Numerical Analysis of Leray-Tikhonov Deconvolution Models of Fluid Motion

Carolina C. Manica* and Iuliana Stanculescu†

Abstract

This report develops and studies a new family of NSE-regularizations, Tikhonov Leray Regularization with Time Relaxation Models. This new family of turbulence models is based on a modification (consistent with the large scales) of Tikhonov-Lavrentiev regularization. With this approach, we obtain an approximation of the unfiltered solution by one filtering step. We introduce the modified Tikhonov deconvolution operator and study its mathematical properties. We also perform rigorous numerical analysis of a computational attractive algorithm for the considered family of models. Numerical experiments that support our theoretical results are presented.

1 Introduction

The Navier-Stokes equations (NSE), given by

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0\,, \qquad \text{in } \Omega\times(0,T) \tag{1.1}$$

is an exact model for the flow of a viscous, incompressible fluid, [15]. Their solution contains so much information that they become impractical for many problems within typical time and resource limitations. Various models and tools have been developed seeking to give a reasonable treatment of this richness of information. The key is to capture all the relevant information with less (computational) work than that involved in solving the NSE. One of the ideas is to use regularizations of (1.1). Leray, [29], proposed the following:

$$\mathbf{u}_t + \overline{\mathbf{u}}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ and } \nabla\cdot\mathbf{u} = 0\,, \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

where $\overline{\mathbf{u}} = G\mathbf{u}$ is a smooth/averaged velocity. He selected G to be the Gaussian filter associated with a length scale δ. He proved existence and uniqueness of strong solutions to (1.2) and showed that a subsequence $\mathbf{u}_{\delta_j}$ converges to a weak solution of the NSE as $\delta_j \to 0$. If that weak solution is a smooth, strong solution it is not difficult to prove additionally that $||\mathbf{u}_{NSE} - \mathbf{u}_{LerayModel}|| = O(\delta^2)$ using only $||\mathbf{u} - \overline{\mathbf{u}}|| = O(\delta^2)$.

Continuing his idea, new regularization models can be derived every time a suitable regularization operator is chosen. One modification is to replace the Gaussian filter by a differential filter, $\overline{\mathbf{u}} := (-\delta^2\Delta + 1)^{-1}\mathbf{u}$. Properties of the resulting Leray-α model (1.2) are derived by Geurts and Holm [22, 21] together with some tests in turbulent flow simulations and by Dunca [9] in a shape design problem. Experiments have shown that, due to its low accuracy, (1.2) with the above differential filter can have catastrophic error growth and not adequately conserve physically important integral invariants. They also indicate that the increase in accuracy resulting from using deconvolution (replacing $\overline{\mathbf{u}}$ by $D\overline{\mathbf{u}}$:= higher order approximation of $\mathbf{u}$) decreases error growth and improves conservation properties. The theory of the Leray-α model is developed in [6, 5, 25, 41].

†Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Brazil; email: carolina.manica@ufrgs.br, www.chasqueweb.ufrgs.br/ carolina.manica

†Farquhar College of Arts and Sciences, Nova Southeastern University, Ft. Lauderdale, FL, 33314, U.S.A.; email: is162@nova.edu, http://cs.ncs.nova.edu/iuliana; partially supported by NSF grant 0508260

The deconvolution problem is central in image processing, [2] and many algorithms can be adapted to give possible better regularizations of the NSE. The van Cittert deconvolution algorithm, see [9, 31, 37] is one such example. It is time to explore other operators, which can lead to possibly more accurate models. Based on the theory of inverse problems, we study a **new**, modified Tikhonov regularization operator:

$$D_\mu(\overline{\mathbf{u}}) = \text{ approximation of } \mathbf{u}.$$

The operator D_μ, $0 \le \mu \le 1$, is a Tikhonov-Lavrentiev regularization of the formal filter inverse *adapted to turbulence*, i.e. designed to accurately capture the large scales of a flow, while modeling the small (or under resolved) scales (and truncating). The case $\mu = 0$, would result in $D_\mu(\overline{\mathbf{u}}) = \mathbf{u}$ (no regularization), whereas $\mu = 1$, leads to $D_\mu(\overline{\mathbf{u}}) = \overline{\mathbf{u}}$ (the Leray/Leray-α model (1.2)). For smooth velocity fields $\mathbf{u}$, we have $D_\mu(\overline{\mathbf{u}}) = \mathbf{u} + O(\mu\,\delta^2)$.

Replacing $\overline{\mathbf{u}}$ by $D_\mu(\overline{\mathbf{u}})$ in (1.2), we study the following Leray-Tikhonov Model with Time Relaxation:

$$\begin{aligned} \mathbf{u}_t + D_\mu(\overline{\mathbf{u}}) \cdot \nabla\, \mathbf{u} - \nu \triangle \mathbf{u} + \nabla q + \chi(\mathbf{u} - D_\mu(\overline{\mathbf{u}})) &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0. \end{aligned} \tag{1.3}$$

The term $\chi(\mathbf{u} - D_\mu(\overline{\mathbf{u}}))$ is included to damp unresolved fluctuations over time, where $\chi \ge 0$ is the time relaxation parameter. It is a generalized fluctuation term (often included in Approximate Deconvolution Models of turbulence to damp marginally unresolved scales, see [37] and [33]).

Our goal is to perform a convergence analysis of a discretization of (1.3), when $\mu, \delta, h \to 0$. The discretization consists of the finite element method in space, combined with the Crank-Nicolson algorithm in time. The notation and definitions necessary for the scheme and for the numerical analysis are in Section 2, where we also give a detailed mathematical theory of the Modified Tikhonov deconvolution operator. Section 3 develops the theory for the scheme, showing stability, existence of solutions, and analysis of convergence. Numerical experiments are presented in Section 4, followed by conclusions.

