \quad [h]_{\widetilde{\rm res}} = h1_{\tilde{\mathcal{M}}_{\rm res}}.\end{aligned} \tag{3.4}$$

We shall collect the estimates in the following two lemmas. Lemma 3.2 deals with the estimates of "real-time" quantities while Lemma 3.3 deals with their "fast-time" counterparts.

Lemma 3.2. *Under assumptions of Theorem 2.1 we have the following uniform estimates uniformly with respect to ϵ.*

$$\operatorname*{ess\ sup}_{t\in(0,T)} |\mathcal{M}_{\mathrm{res}}| \leq c\varepsilon^2, \tag{3.5}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\varrho_\varepsilon (t) \right]_{\mathrm{res}} \right\|_{L^\gamma(\Omega)} \leq c\varepsilon^{2/\gamma}, \tag{3.6}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\varrho_\varepsilon(t) - \overline{\varrho} \right]_{\mathrm{res}} \right\|_{L^\gamma(\Omega)} \leq c\varepsilon^{2/\gamma}, \tag{3.7}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} (t) \right]_{\mathrm{ess}} \right\|_{L^2(\Omega)} \leq c, \tag{3.8}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} (t) \right]_{\mathrm{res}} \right\|_{L^p(\Omega)} \leq c\varepsilon^{\frac{2}{p}-1},\ p \leq \gamma, \tag{3.9}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \varrho_\varepsilon (t) \right\|_{L^\gamma(\Omega)} \leq c, \tag{3.10}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon^2 (t)\|_{L^1(\Omega)} \leq c, \tag{3.11}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon (t)\|_{L^{\frac{2\gamma}{2+\gamma}}(\Omega)} \leq c, \tag{3.12}$$

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c \tag{3.13}$$

Lemma 3.3. *Under assumptions of Theorem 2.1 we have the following uniform estimates uniformly with respect to ϵ.*

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} |\tilde{\mathcal{M}}_{\mathrm{res}}| \leq c\varepsilon^2, \tag{3.14}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \left\| \left[r_\varepsilon (t) \right]_{\widetilde{\mathrm{res}}} \right\|_{L^\gamma(\Omega)} \leq c\varepsilon^{2/\gamma}, \tag{3.15}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \left\| \left[r_\varepsilon(t) - \overline{\varrho} \right]_{\widetilde{\mathrm{res}}} \right\|_{L^\gamma(\Omega)} \leq c\varepsilon^{2/\gamma}, \tag{3.16}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \left\| \left[\frac{r_\varepsilon - \overline{\varrho}}{\varepsilon} (t) \right]_{\widetilde{\mathrm{ess}}} \right\|_{L^2(\Omega)} \leq c, \tag{3.17}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \left\| \left[\frac{r_\varepsilon - \overline{\varrho}}{\varepsilon} (t) \right]_{\widetilde{\mathrm{res}}} \right\|_{L^p(\Omega)} \leq \varepsilon^{\frac{2}{p}-1},\ p \leq \gamma, \tag{3.18}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \left\| r_\varepsilon (t) \right\|_{L^\gamma(\Omega)} \leq c, \tag{3.19}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \|r_\varepsilon \mathbf{v}_\varepsilon^2 (t)\|_{L^1(\Omega)} \leq c, \tag{3.20}$$

$$\operatorname*{ess\ sup}_{t\in(0,T/\varepsilon)} \|\mathbf{q}_\varepsilon (t)\|_{L^{\frac{2\gamma}{2+\gamma}}(\Omega)} \leq c, \tag{3.21}$$

$$\|\mathbf{v}_\varepsilon\|_{L^2(0,T/\varepsilon;W^{1,2}(\Omega))} \leq \frac{c}{\sqrt{\varepsilon}}. \tag{3.22}$$

4 Limit passage in fast-time variables and homogenous wave equation

Lemma 4.1. *Under assumptions of Theorem 2.1 we have at least for a chosen subsequence of* $\varepsilon \to 0+$,

$$\forall \mathcal{T} > 0, \quad r_\varepsilon^{(1)} \to r^{(1)} \quad \textit{in } C_{\rm weak}([0,\mathcal{T}]; L^{s_\gamma}(\Omega)),\ s_\gamma = \min\{\gamma, 2\}, \tag{4.1}$$

$$\forall \mathcal{T} > 0, \quad \mathbf{q}_\varepsilon \to \mathbf{q} \quad \textit{weakly-}* \textit{ in } L^\infty(0,\mathcal{T}; L^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$

where the fast-time variables $r_\varepsilon^{(1)}$ *and* $\mathbf{q}_\varepsilon$ *have been defined in (2.40). Moreover, the weak limits* $r^{(1)}$, $\mathbf{q}$ *satisfy the homogenous wave equation*

$$\begin{gathered}\forall \varphi \in C_c^\infty([0,\infty) \times \overline{\Omega})), \\ \int_0^\infty \int_\Omega \Big(r^{(1)} \partial_t \varphi + \mathbf{q} \cdot \nabla_x \varphi \Big) {\rm d}x{\rm d}t = - \int_\Omega \varrho_0^{(1)} \varphi(0) {\rm d}x,\end{gathered} \tag{4.2}$$

$$\begin{gathered}\forall \varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}); R^3), \quad \varphi \cdot \mathbf{n} = 0 \quad \textit{on } (0,\infty) \times \partial\Omega, \\ \int_0^\infty \int_\Omega \Big(\mathbf{q} \partial_t \varphi + p'(\overline{\varrho}) r^{(1)} {\rm div} \varphi \Big) {\rm d}x{\rm d}t = - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0) {\rm d}x.\end{gathered} \tag{4.3}$$

where $\varrho_0^{(1)}$ *and* $\mathbf{u}_0$ *are defined in (2.38),* $\varrho_0 = \overline{\varrho} + \varrho_0^{(1)}$ *and*

$$r^{(1)}(0) = \varrho_0^{(1)}. \tag{4.4}$$

Proof of Lemma 4.1:

Let $\varphi \in C_c^\infty([0,\infty) \times \Omega; R^3)$, $\varphi \cdot \mathbf{n}|_{[0,\infty)\times\partial\Omega} = 0$; then there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, $\varphi \in C_c^\infty([0, T/\varepsilon) \times \overline{\Omega})$. We rewrite (2.19) with the test function $\varphi_\varepsilon(t,x) = \varphi(t/\varepsilon, x)$, where φ_ε is compactly supported in $[0,T) \times \overline{\Omega}$. After the change of variables to the fast time variable $\tau = \frac{t}{\varepsilon}$, we obtain

$$\begin{gathered}\int_0^{T/\varepsilon} \int_\Omega r_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \varphi \ {\rm d}x{\rm d}t + \\ \varepsilon \int_0^{T/\varepsilon} \int_\Omega (r_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) : \nabla_x \varphi \ {\rm d}x{\rm d}t + \frac{1}{\varepsilon} \int_0^{T/\varepsilon} \int_\Omega (r_\varepsilon^\gamma - \overline{\varrho}^\gamma) \ {\rm div}\varphi \ {\rm d}x{\rm d}t = \\ \varepsilon \int_0^{T/\varepsilon} \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{v}_\varepsilon) : \nabla_x \varphi + r_\varepsilon \mathbf{f}_\varepsilon \cdot \varphi \Big) \ {\rm d}x \ {\rm d}t - \int_\Omega \varrho_{\varepsilon,0} \mathbf{u}_{\varepsilon,0} \cdot \varphi(0,\cdot) \ {\rm d}x,\end{gathered} \tag{4.5}$$

where

$$\mathbf{f}_\varepsilon(t,x) = \mathbf{f}(\varepsilon t, x).$$

Similarly, continuity equation (2.17), where we take $b = 0$, rescaled to the fast time yields

$$\int_0^{T/\varepsilon} \int_\Omega \Big(r_\varepsilon^{(1)} \partial_t \varphi + r_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla_x \varphi \Big) {\rm d}x{\rm d}t = - \int_\Omega \varrho_{\varepsilon,0}^{(1)} \varphi(0,\cdot) {\rm d}x \tag{4.6}$$

where $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}))$ and $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is so small that φ is supported in $[0, T/\varepsilon) \times \overline{\Omega}$.