2 Notation and Preliminaries

Throughout this report we use standard notation for Lebesgue and Sobolev spaces and their norms. Let $\|\cdot\|$ and $(\cdot,\cdot)$ be the L^2 norm and inner product respectively. The $L^p(\Omega)$ norm and the Sobolev $W_p^k(\Omega)$ norm are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$. The semi-norm in $W_p^k(\Omega)$ is denoted by $|\cdot|_{W_p^k}$. The space H^k represents the Sobolev space $W_2^k(\Omega)$ and $\|\cdot\|_k$ denotes the norm in H^k. For time dependent functions $\mathbf{v}(\mathbf{x},t)$, with $t \in (0,T)$, we define the norm

$$\|\mathbf{v}(\mathbf{x},t)\|_{m,k} = \begin{cases} \left(\int_0^T \|\mathbf{v}(\cdot,t)\|_k^m \, dt\right)^{1/m}, & \text{if } 1 \le m < \infty \\ \operatorname{ess\,sup}_{0<t<T} \|\mathbf{v}(\cdot,t)\|_k\,, & \text{if } m = \infty. \end{cases}$$

The flow domain Ω is a regular, bounded, polyhedral domain in $\mathbb{R}^n$. The pressure and velocity spaces are

$$\begin{aligned} Q &= L_0^2(\Omega), \\ X &= H_0^1(\Omega). \end{aligned}$$

The dual space of X is $X^\star$ and the corresponding norm is $\|\cdot\|_\star$. For the variational formulation we define the space of divergence free functions

$$V := \{\mathbf{v} \in X,\ (\nabla \cdot \mathbf{v}, q) = 0 \ \ \forall q \in Q\}\,.$$

The velocity-pressure finite element spaces $X_h \subset X$, $Q_h \subset Q$ are assumed to be conforming and satisfy the discrete inf-sup condition. Taylor-Hood elements are one common example of such a choice for (X_h, Q_h) (see [18] and [17]). The discretely divergence free subspace of X_h is defined as

$$V_h = \{\mathbf{v}_h \in X_h,\ (\nabla \cdot \mathbf{v}_h, q_h) = 0 \ \ \forall q_h \in Q_h\}.$$

For the convective term, we consider the following trilinear form.

Definition 2.1. [*The skew symmetric operator b^**] *The skew-symmetric trilinear form $b^* : X \times X \times X \to \mathbb{R}$ is defined as*

$$b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}). \tag{2.1}$$

Lemma 2.1. *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ such that $\mathbf{v} \in L^\infty(\Omega)$ and $\nabla \mathbf{v} \in L^\infty(\Omega)$, where indicated. The trilinear term $b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})$ can be bounded in the following ways*

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le \frac{1}{2}\left(\|\mathbf{u}\| \|\nabla \mathbf{v}\|_\infty \|\mathbf{w}\| + \|\mathbf{u}\| \|\mathbf{v}\|_\infty \|\nabla \mathbf{w}\|\right) \tag{2.2}$$

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C_0(\Omega) \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\| \tag{2.3}$$

$$|b^*(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C_0(\Omega) \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|. \tag{2.4}$$

Proof. For the proof see [32]. □

We also use the following approximation properties, see [4]:

$$\begin{aligned}
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\| &\le Ch^{k+1} \|\mathbf{u}\|_{k+1}, \ \ \mathbf{u} \in H^{k+1}(\Omega)^d, \\
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_1 &\le Ch^{k} \|\mathbf{u}\|_{k+1}, \ \ \mathbf{u} \in H^{k+1}(\Omega)^d, \\
\inf_{r \in Q_h} \|p - r\| &\le Ch^{s+1} \|p\|_{s+1}, \ \ p \in H^{s+1}(\Omega).
\end{aligned} \tag{2.5}$$

We often use some well known inequalities:

- Cauchy-Schwarz inequality: $|(f, g)| \le \|f\| \|g\|$, for all f and $g \in L^2(\Omega)$,
- Young's inequality: $ab \le \frac{\epsilon}{p} a^p + \frac{\epsilon^{-q/p}}{q} b^q$, where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\epsilon > 0$ and $a, b \ge 0$,
- Poincaré-Friedrich's inequality: $\|\mathbf{v}\| \le C_{PF} \|\nabla \mathbf{v}\|$, for all $\mathbf{v} \in X$.

2.1 Differential Filters

In our analysis we use differential filters. Differential filters are well-established in LES, starting with the work of Germano [20] and continuing with [19], [35]. They have many connections to regularization processes such as the Yoshida regularization of semigroups and the very interesting work of Foias, Holm, Titi [13] (and others) on Lagrange averaging of the Navier-Stokes equations. We also define the discrete differential filter, following Manica and Kaya Merdan [34].

Definition 2.2. [*Continuous differential filter*] *For $\boldsymbol{\phi} \in L^2(\Omega)$ and $\delta > 0$ fixed, denote the filtering operation on $\boldsymbol{\phi}$ by $\overline{\boldsymbol{\phi}}$, where $\overline{\boldsymbol{\phi}}$ is the unique solution (in X) of*

$$-\delta^2 \Delta \overline{\boldsymbol{\phi}} + \overline{\boldsymbol{\phi}} = \boldsymbol{\phi}. \tag{2.6}$$

Set $A := -\delta^2 \Delta + I$, thus $\overline{\boldsymbol{\phi}} := A^{-1} \boldsymbol{\phi}$.

Following Manica and Kaya Merdan [34], at this point, we define the discrete counterpart of the above differential filter.

Definition 2.3. [*Discrete differential filter*] *Given $\mathbf{v} \in L^2(\Omega)$, for a given filtering radius $\delta > 0$, $\overline{\mathbf{v}}^h = A_h^{-1} \mathbf{v}$ is the unique solution in X_h of*

$$\delta^2 (\nabla \overline{\mathbf{v}}^h, \nabla \boldsymbol{\chi}) + (\overline{\mathbf{v}}^h, \boldsymbol{\chi}) = (\mathbf{v}, \boldsymbol{\chi}) \ \ \forall \boldsymbol{\chi} \in X_h. \tag{2.7}$$

Definition 2.4. *The L^2 projection $\Pi_h : L^2(\Omega) \to X_h$ and discrete Laplacian operator $\Delta_h : X \to X_h$ are defined by*

$$(\Pi_h \mathbf{v} - \mathbf{v}, \boldsymbol{\chi}) = 0\,, \qquad (\Delta_h \mathbf{v}, \boldsymbol{\chi}) = -(\nabla \mathbf{v}, \nabla \boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in X_h. \tag{2.8}$$

Remark 2.1. *The extension from X_h to X in the definition of Δ_h is the extension by zero on the orthogonal complement of X_h w.r.t. $(\nabla\cdot, \nabla\cdot)$. With Δ_h, we can rewrite $\overline{\mathbf{v}}^h = (-\delta^2\Delta_h + \Pi_h)^{-1}\mathbf{v}$ and $A_h = (-\delta^2\Delta_h + \Pi_h)$.*

2.2 Why a modification of Tikhonov-Lavrentiev is needed?

The deconvolution problem is central in image processing, see [2]. The basic problem in deconvolution is: given $\overline{\mathbf{u}}$ solve for $\mathbf{u}$ the following equation

$$G\mathbf{u} = \overline{\mathbf{u}}, \tag{2.9}$$

where G is not invertible and thus exact deconvolution is typically ill-posed.

Throughout the years, many approaches have been used to address and answer questions concerning ill-posed problems. Some were based on constrained least-square solutions, others determined the smoothest approximate solution compatible with the data within a given noise level. Tikhonov proposed a general approach, called *regularization*, which is a unification of the these two methods, [40]. The basic idea of regularization consists of constructing a family of approximate solutions depending on a positive *regularization parameter*, μ. Using Tikhonov regularization, we compute a family of approximate solutions to the ill-posed deconvolution problem (2.9) as

$$\mathbf{u}_\mu = \arg\min_{\mathbf{u}} \left[||G\mathbf{u} - \overline{\mathbf{u}}||^2 + \mu ||\mathbf{u}||^2 \right].$$

The main property of regularization is that, for a non-zero value of μ, one can obtain an optimal approximation of the exact solution of the problem (2.9). Lavrentiev adapted Tikhonov's idea to symmetric positive definite (SPD) operators G. In this case the regularization process is called Tikhonov-Lavrentiev and leads to a family of approximate solutions given by

$$\mathbf{u}_\mu = \arg\min_{\mathbf{u}} \left[\frac{1}{2}(G\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{u}) + \frac{\mu}{2}(\mathbf{u}, \mathbf{u}) \right].$$

Remark 2.2. *The Tikhonov-Lavrentiev regularization process gives an approximate solution to the deconvolution problem as follows: let $\mu > 0$ and let G be SPD. Then,*

$$\mathbf{u}_\mu = (G + \mu I)^{-1}\overline{\mathbf{u}} \tag{2.10}$$

solves (2.9) approximately as $\mu \to 0$.

Lemma 2.2. [*Error in Tikhonov-Lavrentiev approximation*] *With the differential filter (2.6), we have*

$$\|\mathbf{u} - \mathbf{u}_\mu\| \le \mu(\delta^2 \, \|\triangle \mathbf{u}\| + \|\mathbf{u}\|). \tag{2.11}$$

Proof. Note that $\mathbf{u}, \triangle\mathbf{u} \in L^2(\Omega)$. Also, $G = (-\delta^2 \triangle + I)^{-1}$ and $A = -\delta^2 \triangle + I$. Now,

$$\begin{aligned} \mathbf{u} - \mathbf{u}_\mu &= \mathbf{u} - (G + \mu\, I)^{-1}\, G\mathbf{u} = \mathbf{u} - (A^{-1} + \mu\, I)^{-1}\, A^{-1}\mathbf{u} \\ &= \mathbf{u} - (A(A^{-1} + \mu\, I))^{-1}\mathbf{u} = \mathbf{u} - (I + \mu\, A)^{-1}\mathbf{u} \\ &= (I + \mu\, A)^{-1}\, ((I + \mu\, A) - I)\mathbf{u} = \mu\, (I + \mu\, A)^{-1}\, A\mathbf{u}\,. \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\mu\| &= \mu\, \|(I + \mu\, A)^{-1}\, A\mathbf{u}\| \\ &\le \mu\, \|(I + \mu\, A)^{-1}\|\, \|A\mathbf{u}\|\,. \end{aligned}$$

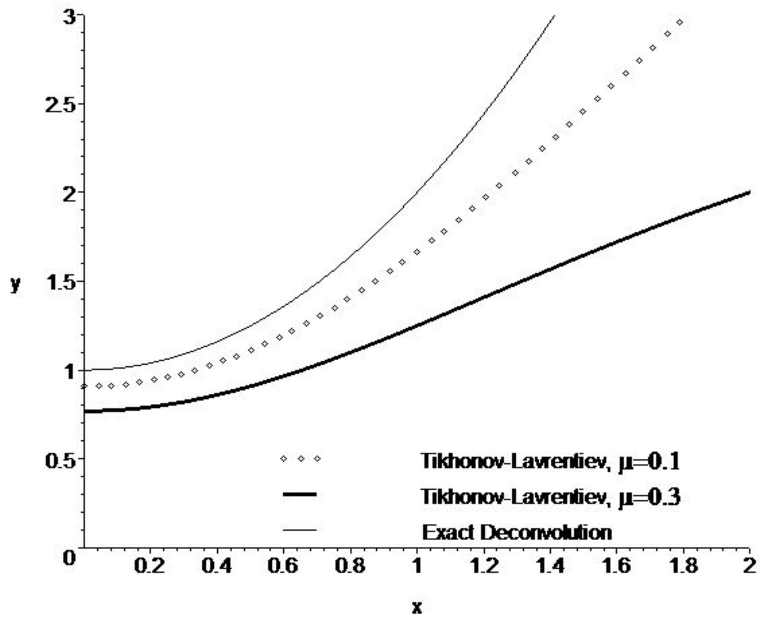

Figure 1: Tikhonov-Lavrentiev and Exact Deconvolution

All of our operators are self-adjoint. The spectrum of $I+\mu\, A$ is contained in the interval $[1+\mu,\infty)$; and so the spectrum of $(I+\mu\, A)^{-1}$ is contained in the interval $(0,(1+\mu)^{-1}]$. Hence,

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_\mu\| &\leq \frac{\mu}{1+\mu}\,\|A\mathbf{u}\| \leq \mu\,\|(-\delta^2\,\triangle + I)\mathbf{u}\| \\ &\leq \mu\,(\delta^2\,\|\triangle\mathbf{u}\| + \|\mathbf{u}\|)\ .\end{aligned}$$

□

With the differential filter (2.6), the transfer function of exact deconvolution is $\widehat{G}(k) = 1/(1+\delta^2k^2)$, where k is the wave number. Then, the transfer function of $(G+\mu I)^{-1}$ is

$$(\widehat{G+\mu I})^{-1}(k) = \frac{1}{\mu+\widehat{G}(k)}. \tag{2.12}$$

This (for $\mu = 0.1$ and 0.3) and exact deconvolution are plotted in Figure 1. It is clear that, for large scales (small wave number k), the operator $(G+\mu I)^{-1}$ is not a good approximation of the inverse of G. Thus, it is sensible to consider a modified Tikhonov-Lavrentiev operator, which is more consistent for large scales. With this, we obtain a new approximation of $\mathbf{u}$ and improve (2.11).

Definition 2.5. [*Modified Tikhonov Approximate Deconvolution Operator*] *Let G be a symmetric positive-definite operator and let $0<\mu<1$. Given $\overline{\mathbf{u}}$, an approximate solution to the deconvolution problem (2.9) is given by*

$$\mathbf{u}_\mu = ((1-\mu)G+\mu I)^{-1}\overline{\mathbf{u}}. \tag{2.13}$$

Set $D_\mu = ((1-\mu)G+\mu I)^{-1}$. The operator D_μ is the modified Tikhonov deconvolution operator.