In virtue of (3.21) and thanks to (4.6), the sequence

$$\Big\{ \int_\Omega r_\varepsilon^{(1)} \varphi {\rm d}x \Big\}_{\varepsilon > 0}, \quad \text{where } \varphi \in C_c^\infty(\overline{\Omega}),$$

is bounded and equi-uniformly continuous subset of $C[0,\mathcal{T}]$ with any $\mathcal{T} \in (0,\infty)$. Arguing by the Arzela-Ascoli theorem, the separability of $L^{s_\gamma}(\Omega)$ and the diagonalization, keeping in mind estimates (3.17), (3.18), we conclude that

$$\forall \mathcal{T} > 0, \quad r_\varepsilon^{(1)} \to r^{(1)} \quad \text{in } C_{\rm weak}([0,\mathcal{T}]; L^{s_\gamma}(\Omega)), \tag{4.7}$$

at least for a chosen subsequence. Moreover, (4.4) holds.

By virtue of (3.21) we also have

$$\forall \mathcal{T} > 0, \quad \mathbf{q}_\varepsilon = r_\varepsilon \mathbf{v}_\varepsilon \to \mathbf{q} \quad \text{weakly-}* \text{ in } L^\infty(0,\mathcal{T}; L^{\frac{2\gamma}{\gamma+1}}(\Omega)). \tag{4.8}$$

This allows to pass to the limit in (4.6) and to get equation (4.2).

Writing

$$\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon} = \Big[\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon}\Big]_{\widetilde{\rm ess}} + \Big[\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon}\Big]_{\widetilde{\rm res}} =$$

$$\gamma \overline{\varrho}^{\gamma-1} \Big[\frac{r - \overline{\varrho}}{\varepsilon}\Big]_{\widetilde{\rm ess}} + \varepsilon\gamma(\gamma-1) z^{\gamma-2} \Big[\Big(\frac{r - \overline{\varrho}}{\varepsilon}\Big)^2\Big]_{\widetilde{\rm ess}} + \Big[\frac{r^\gamma - \overline{\varrho}^\gamma}{\varepsilon}\Big]_{\widetilde{\rm res}},$$

where $\overline{\varrho}/2 \le z \le 2\overline{\varrho}$, and exploiting (3.14), (3.16), (3.17–3.18), (3.20), we obtain

$$\forall \mathcal{T} > 0, \quad \frac{r_\varepsilon^\gamma - \overline{\varrho}^\gamma}{\varepsilon} \to r^{(1)} \quad \text{weakly in } L^1(0,\mathcal{T}; L^1(\Omega)). \tag{4.9}$$

For a given test function φ the upper bound of the time integrals in (4.5) and (4.6) are independent of $\epsilon \to 0+$. With this observation and with estimates (3.19), (3.20) and (3.22) in mind, we verify that

$$\varepsilon \int_0^{T/\varepsilon} \int_\Omega (r_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) : \nabla_x \varphi \; {\rm d}x{\rm d}t \to 0$$

$$\varepsilon \int_0^{T/\varepsilon} \int_\Omega \mathbb{S}(\nabla_x \mathbf{v}_\varepsilon) : \nabla_x \varphi \, {\rm d}x{\rm d}t \to 0$$

$$\varepsilon \int_0^{T/\varepsilon} \int_\Omega r_\varepsilon \mathbf{f}_\varepsilon \cdot \varphi \; {\rm d}x \; {\rm d}t \to 0.$$

Now, we are ready to let $\varepsilon \to 0+$ in (4.5) to get (4.3). The proof of Lemma 4.1 is complete.

□

5 Limit passage in the real-time variables and the Navier-Stokes equations

Lemma 5.1. *Under assumptions of Theorem 2.1 we have at least for a chosen subsequence of $\varepsilon \to 0+$,*

(i)

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)). \tag{5.1}$$

Moreover,

$$\mathbf{U} \in L^\infty(0,T; L^2(\Omega; R^3)) \cap C_{\rm weak}([0,T]; L^2(\Omega; R^3)), \tag{5.2}$$

$$\mathrm{div}\mathbf{U} = 0 \ a.a. \ on \ (0,T)\times\Omega, \ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0 \tag{5.3}$$

and the integral identity

$$\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \overline{\varrho}(\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi\Big) \, \mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega \overline{\varrho}\mathbf{f}\cdot\varphi \mathrm{d}x\mathrm{d}t + \tag{5.4}$$

$$\int_0^T \int_\Omega \mu(\nabla_x\mathbf{U} + \nabla_x^\perp\mathbf{U}) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{H}(\mathbf{u}_0)\cdot\varphi(0,\cdot) \ \mathrm{d}x$$

holds for any test function

$$\varphi \in C_c^\infty([0,T)\times\overline{\Omega}; R^3), \ \mathrm{div} \ \varphi = 0, \ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0. \tag{5.5}$$

In addition, $\mathbf{u}$ *satisfies the energy inequality*

$$\begin{gathered}\Big[\tfrac{1}{2}\overline{\varrho}\int_\Omega |\mathbf{u}|^2\mathrm{d}x\Big](\tau) + \int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}) : \nabla_x\mathbf{u}\mathrm{d}x\mathrm{d}t \\ \le \liminf_{\varepsilon\to0+}\Big[\int_\Omega\Big(\tfrac{1}{2}\varrho_\varepsilon\mathbf{u}_{\varepsilon,0}^2 + \tfrac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\Big] + \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t\end{gathered} \tag{5.6}$$

for a.a. $\tau\in(0,T)$.

In the other words, $\mathbf{U}$ *is a weak solution with bounded energy (in the sense of Definition 1.3) to the Navier-Stokes equations (2.21) with slip boundary conditions (2.7) and initial conditions* $\mathbf{U}(0) = \mathbf{H}(\mathbf{u}_0)$.

(ii) *Moreover, if* $\mathbf{H}(u_0)$ *satisfies conditions (2.46) then* $\mathbf{U}\in W^{1,\frac54}(0,T; L^{\frac54}(\Omega;R^3))\cap L^{\frac54}(0,T;W^{2,\frac54}(\Omega))$ *and there exists a function* $\Pi\in L^{\frac54}(0,T;W^{1,\frac54}(\Omega))$ *such that the couple* $(\Pi,\mathbf{U})$ *verifies*

$$\overline{\varrho}\partial_t\mathbf{U} + \overline{\varrho}\mathrm{div}(\mathbf{U}\otimes\mathbf{U}) + \nabla\Pi = \overline{\mu}\mathrm{div}(\nabla_x\mathbf{U} + \nabla_x^T\mathbf{U}) + \overline{\varrho}\mathbf{f} \tag{5.7}$$

almost everywhere in $(0,T)\times\Omega$.

Lemma 5.2. *Let assumptions of Theorem 5.1 be satisfied. Suppose that the initial data satisfy in addition conditions (2.30–2.32). Then we have:*

(i) The energy inequality (5.6) is replaced by

$$\begin{gathered}\tfrac12\overline{\varrho}\int_\Omega|\mathbf{U}|^2(\tau)\,\mathrm{d}x + \int_0^\tau\int_\Omega\tfrac{\overline{\mu}}{2}(\nabla_x\mathbf{U}+\nabla_x^T\mathbf{U}) : (\nabla_x\mathbf{U}+\nabla_x^T\mathbf{U})\,\mathrm{d}x\mathrm{d}t \\ \le \overline{\varrho}\int_0^\tau\int_\Omega \mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t + \tfrac12\overline{\varrho}\int_\Omega|\mathbf{u}_0|^2\,\mathrm{d}x \ \textit{for a.a.} \ \tau\in(0,T).\end{gathered} \tag{5.8}$$

In the other words, $\mathbf{U}$ *is a Leray-Hopf weak solution (Definition 1.3) to the Navier-Stokes equations (2.21) with slip boundary conditions (2.7) and initial conditions* $\mathbf{U}_0 = \mathbf{u}_0$.

(ii) Moreover $\mathbf{U} = \mathbf{U}_0 \in W^{2,\frac32}(\Omega)$ *and there exists a unique function* $\Pi\in W^{1,\frac32}(\Omega)$, $\int_\Omega \Pi\,\mathrm{d}x = 0$ *such that*

$$\overline{\varrho}\mathrm{div}(\mathbf{U}\otimes\mathbf{U}) - \overline{\mu}\mathrm{div}(\nabla_x\mathbf{U}+\nabla_x^T\mathbf{U}) + \nabla_x\Pi = \overline{\varrho}\mathbf{f}, \tag{5.9}$$

almost everywhere in Ω.

Lemma 5.1 will be proved throughout Sections 6.1–6.5. Lemma 5.2 will be proved in Section 7.

6 Proof of Lemma 5.1 and Theorem 2.1

6.1 Limits in the density, velocity and momentum

Since ϱ_ε satisfies (2.17) with $\mathbf{u}_\varepsilon$ on place of $\mathbf{u}$ and (3.6–3.10) it is a routine matter to establish that $\varrho_\varepsilon \in C([0,T]; L^r(\Omega)$, $1 \le r < \gamma$ and that

$$\varrho_\varepsilon \to \overline{\varrho} \quad \text{in } L^\infty(0,T; L^\gamma(\Omega) \text{ and } C([0,T]; L^r(\Omega)),\ r \in [1,\gamma), \tag{6.1}$$

where the second convergence is deduced from the fact that $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ satisfies the renormalized continuity equation (2.17) with $\mathbf{u}_\varepsilon$ obeying the bound (3.13). In accordance with this bound,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{6.2}$$

By virtue of (6.1), (6.2) and (3.12),

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \quad \text{weakly-}* \text{ in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)). \tag{6.3}$$

Thus the limit $\varepsilon \to 0+$ in the continuity equation (2.17) yields

$$\mathrm{div}\mathbf{U} = 0. \tag{6.4}$$

Due to the continuity properties of the projections $\mathbf{H}$, $\mathbf{H}^\perp$ (see (2.33–2.35)), we conclude from (6.2) and (6.4) that

$$\begin{aligned} &\mathbf{H}(\mathbf{u}_\varepsilon) \to \mathbf{U} \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),\\ &\mathbf{H}^\perp(\mathbf{u}_\varepsilon) \to 0 \quad \text{weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)). \end{aligned} \tag{6.5}$$

We deduce from (2.19) that the sequence of functions

$$t \to \Big[\int_\Omega \mathbf{H}(\varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot\varphi \mathrm{d}x\Big](t), \quad \varphi \in C^\infty(\overline{\Omega}),\ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$$

is bounded and equi-uniformly continuous in $C[0,T]$. Then, using the Arzela-Ascoli theorem, separability of $L^{[\frac{2\gamma}{2\gamma+1}]'}(\Omega; R^3)$ and density plus diagonalization argument, we obtain

$$\mathbf{H}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \to \overline{\varrho}\mathbf{H}(\mathbf{U}) = \overline{\varrho}\mathbf{U} \quad \text{in } C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)). \tag{6.6}$$

For $\gamma > \frac{3}{2}$, $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{2\gamma}{\gamma+1}}(\Omega)$; standard compactness argument then yields

$$\mathbf{H}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \to \overline{\varrho}\mathbf{U} \quad \text{in } L^2(0,T; [W^{1,2}(\Omega; R^3)]^*). \tag{6.7}$$

We also observe that

$$\begin{aligned} &(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon \to 0, \quad \mathbf{H}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) \to 0,\\ &\mathbf{H}^\perp\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) \to 0 \quad \text{in } L^2(0,T; L^{\frac{6}{5}}(\Omega; R^3)), \end{aligned} \tag{6.8}$$

where we have used (6.1-6.2) and continuity of $\mathbf{H}, \mathbf{H}^\perp$. Writing

$$\overline{\varrho}\Big(\mathbf{H}(\mathbf{u}_\varepsilon)\Big)^2 = \mathbf{H}\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big)\cdot\mathbf{H}(\mathbf{u}_\varepsilon) + \mathbf{H}(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\mathbf{H}(\mathbf{u}_\varepsilon)$$

we infer due to (6.5), (6.7) and (6.8) that

$$\mathbf{H}(\mathbf{u})_\varepsilon \to \mathbf{H}(\mathbf{U}) = \mathbf{U} \quad \text{in } L^2(0,T;L^2(\Omega;R^3)). \tag{6.9}$$

The projection of the convective term in (2.19) on the divergenceless vector fields can be written as

$$\int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \mathbf{H}^\perp \Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{H}\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t +$$
$$\int_0^T \int_\Omega \mathbf{H}\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{H}^\perp\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t, \tag{6.10}$$

where

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3), \ \mathrm{div}\ \varphi = 0, \ \varphi \cdot \mathbf{n}|_{[0,T)\times\partial\Omega} = 0. \tag{6.11}$$

By virtue of (6.2) and (6.7)

$$\int_0^T \int_\Omega \mathbf{H}\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega \overline{\varrho} \mathbf{U} \otimes \mathbf{U} : \nabla_x \varphi \mathrm{d}x\mathrm{d}t.$$

Further

$$\int_0^T \int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{H}\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t =$$
$$\int_0^T \int_\Omega \mathbf{H}^\perp\Big((\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon\Big) \otimes \mathbf{H}\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t + \overline{\varrho} \int_0^T \int_\Omega \mathbf{H}^\perp(\mathbf{u}_\varepsilon) \otimes \mathbf{H}(\mathbf{u}_\varepsilon) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t,$$

where the first term tends to 0 due to (6.5) and (6.8), while the second one converges to 0 by virtue of (6.5) and (6.9).

Thus, we have

$$\int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega \overline{\varrho} \mathbf{U} \otimes \mathbf{U} : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \tag{6.12}$$

for all φ belonging to (6.11) provided we show

$$\int_0^T \int_\Omega \mathbf{H}^\perp\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{H}^\perp\Big(\mathbf{u}_\varepsilon\Big) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \to 0 \tag{6.13}$$

with any φ in (6.11). In classical interpretation the last identity means that

$$\mathrm{div}\Big[\mathbf{H}^\perp\Big(\varrho_\varepsilon \mathbf{u}_\varepsilon\Big) \otimes \mathbf{H}^\perp\Big(\mathbf{u}_\varepsilon\Big)\Big]$$

as $\varepsilon \to 0$ becomes a gradient.

We shall devote to the proof of (6.13) the next two paragraphs.