Remark 2.3. *With the differential filter (2.6), the modified Tikhonov deconvolution operator can be defined variationally as: given $\boldsymbol{\phi}\in X$, for a given filtering radius $\delta>0$ and $0<\mu<1$, $\boldsymbol{\psi} = D_\mu\overline{\boldsymbol{\phi}}$ is the unique solution in X of the problem*

$$\mu\delta^2(\nabla\boldsymbol{\psi},\nabla\boldsymbol{\chi}) + (\boldsymbol{\psi},\boldsymbol{\chi}) = (\boldsymbol{\phi},\boldsymbol{\chi}), \quad \forall\boldsymbol{\chi}\in X, \tag{2.14}$$

where $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$ satisfy $\overline{\boldsymbol{\phi}} = A^{-1}\boldsymbol{\phi}$ in X, i.e. $\delta^2(\nabla\overline{\boldsymbol{\phi}},\nabla\boldsymbol{\chi}) + (\overline{\boldsymbol{\phi}},\boldsymbol{\chi}) = (\boldsymbol{\phi},\boldsymbol{\chi})$.

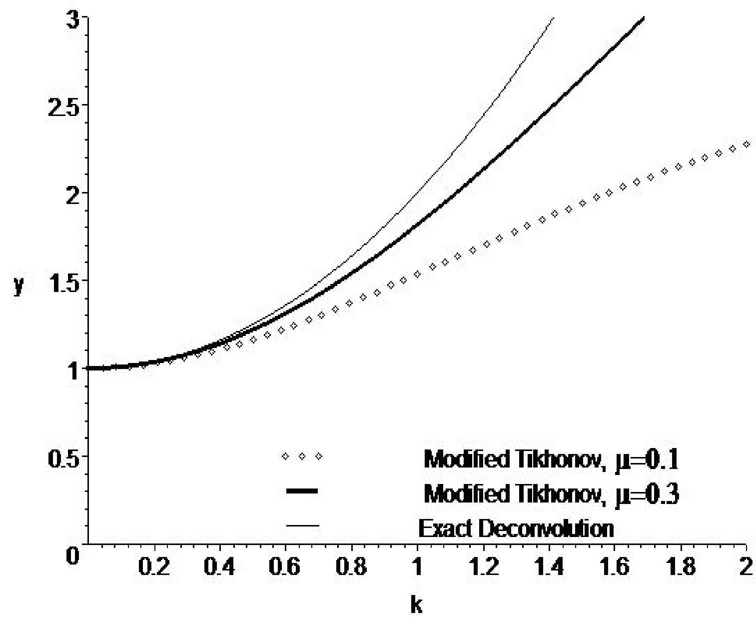

Figure 2: Modified Tikhonov and Exact Deconvolution

With the differential filter (2.6), the transfer function of D_μ is

$$\widehat{D}_\mu(k) = \frac{1}{\mu + (1-\mu)\widehat{G}(k)}. \tag{2.15}$$

Exact deconvolution and $\widehat{D}_\mu$ (for $\mu = 0.1$ and 0.3) are plotted in Figure 2. The figure reflects a high order contact of the graphs for wave numbers near 0. Thus, D_μ leads to a very accurate solution of the deconvolution problem.

2.3 Continuous and Discrete Modified Tikhonov Deconvolution Operator

In this section we analyze in more detail properties of the modified Tikhonov deconvolution operator. In particular, we prove that it is a bounded, self adjoint, and positive definite operator. We also introduce and study a discrete version of the modified Tikhonov operator.

Theorem 2.1. *Let G be SPD and $0 < \mu < 1$. Given $\overline{\mathbf{u}}$, the solution of (2.13) is given by the unique minimizer in $L^2(\Omega)$ of the functional*

$$F_\mu(\mathbf{u}) = \frac{1}{2}(G\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{u}) + \frac{\mu}{2}(\mathbf{u} - G\mathbf{u}, \mathbf{u}). \tag{2.16}$$

Proof. The unique minimizer in $L^2(\Omega)$ of the functional (2.16) is calculated as the minimum, when $t = 0$, of the function

$$F_\mu(\mathbf{u}+t\mathbf{w}) = \frac{1}{2}(G(\mathbf{u}+t\mathbf{w}), \mathbf{u}+t\mathbf{w}) - (\overline{\mathbf{u}}, \mathbf{u}+t\mathbf{w}) + \frac{\mu}{2}(\mathbf{u}+t\mathbf{w} - G(\mathbf{u}+t\mathbf{w}), \mathbf{u}+t\mathbf{w}), \ \ \forall \mathbf{w} \in L^2(\Omega). \tag{2.17}$$

Differentiating and setting $\frac{d}{dt}F_\mu(\mathbf{u}+t\mathbf{w})\big|_{t=0} = 0$ we obtain

$$((1-\mu)G\mathbf{u} + \mu\mathbf{u}, \mathbf{w}) = (\overline{\mathbf{u}}, \mathbf{w}), \ \ \forall \mathbf{w} \in L^2(\Omega). \tag{2.18}$$

Thus, if $\mathbf{u}$ is a solution of (2.18), then $\mathbf{u}$ is also a solution of (2.13). □

Proposition 2.1. *Let the averaging operator be the differential filter* $G\mathbf{u} := (-\delta^2\triangle + I)^{-1}\mathbf{u}$. *Let* $0 < \mu < 1$ *be fixed. The operator* $D_\mu : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ *is one-to-one and onto, bounded, self-adjoint and positive definite.*

Proof. We first recall that G is a linear, self-adjoint positive definite operator, with spectrum contained in $[0, 1]$. Thus the spectrum of $(1-\mu)\,G + \mu\,I$ is contained in $[\mu, 1]$, and consequently, the spectrum of $D_\mu = ((1-\mu)\,G + \mu\,I)^{-1}$ is a subset of the interval $[1, \mu^{-1}]$. Consequently, we have that D_μ is one-to-one and onto, bounded, self-adjoint and positive definite. □

Remark 2.4. *Both* $A^{-1} := (-\delta^2\triangle + I)^{-1}$, *given by Definition 2.2, and* D_μ *are linear combinations of the Laplace operator. Thus, they commute with the Laplace operator and with each other.*

Lemma 2.3. *With the differential filter given by Definition 2.2 and for smooth* $\mathbf{u}$ *we have*

$$\|D_\mu \triangle \overline{\mathbf{u}}\| \leq \|\triangle \mathbf{u}\|\,. \tag{2.19}$$

Proof. We have that

$$\begin{aligned} D_\mu\, G &= ((1-\mu)\,A^{-1} + \mu\,I)^{-1}\,A^{-1} \\ &= (A\,((1-\mu)\,A^{-1} + \mu\,I))^{-1} = ((1-\mu)\,I + \mu\,A))^{-1}\,. \end{aligned}$$

The spectrum of A is contained in $[1, \infty)$, and thus the spectrum of $(1-\mu)\,I + \mu\,A$ is a subset of $[(1-\mu)+\mu, \infty) = [1, \infty)$, too. Hence, the spectrum of the self-adjoint operator $D_\mu\,G$ is contained in the interval $(0, 1]$; which implies that $\|D_\mu\,G\| \leq 1$.