6.2 Wave equation in real time and its spectral analysis

Following Schochet [41] and Lions-Masmoudi [30] we rewrite equations (2.17) (with $b = 0$) and (2.19) as a wave equation

$$\forall \varphi \in C_c^\infty((0,T) \times \overline{\Omega}),$$
$$\int_0^T \int_\Omega \Big(\varepsilon \varrho_\varepsilon^{(1)} \partial_t \varphi + \mathbf{z}_\varepsilon \cdot \nabla_x \varphi\Big) \mathrm{d}x\mathrm{d}t = 0 \tag{6.14}$$

$$\forall \varphi \in C_c^\infty((0,T)\times\overline{\Omega};R^3), \quad \varphi\cdot\mathbf{n}=0 \quad \text{on } (0,T)\times\partial\Omega,$$

$$\int_0^\infty\int_\Omega\Big(\varepsilon\mathbf{z}_\varepsilon\partial_t\varphi+p'(\overline{\varrho})\varrho_\varepsilon^{(1)}\mathrm{div}\varphi\Big)\mathrm{d}x\mathrm{d}t= \tag{6.15}$$

$$\varepsilon\int_0^T\int_\Omega\Big(-\mathbb{T}_\varepsilon:\nabla\varphi-\mathbf{F}_\varepsilon\cdot\varphi+g_\varepsilon\mathrm{div}\varphi\Big)\mathrm{d}x\mathrm{d}t,$$

where we have set

$$\varrho_\varepsilon^{(1)}=\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}, \quad \mathbf{z}_\varepsilon=\varrho_\varepsilon\mathbf{u}_\varepsilon \tag{6.16}$$

and

$$\mathbb{T}_\varepsilon=\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon-\mathbb{S}(\nabla\mathbf{u}_\varepsilon), \quad \mathbf{F}_\varepsilon=\varrho_\varepsilon\mathbf{f},$$

$$g_\varepsilon=\Big(\gamma\overline{\varrho}^{\gamma-1}\tfrac{1}{\varepsilon}\Big[\tfrac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\mathrm{res}}-\gamma(\gamma-1)z^{\gamma-2}\Big[\Big(\tfrac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)^2\Big]_{\mathrm{ess}}- \tag{6.17}$$

$$\tfrac{1}{\varepsilon}\Big[\tfrac{\varrho_\varepsilon^\gamma-\overline{\varrho}^\gamma}{\varepsilon}\Big]_{\mathrm{res}}\Big), \quad z\in(\overline{\varrho}/2,2\overline{\varrho}).$$

To identify the basic modes of (6.14), (6.15), we are naturally led to the eigenvalue problem

$$\nabla_x\omega=\lambda\mathbf{V}, \quad \mathrm{div}\mathbf{V}=\lambda\omega \quad \text{in } \Omega, \quad \mathbf{V}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{6.18}$$

which is equivalent to the eigenvalue problem for the Laplace operator

$$\Delta\omega=-\Lambda\omega, \quad \text{in } \Omega, \quad \nabla_x\omega\cdot\mathbf{n}|_{\partial\Omega}=0, \quad \lambda^2=-\Lambda. \tag{6.19}$$

The latter problem admits in $L^2(\Omega)$ a system of real eigenfunctions $\{\omega_{j,m}\}_{j=0,m=1}^{\infty,m_j}$, which forms an orthonormal basis in $L^2(\Omega)$, corresponding to real eigenvalues

$$\begin{gathered}(\Lambda_0=)\Lambda_{0,1}=0, \quad \omega_0=\omega_{0,1}=1/|\Omega| \text{ and } m_0=1,\\ 0<\Lambda_{1,1}=\ldots=\Lambda_{1,m_1}=(\Lambda_1)<\Lambda_{2,1}=\ldots=\Lambda_{2,m_2}(=\Lambda_2)<\ldots,\end{gathered} \tag{6.20}$$

where m_j denotes the multiplicity of the eigenvalue Λ_j.

Accordingly, the original system (6.18) admits solutions

$$\begin{gathered}\mathbf{V}_{j,m}=-\mathrm{i}(\sqrt{\Lambda_j})^{-1}\nabla\omega_{j,m}, \quad \lambda_j=\mathrm{i}\sqrt{\Lambda_j}, \quad \text{where } j=1,2,\ldots,\\ \lambda_0=0, \quad \text{with eigenspace } \dot{\mathbf{L}}(\Omega)=\mathbf{H}(L^2(\Omega;R^3)),\end{gathered} \tag{6.21}$$

meaning that

$$\begin{gathered}L^2(\Omega;R^3)=\mathbf{H}(L^2(\Omega;R^3))\oplus\mathbf{H}^\perp(L^2(\Omega;R^3)),\\ \text{where } \mathbf{H}^\perp(L^2(\Omega;R^3))=\overline{\{\mathrm{span}\{\mathrm{i}\mathbf{V}_{j,m}\}_{j=1,m=1}^{\infty,m_j}\}}^{L^2(\Omega;R^3)}.\end{gathered} \tag{6.22}$$

In the sequel we denote, for $a\in L^1(\Omega)$, $\mathbf{z}\in L^1(\Omega;R^3)$,

$$[a]_{j,m}=\int_\Omega a\omega_{j,m}\mathrm{d}x, \quad [\mathbf{z}]_{j,m}=\int_\Omega\mathbf{z}\cdot\mathbf{V}_{j,m}\mathrm{d}x \tag{6.23}$$

and

$$\{a\}_M=\sum_{\{j>0|\Lambda_j\le M\}}\sum_{m=1}^{m_j}[a]_{j,m}\omega_{j,m}, \quad \{\mathbf{z}\}_M=\sum_{\{j>0|\Lambda_j\le M\}}\sum_{m=1}^{m_j}[\mathbf{z}]_{j,m}\mathbf{V}_{j,m}, \tag{6.24}$$

where M is a fixed integer.

Since $\omega_{j,m}$, $\mathbf{V}_{j,m}$ are smooth functions on $\overline{\Omega}$ and $\mathbf{V}_{j,m}\cdot\mathbf{n}|_{\partial\Omega}=0$, we can use them as test functions in (6.14–6.15) to obtain

$$\begin{aligned}\varepsilon\partial_t[\varrho_\varepsilon^{(1)}]_{j,m}+\mathrm{i}\sqrt{\Lambda_j}[\mathbf{z}_\varepsilon]_{j,m}=0,\\ \varepsilon\partial_t[\mathbf{z}_\varepsilon]_{j,m}+\mathrm{i}\sqrt{\Lambda_j}p'(\overline{\varrho})[\varrho_\varepsilon^{(1)}]_{j,m}=\varepsilon A_\varepsilon^{j,m}\end{aligned}\tag{6.25}$$

where

$$A_\varepsilon^{j,m}=\int_\Omega\Big(-\mathbb{T}_\varepsilon:\nabla\mathbf{V}_{j,m}-\varrho_\varepsilon\mathbf{f}\cdot\mathbf{V}_{j,m}+g_\varepsilon\mathrm{div}\mathbf{V}_{j,m}\Big)\mathrm{d}x\quad\text{is bounded in } L^2(0,T).$$

Finally multiplying the first equation in (6.25) by $\omega_{j,m}$ and the second one by $\mathbf{V}_{j,m}$, we get after some calculus

$$\left\{\begin{aligned}\varepsilon\partial_t\{\varrho_\varepsilon^{(1)}\}_M+\mathrm{div}\{\mathbf{z}_\varepsilon\}_M=0,\\ \varepsilon\partial_t\{\mathbf{z}_\varepsilon\}_M+p'(\overline{\varrho})\nabla_x\{\varrho_\varepsilon^{(1)}\}_M=\varepsilon\mathbf{a}_{\varepsilon,M},\end{aligned}\right\}\quad\text{a.e. in } (0,T)\times\Omega,\tag{6.26}$$

where

$$\mathbf{a}_{\varepsilon,M}=\sum_{\{j>0|\Lambda_j\le M\}}\sum_{m=1}^{m_j}A_\varepsilon^{j,m}\mathbf{V}_{j,m}\quad\text{is bounded in } L^2(0,T;C^k(\overline{\Omega};R^3)),\ k\in N\tag{6.27}$$

for any fixed natural number M, uniformly with respect to $\varepsilon\to 0$.

6.3 Treatement of the convective term

As far as term (6.13) is concerned, we can write

$$\begin{aligned}\mathbf{H}^\perp(\mathbf{z}_\varepsilon)\otimes\mathbf{H}^\perp(\mathbf{u}_\varepsilon)=\Big[\Big\{\mathbf{H}^\perp(\mathbf{z}_\varepsilon)\Big\}_M+\Big[\mathbf{H}^\perp(\mathbf{z}_\varepsilon)-\Big\{\mathbf{H}^\perp(\mathbf{z}_\varepsilon)\Big\}_M\Big]\Big]\\ \otimes\Big[\Big\{\mathbf{H}^\perp(\mathbf{u}_\varepsilon)\Big\}_M+\Big[\mathbf{H}^\perp(\mathbf{u}_\varepsilon)-\Big\{\mathbf{H}^\perp(\mathbf{u}_\varepsilon)\Big\}_M\Big]\Big].\end{aligned}\tag{6.28}$$

We have

$$\mathbf{H}^\perp(\mathbf{z}_\varepsilon)-\Big\{\mathbf{H}^\perp(\mathbf{z}_\varepsilon)\Big\}_M=$$

$$\Big[\mathbf{H}^\perp\Big((\varrho_\varepsilon-\overline{\varrho})\mathbf{u}_\varepsilon\Big)-\Big\{\mathbf{H}^\perp\Big((\varrho_\varepsilon-\overline{\varrho})\mathbf{u}_\varepsilon\Big)\Big\}_M\Big]+\overline{\varrho}\mathbf{H}^\perp(\mathbf{u}_\varepsilon)-\overline{\varrho}\Big\{\mathbf{H}^\perp(\mathbf{u}_\varepsilon)\Big\}_M,$$

where, according to (6.8),

$$\Big[\mathbf{H}^\perp\Big((\varrho_\varepsilon-\overline{\varrho})\mathbf{u}_\varepsilon\Big)-\Big\{\mathbf{H}^\perp\Big((\varrho_\varepsilon-\overline{\varrho})\mathbf{u}_\varepsilon\Big)\Big\}_M\Big]\to 0\quad\text{in } L^2(0,T;L^{\frac{6}{5}}(\Omega)).$$

Furthermore, in agreement with (6.18-6.22),

$$\|\mathrm{div}\mathbf{u}_\varepsilon\|^2_{L^2(\Omega)}=\Big\|\sum_{j=1}^\infty\sum_{m=1}^{m_j}[\mathbf{u}_\varepsilon]_{j,m}\mathrm{div}\mathbf{V}_{j,m}\Big\|^2_{L^2(\Omega)}$$

$$= \Big\| \sum_{j=1}^{\infty} \sum_{m=1}^{m_j} [\mathbf{u}_\varepsilon]_{j,m} \sqrt{\Lambda_j} \omega_{j,m} \Big\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} \sum_{m=1}^{m_j} \Lambda_j [\mathbf{u}_\varepsilon]^2_{j,m};$$

whence

$$\|\mathbf{H}^\perp(\mathbf{u}_\varepsilon) - \{\mathbf{H}^\perp(\mathbf{u}_\varepsilon)\}_M\|_{L^2(\Omega)} = \sum_{\{j>0|\Lambda_j>M\}} \sum_{m=1}^{m_j} [\mathbf{u}_\varepsilon]^2_{j,m} \le \frac{1}{M} \|\mathrm{div}\mathbf{u}_\varepsilon\|^2_{L^2(\Omega)}. \quad (6.29)$$

In view of these observations, the proof of (6.13) reduces to showing

$$\int_0^T \int_\Omega \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M \otimes \Big\{\mathbf{H}^\perp\Big(\mathbf{u}_\varepsilon\Big)\Big\}_M : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \to 0$$

or equivalently, due to (6.1), (6.8)

$$\int_0^T \int_\Omega \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M \otimes \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \to 0, \quad (6.30)$$

for any divergenceless φ as in (6.11).

By virtue of (6.21-6.22), $\{\mathbf{H}^\perp(\mathbf{z}_\varepsilon)\}_M = \nabla_x \Psi_\varepsilon$ and identity (6.26) can be rewritten

$$\begin{aligned} \varepsilon \partial_t d_\varepsilon + \Delta \Psi_\varepsilon = 0, \\ \varepsilon \partial_t \nabla \Psi_\varepsilon + p'(\overline{\varrho}) \nabla d_\varepsilon = \varepsilon \mathbf{a}_{\varepsilon,M}, \end{aligned} \quad (6.31)$$

where

$$\Psi_\varepsilon = \mathrm{i} \sum_{\{j>0|\Lambda_j \le M\}} \sum_{m=1}^{m_j} [\mathbf{z}_\varepsilon]_{j,m} \sqrt{\Lambda_j} \omega_{j,m}, \quad d_\varepsilon = \sum_{\{j>0|\Lambda_j \le M\}} \sum_{m=1}^{m_j} [\varrho_\varepsilon^{(1)}]_{j,m} \omega_{j,m}. \quad (6.32)$$

With this notation, and recalling that d_ε is bounded in $W^{1,\infty}(0,T;C^k(\overline{\Omega}))$ and Ψ_ε in $L^\infty(0,T;C^k(\overline{\Omega})) \cap W^{1,2}(0,T;C^k(\overline{\Omega}))$, $k \in N$, as a consequence of (6.23), (6.25), we can rewrite (6.30) as the following chain of identities:

$$\int_0^T \int_\Omega \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M \otimes \Big\{\mathbf{H}^\perp\Big(\mathbf{z}_\varepsilon\Big)\Big\}_M : \nabla_x \varphi \mathrm{d}x\mathrm{d}t = \sum_{j,k=1}^3 \int_0^T \int_\Omega \partial_k \Psi_\varepsilon \partial_j \Psi_\varepsilon \partial_j \varphi_k \mathrm{d}x\mathrm{d}t$$

$$= \tfrac{1}{2} \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k |\nabla \Psi_\varepsilon|^2 \varphi_k \mathrm{d}x\mathrm{d}t + \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k \Psi_\varepsilon \Delta \Psi_\varepsilon \varphi_k \mathrm{d}x\mathrm{d}t$$

$$= \tfrac{1}{2} \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k |\nabla \Psi_\varepsilon|^2 \varphi_k \mathrm{d}x\mathrm{d}t - \varepsilon \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k \Psi_\varepsilon \partial_t d_\varepsilon \varphi_k \mathrm{d}x\mathrm{d}t$$

$$= \tfrac{1}{2} \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k |\nabla \Psi_\varepsilon|^2 \varphi_k \mathrm{d}x\mathrm{d}t - \varepsilon \sum_{k=1}^3 \int_0^T \int_\Omega \partial_t \Big(\partial_k \Psi_\varepsilon d_\varepsilon\Big) \varphi_k \mathrm{d}x\mathrm{d}t$$

$$+ \varepsilon \sum_{k=1}^3 \int_0^T \int_\Omega \partial_t (\partial_k \Psi_\varepsilon) d_\varepsilon \varphi_k \mathrm{d}x\mathrm{d}t =$$

$$= \tfrac{1}{2} \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k |\nabla \Psi_\varepsilon|^2 \varphi_k \mathrm{d}x\mathrm{d}t + \varepsilon \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k \Psi_\varepsilon d_\varepsilon \partial_t \varphi_k \mathrm{d}x\mathrm{d}t$$

$$- \tfrac{1}{2} p'(\overline{\varrho}) \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k |d_\varepsilon|^2 \varphi_k \mathrm{d}x\mathrm{d}t + \varepsilon \int_0^T \int_\Omega d_\varepsilon \mathbf{a}_{\varepsilon,M} \cdot \varphi \mathrm{d}x\mathrm{d}t$$

$$= \varepsilon \sum_{k=1}^3 \int_0^T \int_\Omega \partial_k \Psi_\varepsilon d_\varepsilon \partial_t \varphi_k + \varepsilon \int_0^T \int_\Omega d_\varepsilon \mathbf{a}_{\varepsilon,M} \cdot \varphi \mathrm{d}x\mathrm{d}t, \quad (6.33)$$

where we have used several times equations (6.31) and the properties (6.11) of φ. By virtue of (6.27) the right hand side of this chain of identities tends to 0 as $\varepsilon \to 0$. Proof of the limit (6.13) and of the part (i) of Lemma 5.1 is complete.