Using Remark 2.4 and the Cauchy-Schwarz inequality, it follows that

$$\|D_\mu\,\triangle\overline{\mathbf{u}}\| = \|D_\mu\,G\,\triangle\mathbf{u}\| \leq \|D_\mu\,G\|\,\|\triangle\mathbf{u}\| \leq \|\triangle\mathbf{u}\|.$$

□

Lemma 2.4. [*Error in Approximate deconvolution*] *Consider the differential filter given by Definition 2.2. Then, for smooth* $\mathbf{u}$

$$\|\mathbf{u} - D_\mu\overline{\mathbf{u}}\| \leq \mu\delta^2\,\|\triangle\mathbf{u}\|\,. \tag{2.20}$$

Proof. Indeed, by algebraic manipulation, we have:

$$\begin{aligned} \mathbf{u} - D_\mu\overline{\mathbf{u}} &= (I - D_\mu A^{-1})\mathbf{u} \\ &= D_\mu(D_\mu^{-1} - A^{-1}\ )\mathbf{u} \\ &= \mu D_\mu(I - A^{-1})\mathbf{u}. \end{aligned}$$

But, $(I - A^{-1})\mathbf{u} = \mathbf{u} - \overline{\mathbf{u}} = -\delta^2\triangle\overline{\mathbf{u}}$. Thus,

$$\mathbf{u} - D_\mu\overline{\mathbf{u}} = -\mu\delta^2 D_\mu\triangle\overline{\mathbf{u}}.$$

Using the proof of Lemma 2.3 and taking the L^2 norm of both sides we obtain the desired result. □

With the discrete filter (2.7), we define the discrete modified Tikhonov deconvolution operator.

Definition 2.6. [*Discrete Modified Tikhonov Deconvolution Operator*] *For a given filtering radius* $\delta > 0$ *and* $0 < \mu < 1$, *the discrete counterpart of* D_μ *is denoted by* D_μ^h, *and is defined by*

$$D_\mu^h = \left((1-\mu)A_h^{-1} + \mu I\right)^{-1}.$$

With the discrete differential filter in Definition 2.3 we define D_μ^h *precisely. Given* $\Psi \in X$, $\boldsymbol{\psi}^h = D_\mu^h\overline{\Psi}^h$ *is the solution in* X_h *of the problem*

$$\mu\delta^2(\nabla\boldsymbol{\psi}^h, \nabla\boldsymbol{\chi}) + (\boldsymbol{\psi}^h, \boldsymbol{\chi}) = (\Psi, \boldsymbol{\chi}), \quad \forall\boldsymbol{\chi} \in X_h, \tag{2.21}$$

where Ψ *and* $\overline{\Psi}^h$ *satisfy* $\delta^2(\nabla\overline{\Psi}^h, \nabla\boldsymbol{\chi}) + (\overline{\Psi}^h, \boldsymbol{\chi}) = (\Psi, \boldsymbol{\chi})$.

Proposition 2.2. *The operator D_μ^h is bounded self-adjoint and positive definite on X_h.*

Proof. The operator D_μ^h is the inverse of a convex combination of A_h^{-1} and I. Both these operators are SPD on X_h. Thus, so is D_μ^h. □

Lemma 2.5. *For $\mathbf{v} \in X$, we have the following bounds for the discretely filtered and approximately deconvolved $\mathbf{v}$*

$$\begin{aligned} \left\|\overline{\mathbf{v}}^h\right\| &\leq \|\mathbf{v}\|, &(2.22)\\ \left\|\nabla\overline{\mathbf{v}}^h\right\| &\leq \|\nabla\mathbf{v}\|, &(2.23)\\ \left\|D_\mu^h\overline{\mathbf{v}}^h\right\| &\leq \|\mathbf{v}\|, &(2.24)\\ \left\|\nabla D_\mu^h\overline{\mathbf{v}}^h\right\| &\leq \|\nabla\mathbf{v}\|. &(2.25)\end{aligned}$$

Proof. Let $\boldsymbol{\chi} = \overline{\mathbf{v}}^h$ in (2.7) and apply the Cauchy-Schwarz inequality in the right hand side. We have

$$\delta^2\left\|\nabla\overline{\mathbf{v}}^h\right\|^2 + \left\|\overline{\mathbf{v}}^h\right\|^2 \leq \left\|\overline{\mathbf{v}}^h\right\|\|\mathbf{v}\|.$$

Thus

$$\left\|\overline{\mathbf{v}}^h\right\|^2 \leq \left\|\overline{\mathbf{v}}^h\right\|\|\mathbf{v}\|$$

and (2.22) follows. For the proof of (2.23) we proceed in a similar way, we set $\boldsymbol{\chi} = \triangle_h\overline{\mathbf{v}}^h$ in (2.7) and obtain

$$\delta^2\left\|\triangle_h\overline{\mathbf{v}}^h\right\|^2 + \left\|\nabla\overline{\mathbf{v}}^h\right\|^2 \leq \left\|\nabla\overline{\mathbf{v}}^h\right\|\|\nabla\mathbf{v}\|.$$

To prove (2.24), let $\Psi = \mathbf{v}$ and $\boldsymbol{\chi} = D_\mu^h\overline{\mathbf{v}}^h$ in (2.21). Definition 2.3 and the Cauchy-Schwarz inequality give

$$\mu\delta^2\left\|\nabla D_\mu^h\overline{\mathbf{v}}^h\right\|^2 + \left\|D_\mu^h\overline{\mathbf{v}}^h\right\|^2 \leq \|\mathbf{v}\|\left\|D_\mu^h\overline{\mathbf{v}}^h\right\|, \tag{2.26}$$

proving (2.24). The proof of (2.25) is similar. Let $\Psi = \mathbf{v}$ and $\boldsymbol{\chi} = \triangle_h D_\mu^h\overline{\mathbf{v}}^h$ in (2.21). The definition of $\triangle_h$ and Cauchy-Schwarz inequality give

$$\mu\delta^2\left\|\triangle_h D_\mu^h\overline{\mathbf{v}}^h\right\|^2 + \left\|\nabla D_\mu^h\overline{\mathbf{v}}^h\right\|^2 \leq \|\nabla\mathbf{v}\|\left\|\nabla D_\mu^h\overline{\mathbf{v}}^h\right\|. \tag{2.27}$$

Now the conclusion follows. □

Theorem 2.2. *Let $\mathbf{v}_h^* := \mathbf{v} - D_\mu^h\overline{\mathbf{v}}^h$, where $\mathbf{v} \in X$. Then, we have*

$$(\mathbf{v}_h^*, \mathbf{v}) > 0. \tag{2.28}$$

Proof. Let $\mathbf{v} \in X$. Poincaré's inequality together with (2.26) gives

$$\left\|D_\mu^h\overline{\mathbf{v}}^h\right\|^2 \leq \frac{1}{(C_{PF}^2\mu\delta^2+1)}\|\mathbf{v}\|^2, \tag{2.29}$$

which leads to

$$\left(\mathbf{v} - D_\mu^h\overline{\mathbf{v}}^h, \mathbf{v}\right) \geq \frac{C_{PF}^2\mu\delta^2}{1+C_{PF}^2\mu\delta^2}\|\mathbf{v}\|^2. \tag{2.30}$$

and (2.28) follows for all $\mathbf{v} \in X$. □

Lemma 2.6. *Let $\mathbf{v} \in X$. Then*

$$(\mathbf{v}_h^*, \boldsymbol{\chi}_h) \leq \mu\delta^2\|\nabla\mathbf{v}\|\|\nabla\boldsymbol{\chi}_h\|, \ \forall\boldsymbol{\chi}_h \in X_h. \tag{2.31}$$

Proof. Definition 2.6 leads to

$$(\mathbf{v}_h^*, \boldsymbol{\chi}_h) = \mu\delta^2(\nabla D_\mu^h \overline{\mathbf{v}}^h, \nabla \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in X_h.$$

Applying the Cauchy-Schwarz inequality and Lemma 2.5 in the RHS, we obtain (2.31). □

From Theorem 2.2 follows that $I - D_\mu^h A_h^{-1}$ is SPD. Fundamental in deriving energy estimates for the scheme outlined in the next section is the norm of $\mathbf{v}_h^*$ defined as

$$\|\mathbf{v}_h^*\|^2 := (\mathbf{v}_h^*, \mathbf{v})\,. \tag{2.32}$$

Lemma 2.7. *For all $\mathbf{v} \in X$ with $\triangle \mathbf{v} \in L^2(\Omega)$, we have*

$$\left\|\mathbf{v} - D_\mu^h \overline{\mathbf{v}}^h\right\| \le \mu\delta^2 \|\mathbf{v}\|_2 + C\,(\delta h^k + h^{k+1})\,\|\mathbf{v}\|_{k+1} + (\mu^{1/2}\delta h^k + h^{k+1})\,\|D_\mu \overline{\mathbf{v}}\|_{k+1}\,. \tag{2.33}$$

Proof. Applying the triangle inequality, we obtain

$$\left\|\mathbf{v} - D_\mu^h \overline{\mathbf{v}}^h\right\| \le \|\mathbf{v} - D_\mu \overline{\mathbf{v}}\| + \left\|D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}\right\| + \left\|D_\mu^h \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}^h\right\|. \tag{2.34}$$

We now look at each term in the right hand side separately. Lemma 2.4 gives

$$\|\mathbf{v} - D_\mu \overline{\mathbf{v}}\| \le \mu\delta^2 \|\mathbf{v}\|_2\,. \tag{2.35}$$

For the second term, let $\boldsymbol{\chi} \in X_h$ in (2.14) and subtract it from (2.21). Let $\mathbf{e} := D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}} = D_\mu \overline{\mathbf{v}} - \mathbf{v}^h - D_\mu^h \overline{\mathbf{v}} + \mathbf{v}^h$ for all $\mathbf{v}^h \in X_h$. Using Galerkin orthogonality, we obtain

$$\mu\delta^2 \left\|\nabla D_\mu \overline{\mathbf{v}} - \nabla D_\mu^h \overline{\mathbf{v}}\right\|^2 + \left\|D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}\right\|^2 \le \inf_{\mathbf{v}^h \in X_h} (\mu\delta^2 \left\|\nabla(D_\mu \overline{\mathbf{v}} - \mathbf{v}^h)\right\|^2 + \left\|D_\mu \overline{\mathbf{v}} - \mathbf{v}^h\right\|^2).$$

The approximation results (2.5) lead to

$$\left\|D_\mu \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}\right\| \le (\mu^{1/2}\delta h^k + h^{k+1})\,\|D_\mu \overline{\mathbf{v}}\|_{k+1}\,. \tag{2.36}$$

To bound the last term we first apply Lemma 2.5

$$\left\|D_\mu^h \overline{\mathbf{v}} - D_\mu^h \overline{\mathbf{v}}^h\right\| \le \left\|\overline{\mathbf{v}} - \overline{\mathbf{v}}^h\right\|.$$

From Definitions 2.2 and 2.3, Galerkin orthogonality and then using the approximation results (2.5) we get

$$\left\|D_\mu \overline{\mathbf{v}} - D_\mu \overline{\mathbf{v}}^h\right\| \le C\,(\delta h^k + h^{k+1})\,\|\overline{\mathbf{v}}\|_{k+1}\,. \tag{2.37}$$

The final conclusion then follows from (2.35), (2.36), and (2.37). □

In the next section we study a Crank-Nicolson Finite Element Scheme for the Modified Tikhonov Approximate Deconvolution Model (1.3).

3 Convergence of the Discrete Model

To begin, we define the scheme and show that its solutions are well defined, unconditionally stable, and optimally convergent to solutions of the NSE.

A strong solution of the Navier-Stokes equations satisfies $\mathbf{u} \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;X)$, $p \in L^2(0,T;Q)$ with $\mathbf{u}_t \in L^2(0,T;X^*)$ such that

$$(\mathbf{u}_t, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v})\,,\ \forall \mathbf{v} \in X\,, \tag{3.1}$$

$$(q, \nabla \cdot \mathbf{u}) = 0\,,\ \forall q \in Q\,. \tag{3.2}$$

Throughout the analysis we use the following notation $\mathbf{v}(t_{n+1/2}) := \mathbf{v}((t_n + t_{n+1})/2)$ for the continuous variables and $\mathbf{v}_{n+1/2} := (\mathbf{v}_n + \mathbf{v}_{n+1})/2$ for both, continuous and discrete variables.