6.4 Incompressible energy inequality

We can rewrite (3.1) to the form

$$\begin{aligned}&\int_0^T \eta'(s)\int_\Omega \frac{1}{2}\varrho_\varepsilon \mathbf{u}_\varepsilon^2 \mathrm{d}x\mathrm{d}s+\int_0^T \eta'(s)\int_0^s\int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon):\nabla \mathbf{u}_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}s\\ \le&\int_0^T \eta'(s)\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon \mathbf{u}_{\varepsilon,0}^2+\frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\mathrm{d}s+\int_0^T \eta'(s)\int_0^s\int_\Omega \varrho_\varepsilon \mathbf{f}\cdot\mathbf{u}_\varepsilon \mathrm{d}x\mathrm{d}t\mathrm{d}s,\end{aligned}\tag{6.34}$$

where $\eta \in C_c^\infty([0,T))$, $\eta\le 0$, $\eta'\ge 0$. Due to (6.1–6.2) and (3.11),

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\mathbf{U}\quad \text{weakly in } L^\infty(0,T;L^2(\Omega)).\tag{6.35}$$

Letting $\varepsilon\to 0+$ in (6.34), taking advantage of (6.2), (6.35), the lower weak continuity at the left hand side and (6.3) at the right hand side, we get

$$\begin{aligned}&\int_0^T \eta'\int_\Omega \frac{1}{2}\overline{\varrho}\mathbf{U}^2\mathrm{d}x\mathrm{d}s+\int_0^T \eta'(s)\int_0^s\int_\Omega \mathbb{S}(\nabla_x \mathbf{U}):\nabla \mathbf{U}\mathrm{d}x\mathrm{d}t\mathrm{d}s\\ \le& \liminf_{\varepsilon\to 0+}\Big[\int_0^\infty \eta'\int_\Omega \Big(\frac{1}{2}\varrho_\varepsilon \mathbf{u}_{\varepsilon,0}^2+\frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\mathrm{d}s\Big]+\int_0^T\eta'\int_0^s\int_\Omega \overline{\varrho}\mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t\mathrm{d}s.\end{aligned}\tag{6.36}$$

By choosing properly η's we deduce by standard arguments (5.6).

6.5 Reconstruction of pressure in the non steady case

In this Section we complete the proof of Lemma 5.1 by reconstructing the pressure. The reconstruction of pressure in the non steady case will be based on the *maximal $L^p - L^p$ regularity to the non stationary Stokes system*

$$\begin{aligned}\partial_t\mathbf{U}+\nabla\Pi=\mu\mathrm{div}(\nabla_x\mathbf{U}+\nabla_x^\perp\mathbf{U})+\mathbf{F}\ \text{a. e. in } (0,T)\times\Omega\\ \mathrm{div}\mathbf{U}=0,\ \text{a. e. in } (0,T)\times\Omega\end{aligned}\tag{6.37}$$

endowed with the initial conditions

$$\mathbf{U}(0,x)=\mathbf{U}_0(x),\ \ x\in\Omega\tag{6.38}$$

and boundary conditions

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega}=0,\ (\nabla_x\mathbf{U}+\nabla_x^\perp\mathbf{U})\mathbf{n}\times\mathbf{n}|_{\partial\Omega}=0\ \text{a.e. in } (0,T)\tag{6.39}$$

in the sense of traces on $\partial\Omega$. The theorem reads:

Lemma 6.1. *Let $\Omega\subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu\in(0,1)$, $1<p<\infty$, $\mu>0$. Suppose that*

$$\mathbf{F}\in L^p((0,T)\times(\Omega;R^3)),\ \mathbf{U}_0\in W^{2-\frac{2}{p},p}(\Omega;R^3),\ \mathrm{div}\mathbf{U}_0=0,$$
$$\mathbf{U}_0\cdot\mathbf{n}|_{\partial\Omega}=0\ \textit{if}\ 1-\frac{3}{p}<0,$$

where $W^{2-\frac{2}{p},p}(\Omega;R^3)$ denotes the Sobolev-Slobodeckii space.

Then the problem (6.37–6.39) admits a solution $(\Pi,\mathbf{U})$, unique in the class

$$\mathbf{U}\in L^p(0,T;W^{2,p}(\Omega;R^3)),\ \partial_t\mathbf{U}\in L^p(0,T;L^p(\Omega;R^3)),$$

$$\mathbf{U}\in C([0,T];W^{2-\frac{2}{p},p}(\Omega;R^3)),\ \Pi\in L^p(0,T;W^{1,p}(\Omega)),\ \int_\Omega\Pi\mathrm{d}x=0.$$

Moreover, there exists a positive constant $c = c(p, q, \Omega, T, \overline{\mu})$ *such that*

$$
\begin{aligned}
&\|\mathbf{U}(t)\|_{W^{2-\frac{2}{p},p}(\Omega;R^3)} + \|\partial_t U\|_{L^p(0,T;L^p(\Omega;R^3))} \\
&+\|\mathrm{div}(\nabla_x \mathbf{U} + \nabla_x^{\perp}\mathbf{U})\|_{L^p(0,T;L^p(\Omega;R^3))} + \|\nabla\Pi\|_{L^p((0,T)\times\Omega;R^3)} \\
&\leq c(\|\mathbf{F}\|_{L^p((0,T)\times\Omega;R^3)} + \|\mathbf{U}_0\|_{W^{2-\frac{2}{p},p}(\Omega;R^3)}),
\end{aligned}
\tag{6.40}
$$

for any $t \in [0, T]$.

The original formulation and proof of this result in is due to Solonnikov [45]. For the more general $L^p - L^q$-versions see e.g. Shibata, Shimizu [44] or Saal, [40].

Remark: *The reader shall notice that the bounds for* $\mathrm{div}(\nabla_x \mathbf{U} + \nabla_x^{\perp}\mathbf{U})$ *and for* $\mathbf{U}$ *in (6.40) imply in addition*

$$
\|\mathbf{U}\|_{L^p(0,T;W^{2,p}(\Omega;R^3))} \leq c(\|\mathbf{F}\|_{L^p((0,T)\times\Omega;R^3)} + \|\mathbf{U}_0\|_{W^{2-\frac{2}{p},p}(\Omega;R^3)}) \tag{6.41}
$$

via a standard uniqueness Agmon, Douglis, Nirenberg argument and the L^p *version of the Korn inequality.*

Coming back to the proof, we rewrite identity (5.4) in the form

$$
\begin{aligned}
&\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{U}\cdot\partial_t\varphi + \mu\mathbf{U}\cdot\mathrm{div}(\nabla_x\varphi + \nabla_x^{\perp}\varphi)\Big)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \mathbf{G}\cdot\varphi\,\mathrm{d}x\mathrm{d}t - \int_\Omega \overline{\varrho}\mathbf{H}(\mathbf{U}_0)\cdot\varphi(0,x)\,\mathrm{d}x,
\end{aligned}
\tag{6.42}
$$

where

$$
\mathbf{G} = \overline{\varrho}\mathbf{U}\cdot\nabla_x\mathbf{U} - \overline{\varrho}\mathbf{f}
$$

and

$$
\begin{aligned}
&\varphi \in C_c^\infty([0,T)\times\overline{\Omega};R^3),\ \varphi\cdot\mathbf{n}|_{[0,T)\times\partial\Omega} = 0, \\
&(\nabla_x\varphi + \nabla_x^{\perp}\varphi)\mathbf{n}\times\mathbf{n}|_{[0,T)\times\partial\Omega} = 0,\ \mathrm{div}\varphi = 0.
\end{aligned}
\tag{6.43}
$$