Algorithm 3.1. *[Crank-Nicolson Finite Element Scheme for Leray-Tikhonov deconvolution model] Let $\Delta t > 0$, $(\mathbf{w}_0, q_0) \in (X_h, Q_h)$, $\mathbf{f} \in X^*$ and $T := M\,\Delta t$ as M is an integer. For $n = 0, 1, 2, \cdots, M-1$, find $(\mathbf{w}^h_{n+1}, q^h_{n+1}) \in (X_h, Q_h)$ satisfying*

$$\frac{1}{\Delta t}(\mathbf{w}^h_{n+1} - \mathbf{w}^h_n, \mathbf{v}^h) + b^*(D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{w}^h_{n+1/2}, \mathbf{v}^h) - (q^h_{n+1/2}, \nabla \cdot \mathbf{v}^h) + \nu(\nabla \mathbf{w}^h_{n+1/2}, \nabla \mathbf{v}^h)$$
$$+\chi(\mathbf{w}^h_{n+1/2} - D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h), \ \forall \ \mathbf{v}^h \in X_h \tag{3.3}$$

$$(\nabla \cdot \mathbf{w}^h_{n+1}, \boldsymbol{\phi}^h) = 0, \ \forall \boldsymbol{\phi}^h \in Q_h. \tag{3.4}$$

Remark 3.1. *Since (X_h, Q_h) satisfies the discrete inf-sup condition, (3.3)-(3.4) is equivalent to*

$$\frac{1}{\Delta t}(\mathbf{w}^h_{n+1} - \mathbf{w}^h_n, \mathbf{v}^h) + b^*(D^h_N \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{w}^h_{n+1/2}, \mathbf{v}^h) + \nu(\nabla \mathbf{w}^h_{n+1/2}, \nabla \mathbf{v}^h)$$
$$+ \chi(\mathbf{w}^h_{n+1/2} - D^h_\mu \overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h), \ \forall \ \mathbf{v}^h \in V_h. \tag{3.5}$$

In the error analysis we use of the following lemmas and notation.

Lemma 3.1. *Assume $\mathbf{u} \in C^0(t_n, t_{n+1}; L^2(\Omega))$. If $\mathbf{u}$ is twice differentiable in time and $\mathbf{u}_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$ then*

$$\left\|\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2})\right\|^2 \ \leq \ \frac{1}{48}(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|^2 \, dt\,. \tag{3.6}$$

If $\mathbf{u}_t \in C^0(t_n, t_{n+1}; L^2(\Omega))$ and $\mathbf{u}_{ttt} \in L^2((t_n, t_{n+1}) \times \Omega)$ then

$$\left\|\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\right\|^2 \ \leq \ \frac{1}{1280}(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{ttt}\|^2 \, dt\,. \tag{3.7}$$

If $\nabla \mathbf{u} \in C^0(t_n, t_{n+1}; L^2(\Omega))$ and $\nabla \mathbf{u}_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$ then

$$\left\|\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}))\right\|^2 \ \leq \ \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^2 \, dt\,. \tag{3.8}$$

Proof. The proof is based on the Taylor expansion with remainder. □

Lemma 3.2. [*Discrete Gronwall Lemma*] *Let Δt, H, and a_n, b_n, c_n, d_n (for integers $n \geq 0$) be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \quad for\ l \geq 0. \tag{3.9}$$

Suppose that $\Delta t d_n < 1 \ \forall n$. Then,

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq exp\left(\Delta t \sum_{n=0}^{l} \frac{d_n}{1 - \Delta t d_n}\right)\left(\Delta t \sum_{n=0}^{l} c_n + H\right) \quad for\ l \geq 0. \tag{3.10}$$

Proof. The proof follows from [24]. □

In the discrete case we use the analogous norms:

$$|||\mathbf{v}|||_{\infty,k} := \max_{0 \leq n \leq M-1} \|\mathbf{v}_n\|_k\,, \qquad |||\mathbf{v}_{1/2}|||_{\infty,k} := \max_{1 \leq n \leq M-1} \|\mathbf{v}_{n+1/2}\|_k\,,$$
$$|||\mathbf{v}|||_{m,k} := \left(\sum_{n=0}^{M-1} \|\mathbf{v}_n\|^m_k \triangle t\right)^{1/m}, \qquad |||\mathbf{v}_{1/2}|||_{m,k} := \left(\sum_{n=1}^{M-1} \|\mathbf{v}_{n+1/2}\|^m_k \triangle t\right)^{1/m}.$$

Lemma 3.3. [*Existence of Solutions and Stability of the Scheme*] *At each time step, there exists a solution of the approximation scheme (3.5). Also, the scheme is unconditionally stable and satisfies the following á priori bound:*

$$\left\|\mathbf{w}_n^h\right\|^2+\nu\Delta t\sum_{k=0}^{n-1}\left\|\nabla\mathbf{w}_{k+1/2}^h\right\|^2+\chi\sum_{k=0}^{n-1}\left\|{\mathbf{w}_{k+1/2}^h}^*\right\|^2\leq\left\|\mathbf{w}_0^h\right\|^2+\frac{\Delta t}{\nu}\sum_{k=0}^{n-1}\left\|\mathbf{f}_{k+1/2}\right\|_*^2, \tag{3.11}$$

for all integers $1\leq n\leq M$.

Proof. We begin by proving the à priori estimate (3.11). In (3.5), set $\mathbf{v}^h=\mathbf{w}_{k+1/2}^h$. Applying Young's inequality, we obtain

$$\frac{1}{\Delta t}(\left\|\mathbf{w}_{k+1}^h\right\|^2-\left\|\mathbf{w}_k^h\right\|^2)+\nu\left\|\nabla\mathbf{w}_{k+1/2}^h\right\|^2+\chi\left\|{\mathbf{w}_{k+1/2}^h}^*\right\|^2\leq\frac{1}{\nu}\left\|\mathbf{f}_{k+1/2}\right\|_*^2,\quad\text{for every } k. \tag{3.12}$$

Summing from $k=0$ to n, where n is an integer, $1\leq n\leq M$, we obtain the desired result.