We compare $\mathbf{U}$ with $\mathbf{V}$ the unique strong solution of the Stokes problem

$$
\begin{aligned}
&\overline{\varrho}\partial_t\mathbf{V} - \mu\mathrm{div}(\nabla_x\mathbf{V} + \nabla_x^{\perp}\mathbf{V}) + \nabla_x\Pi = -\mathbf{G}, \\
&\mathrm{div}\mathbf{V} = 0,\ \mathbf{V}(0,x) = [(\mathbf{H}(\mathbf{u}_0)](x), \\
&\mathbf{V}\cdot\mathbf{n}|_{\partial\Omega} = 0\ \big((\nabla_x\mathbf{V} + \nabla_x^T\mathbf{V})\mathbf{n}\big)\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e. in } (0,T).
\end{aligned}
\tag{6.44}
$$

Due to (5.1–5.2), $\mathbf{G} \in L^{\frac{5}{4}}((0,T)\times\Omega))$; whence Theorem 6.1 affirms, in particular, that

$$
\begin{aligned}
&\mathbf{V} \in C([0,T];L^{\frac{5}{4}}(\Omega;R^3)) \cap L^{\frac{5}{4}}(0,T;W^{2,\frac{5}{4}}(\Omega;R^3)), \\
&\partial_t\mathbf{V} \in L^{\frac{5}{4}}((0,T)\times\Omega;R^3)),\ \Pi \in L^{\frac{5}{4}}(0,T;L^{\frac{5}{4}}(\Omega)),\ \int_\Omega \Pi\mathrm{d}x = 0.
\end{aligned}
\tag{6.45}
$$

Subtracting (6.42) and (6.44) we obtain

$$
\int_0^T \int_\Omega \Big(\overline{\varrho}(\mathbf{U}-\mathbf{V})\cdot\partial_t\varphi + \mu(\mathbf{U}-\mathbf{V})\mathrm{div}(\nabla_x\varphi + \nabla_x^T\varphi)\Big)\,\mathrm{d}x\mathrm{d}t = 0 \tag{6.46}
$$

with any φ such that

$$\mathrm{div}\varphi = 0,$$

$$\varphi \in \cap_{r\in(1,\infty)} L^r(0,T; W^{2,r}(\Omega;R^3)),\ \partial_t\varphi \in \cap_{r\in(1,\infty)} L^r((0,T)\times\Omega;R^3),$$

$$\varphi(T,x) = 0,\ x\in\Omega, \tag{6.47}$$

$$\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\nabla_x\varphi + \nabla_x^T\varphi)\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e.in } (0,T),$$

where the last two identities are satisfied in the sense of traces. In order to enlarge the set of test functions from (6.43) to (6.47), we have employed (5.1–5.3), (6.45) and density argument.

In view of Theorem 6.1 we can use in (6.46) any test function φ, $\varphi(t,x) = \phi(T-t,x)$, where (p,ϕ) solves the Stokes problem

$$\overline{\varrho}\partial_t\phi - \overline{\mu}\mathrm{div}(\nabla_x\phi + \nabla_x^T\phi) + \nabla_x p = \mathbf{F},\ \mathrm{div}\phi = 0,\ \text{a.e. in } (0,T)\times\Omega,$$

$$\phi(0,x) = 0,\ x\in\Omega,$$

$$\phi\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\nabla_x\phi + \nabla_x^T\phi)\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0 \text{ a.e.in } (0,T),$$

where $\mathbf{F}\in C_c^\infty((0,T)\times\Omega;R^3)$. We thus get

$$\int_0^T\int_\Omega (\mathbf{U}-\mathbf{V})\cdot\mathbf{F}\,\mathrm{d}x\mathrm{d}t = 0 \text{ for all } \mathbf{F}\in C_c^\infty((0,T)\times\Omega;R^3),$$

where we have used the fact that $\int_0^T\int_\Omega \mathbf{U}\cdot\nabla_x p\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \mathbf{V}\cdot\nabla_x p\,\mathrm{d}x\mathrm{d}t = 0$. Therefore $\mathbf{U}=\mathbf{V}$. This completes the proof of identity (5.7) as well as of the whole Lemma 5.1.

6.6 Conclusion

Combining the "fast" time solutions constructed in Lemma 4.1 with the "real" time solutions of Lemma 5.1 finishes the proof of Theorem 2.1.

7 Proof of Lemma 5.2 and Theorem 2.2

7.1 Prodi-Serrin conditions

Sequences ϱ_ε, $\mathbf{u}_\varepsilon$, $r_\varepsilon^{(1)}$, $\mathbf{q}_\varepsilon$ satisfy assumptions of Theorem 2.1. They therefore admit limits $\overline{\varrho}$, $\mathbf{U}$, $r^{(1)}$, $\mathbf{q}$ in the sense (2.41–2.44); $r^{(1)}$, $\mathbf{q}$ satisfy equations (4.2) and (4.3) of Lemma 4.1, while $\mathbf{U}$ verifies equation (5.4) and inequality (5.6), where

$$\liminf_{\varepsilon\to 0+}\Big[\int_\Omega\Big(\frac{1}{2}\varrho_\varepsilon\mathbf{u}_{\varepsilon,0}^2 + \frac{1}{\varepsilon^2}\mathcal{H}(\varrho_{\varepsilon,0})\Big)\mathrm{d}x\Big] + \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t = \int\frac{1}{2}\overline{\varrho}|\mathbf{u}_0|^2\,\mathrm{d}x.$$

Therefore, $\mathbf{U}$ is a Leray-Hopf weak solution of problem (2.21), (2.7) with initial data $\mathbf{U}_0 = \mathbf{u}_0$.

At this place, we recall the celebrated Prodi-Serrin uniqueness conditions for the Navier-Stokes equations (cf. Serrin [43]).

Lemma 7.1. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two weak solutions of Navier-Stokes equations (2.21) with boundary conditions (2.7) and the same initial conditions. Let* $\mathbf{v}$ *be a Leray-Hopf weak solution and*

$$\mathbf{w} \in L^r(0,T; L^s(\Omega; R^3)), \text{ for some } r, s \text{ such that } \tfrac{3}{s} + \tfrac{2}{r} = 1,\ s \in (3,\infty).$$

Then $\mathbf{v} = \mathbf{w}$.

We apply this lemma with $\mathbf{v} = \mathbf{U}$ and $\mathbf{w} = \mathbf{u}_0$. We already know that $\mathbf{U}$ is a Leray-Hopf weak solution with initial conditions $\mathbf{u}_0$. Recall that $\mathbf{u}_0$ verifies $(2.30 - 2.31)$ meaning that $\mathbf{w}$ is a weak solution of the nonstationary problem with initial data $\mathbf{w}(0) = \mathbf{u}_0$. Moreover, being time independent, $\mathbf{w}$ belongs to $L^\infty(0,T; L^6(\Omega; R^3))$ and verifies therefore the Prodi-Serrin conditions. Lemma 7.1 yields $\mathbf{v} = \mathbf{w}$ or equivalently $\mathbf{U} = \mathbf{u}_0$.

7.2 Reconstruction of the pressure in the steady case

At this place make a pause to introduce natural spaces useful for the investigation of pressure in the Navier-Stokes equations (2.21) with boundary conditions (2.7), and to list some of their properties needed in the sequel.

For $1 \le p < \infty$, we set

$$W^{1,p}_{\mathbf{n}}(\Omega; R^3) = \{\mathbf{z} \in W^{1,p}(\Omega; R^3) \,|\, \mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

a closed subspace of $W^{1,p}(\Omega; R^3)$ and by

$$\dot{W}^{1,p}_{\mathbf{n}}(\Omega; R^3) = \{\mathbf{z} \in W^{1,p}(\Omega; R^3) \,|\, \mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathrm{div}\mathbf{z} = 0\}.$$

The set

$$C^{2,\nu}_{c,\mathbf{n}} = \{\mathbf{z} \in C^{2,\nu}(\overline{\Omega}; R^3) \,|\, \mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$$

is dense in $W^{1,p}_{\mathbf{n}}(\Omega; R^3)$ and the set

$$\dot{C}^{2,\nu}_{c,\mathbf{n}} = \{\mathbf{z} \in C^{2,\nu}(\overline{\Omega}; R^3) \,|\, \mathbf{z}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathrm{div}\mathbf{z} = 0\}$$

is dense in $\dot{W}^{1,p}_{\mathbf{n}}(\Omega; R^3)$, see e.g. [18, Section 10.7].

The functional

$$\mathcal{F} : W^{1,2}_{\mathbf{n}}(\Omega) \to R \quad \text{defined by}$$

$$\mathcal{F}_\tau(\varphi) = \int_\Omega \overline{\varrho}(\mathbf{u}(\tau) - \mathbf{u}_0)\cdot\varphi \mathrm{d}x + \tag{7.1}$$

$$\int_0^\tau \int_\Omega \Big[\Big(\mu(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u}) - \overline{\varrho}(\mathbf{u}\otimes\mathbf{u})\Big) : \nabla_x\varphi - \overline{\varrho}\mathbf{f}\cdot\varphi\Big]\, \mathrm{d}x\mathrm{d}t$$

is a continuous linear functional on $W^{1,2}_{\mathbf{n}}(\Omega; R^3)$ vanishing on $\dot{W}^{1,2}_{\mathbf{n}}(\Omega; R^3)$.

Now, we shall recall several facts from functional analysis of (unbounded) linear operators. Let $A : X \to Y$, where X and Y are Banach spaces, be a linear (unbounded) closed and densely defined operator. Denote $A^* : Y^* \to X^*$ its adjoint, where X^* and Y^* are duals to X and Y, respectively. It is well known, that $\ker(A) = (\mathcal{R}(A^*))^\perp$, $\ker(A^*) = (\mathcal{R}(A))^\perp$, and also $(\mathcal{R}(A)) = (\ker(A*))^\perp$ provided $\mathcal{R}(A)$ is closed (in Y), $(\mathcal{R}(A^*)) = (\ker(A))^\perp$ provided $\mathcal{R}(A^*)$ is closed (in X^*), see eg Brezis [7]. Thus, if a continuous linear functional $\mathcal{F} : X \to R$ vanishes on $\ker(A)$ then $\mathcal{F} \in (\ker(A))^\perp = \mathcal{R}(A^*)$, provided the last subspace is closed (in X^*).

Next, we mention some properties of the Bogovskii solution operator to the problem div$\mathbf{z} = g$, [5]. We shall need the following facts. For any bounded Lipschitz domain, there exists a linear operator $\mathcal{B}$ with the following properties:

- $$\mathcal{B} : \overline{L^p}(\Omega) := \{g \in L^p(\Omega) \mid \int_\Omega z \mathrm{d}x = 0\} \to W_0^{1,p}(\Omega, R^3), \quad \forall 1 < p < \infty; \tag{7.2}$$

- $$\mathrm{div}\mathcal{B}(g) = g, \quad \forall g \in \overline{L^p}(\Omega); \tag{7.3}$$

- $$\|\mathcal{B}(g)\|_{W^{1,p}(\Omega)} \le c(p, \Omega)\|g\|_{L^p(\Omega)}, \quad \forall g \in \overline{L^p}(\Omega); \tag{7.4}$$

- $$\text{If } g \in \overline{C_c^\infty}(\Omega) := \{g \in C_c^\infty(\Omega) \mid \textstyle\int_\Omega g dx = 0\} \text{ then } \mathcal{B}(g) \in C_c^\infty(\Omega). \tag{7.5}$$

Such operator can be constructed explicitly; we refer to Galdi [20] or to [38], Section 3.3 for more details and further properties.

The operator $A = \mathrm{div}$ from $X = W^{1,p}_{\mathbf{n}}(\Omega; R^3)$ to $Y = L^p(\Omega)$ with $\mathcal{D}(A) = W^{1,p}_{\mathbf{n}}(\Omega; R^3)$, $\ker(A) = \dot{W}^{1,p}_{\mathbf{n}}(\Omega; R^3)$, $\mathcal{R}(A) = \overline{L^p}(\Omega)$ is a continuous operator (hence $\mathcal{R}(A)$ is closed). (The surjectivity of A onto $\overline{L^p}(\Omega)$ follows form the property (7.3) of the Bogovskii operator.) Therefore, for all $\mathbf{v} \in W^{1,p}_{\mathbf{n}}(\Omega; R^3)$, $\psi \in C_c^\infty(\Omega)$ dense subset of $Y^* = L^{p'}(\Omega)$, we have

$$< \psi\, , A\mathbf{v} >_{Y^*,Y} = \int_\Omega \psi \mathrm{div}\mathbf{v} \mathrm{d}x = \int_\Omega \mathbf{v} \cdot \nabla_x \psi \mathrm{d}x =< A^*\psi\, , \mathbf{v} >_{X^*,X} . \tag{7.6}$$

The last identity implies that $A^* : L^{p'}(\Omega) \to (W^{1,p}_{\mathbf{n}}(\Omega; R^3))^*$ is continuous with $\mathcal{D}(A^*) = L^{p'}(\Omega)$ and a range $\mathcal{R}(A^*)$ closed in $(W^{1,p}_{\mathbf{n}}(\Omega; R^3))^*$.

We apply this result to the linear functional

$$\mathcal{F} : W^{1,2}_{\mathbf{n}}(\Omega) \to R,$$

$$\mathcal{F}(\varphi) = \int_\Omega \Big[\Big(\mu(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u}) - \overline{\varrho}\mathbf{u} \otimes \mathbf{u}\Big) : \nabla_x \varphi - \overline{\varrho}\mathbf{f} \cdot \varphi\Big] \mathrm{d}x.$$

and obtain existence of $\Pi \in L^2(\Omega)$, $\int_\Omega \Pi \mathrm{d}x = 0$ such that $\mathcal{F}(\varphi) = \int_\Omega \Pi \mathrm{div}\varphi \mathrm{d}x$. We thus have

Lemma 7.2. *Let $\mathbf{u}_0$ satisfy (2.30), (2.31). Then there exists a unique*

$$\Pi \in L^2(\Omega), \quad \int_\Omega \Pi \mathrm{d}x = 0$$

such that

$$\forall \varphi \in C_c^\infty((0,T) \times \overline{\Omega}), \quad \varphi \cdot \mathbf{n}|_{(0,T)\times\partial\Omega},$$

$$\int_0^T \int_\Omega \Big(\overline{\varrho}\mathbf{u}_0 \otimes \mathbf{u}_0 - \mu(\nabla_x \mathbf{u}_0 + \nabla_x^\perp \mathbf{u}_0)\Big) : \nabla_x \varphi \mathrm{d}x\mathrm{d}t \tag{7.7}$$

$$+ \int_0^T \int_\Omega \Pi \mathrm{div}\varphi \mathrm{d}x\mathrm{d}t = - \int_0^T \int_\Omega \overline{\varrho}\mathbf{f} \cdot \varphi \mathrm{d}x\mathrm{d}t.$$

Now, regarding (7.7) as a (steady) Stokes problem with the right hand side $-\overline{\varrho}\mathbf{u}_0 \cdot \nabla_x \mathbf{u}_0 \in L^{\frac{3}{2}}(\Omega; R^3)$, we may use the standard regularity results for the Stokes problem combined with the uniqueness (cf. e.g. Galdi [20]) to conclude that $\mathbf{u}_0$ must necessarily belong to $W^{2,\frac{3}{2}}(\Omega; R^3)$ and Π to $W^{1,\frac{3}{2}}(\Omega)$.

7.3 Conclusion

Now, we conclude the proof of Theorem 2.2 by replacing in (4.2) $r^{(1)}$ by $r^{(1)} - (p(\overline{\varrho}))^{-1}\Pi$ (Π being time independent, $\int_0^T \int_\Omega \Pi \partial_t \varphi \mathrm{d}x\mathrm{d}t = 0$) and by subtracting (7.7) from equation (4.3).

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