The existence of a solution $\mathbf{w}_{k+1}^h$ to (3.5) follows from the Leray-Schauder Principle, [42]. We reformulate (3.5) as a fixed point problem, insert a parameter λ and adapt the proof of the à priori bound to give a bound uniform in λ. To do this, we define the operator $T:X^{'}\rightarrow V_h$, by $T(\mathbf{y}):=\mathbf{z}$, where

$$(\mathbf{y},\mathbf{v}):=\triangle t\ \nu(\nabla\left(\frac{\mathbf{z}+\mathbf{w}_k^h}{2}\right),\nabla\mathbf{v})+\triangle t\ \chi(\frac{\mathbf{z}+\mathbf{w}_k^h}{2}-D_\mu^h\overline{\frac{(\mathbf{z}+\mathbf{w}_k^h)}{2}}^h,\mathbf{v}),\quad\text{for all }\mathbf{v}\in V_h.$$

The bilinear form on the above right hand side is coercive. Then, by the Lax-Milgram theorem, the operator T exists and is bounded. Note that T is also linear. We also define the nonlinear operator $N:V_h\rightarrow X^{'}$, via the Riesz Representation theorem,

$$(N(\mathbf{z}),\mathbf{v})=-\triangle t\ b^*(D_\mu^h\overline{\frac{(\mathbf{z}+\mathbf{w}_k^h)}{2}}^h,\frac{\mathbf{z}+\mathbf{w}_k^h}{2},\mathbf{v})+(\mathbf{w}_k^h-\mathbf{z},\mathbf{v})+\triangle t(\mathbf{f}_{k+1/2},\mathbf{v})\quad\text{for all }\mathbf{v}\in V_h.$$

Since V_h is finite dimensional, the operator N is trivially bounded and continuous. Finally, we define $F:V_h\rightarrow V_h$, by $F(\mathbf{z})=T(N(\mathbf{z}))$. Then, $\mathbf{z}$ is a solution of (3.5) if and only if it is a fixed point of F.

To show that F has a fixed point in V_h, we apply the Leray-Schauder Principle. We first note that the operator F is algebraic, hence continuous. Since $\dim V_h<\infty$, F is also compact. By the Leray-Schauder Principle, we need to show that any solution $\mathbf{u}_\lambda$ of the fixed point problem $\mathbf{z}=\lambda F(\mathbf{z})$, where $0\leq\lambda<1$, satisfies $\|\mathbf{u}_\lambda\|_X\leq\gamma$, where γ does not depend on λ. We have

$$\begin{aligned}&\triangle t\ \nu(\nabla\left(\frac{\mathbf{u}_\lambda+\mathbf{w}_k^h}{2}\right),\nabla\mathbf{v})+\triangle t\ \chi(\frac{\mathbf{u}_\lambda+\mathbf{w}_k^h}{2}-D_\mu^h\overline{\frac{(\mathbf{u}_\lambda+\mathbf{w}_k^h)}{2}}^h,\mathbf{v})\\&=-\lambda\triangle t\ b^*(D_\mu^h\overline{\frac{(\mathbf{u}_\lambda+\lambda\mathbf{w}_k^h)}{2}}^h,\frac{\mathbf{u}_\lambda+\mathbf{w}_k^h}{2},\mathbf{v})+\lambda(\mathbf{u}_\lambda-\mathbf{w}_k^h,\mathbf{v})\\&\quad+\lambda\triangle t(\mathbf{f}_{k+1/2},\mathbf{v}),\qquad\text{for all }\mathbf{v}\in V_h.\end{aligned} \tag{3.13}$$

Now, set $\mathbf{v}=\frac{\mathbf{u}_\lambda+\mathbf{w}_k^h}{2}$. Since $0\leq\lambda<1$, proceeding as in the à priori estimate bounded we obtain the desired bound for $\|\nabla\mathbf{u}_\lambda\|$. It means that a solution of (3.5) exists at each time step. □

Remark 3.2. *The same argument works in the infinite dimensional case, when (3.5) is posed in* X *instead of* X_h*. The only modification is that compactness of* F *(which holds) is verified separately using the Raleigh Lemma.*

3.1 Convergence Analysis

We now state and prove our main convergence estimate.

Theorem 3.2. *Let $(\mathbf{u}(t), p(t))$ be a sufficiently smooth, strong solution of the NSE with no-slip boundary conditions. Suppose $(\mathbf{w}_h(0), q_h(0))$ are approximations of $(\mathbf{u}(0), p(0))$ to the accuracy of (2.5), respectively. Then there is a constant $C = C(\mathbf{u}, p)$ such that*

$$|||\mathbf{u} - \mathbf{w}^h|||_{\infty,0} \leq F(\triangle t, h, \delta) + Ch^{k+1}|||\mathbf{u}|||_{\infty,k+1}, \tag{3.14}$$

$$\begin{aligned}\left(\nu\triangle t\sum_{n=0}^{M-1}\|\nabla(\mathbf{u}_{n+1/2} - (\mathbf{w}^h_{n+1} + \mathbf{w}^h_n)/2)\|^2\right)^{1/2} &\leq F(\triangle t, h, \delta) + C\nu^{1/2}(\triangle t)^2\|\nabla \mathbf{u}_{tt}\|_{2,0} \\ &\quad + Ch^k|||\mathbf{u}|||_{2,k+1},\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}F(\triangle t, h, \mu, \delta) &= C^* h^{k+1}\,(\nu^{-1/2} + \chi)(\,\|D_\mu\overline{\mathbf{u}}\|_{2,k+1} + \left\|\mathbf{u}_{n+1/2}\right\|_{k+1}\,) \\ &\quad + \nu^{-1/2}\,h^{k+1/2}\left(|||\mathbf{u}|||^2_{4,k+1} + |||\nabla\mathbf{u}|||^2_{4,0}\right) \\ &\quad + \nu^{-1/2}h^k\left(|||\mathbf{u}|||^2_{4,k+1} + \nu^{-1/2}(\|\mathbf{w}^h_0\| + \nu^{-1}|||\mathbf{f}|||_{2,\star})\right) \\ &\quad + \delta h^k(\nu^{-1/2} + \chi)\left(\mu\,\|D_\mu\overline{\mathbf{u}}\|_{2,k+1} + \|\mathbf{u}\|_{2,k+1}\right) \\ &\quad \nu^{-1/2}h^{s+1}|||p_{1/2}|||_{2,s+1} + (\nu^{-1/2} + \chi)\mu\delta^2\,\|\mathbf{u}\|_{2,2} \\ &\quad + (\Delta t)^2\left(\|\mathbf{u}_{ttt}\|_{2,0} + \|\mathbf{f}_{tt}\|_{2,0} + \nu^{-1/2}\|p_{tt}\|_{2,0}\right. \\ &\quad \left. + \nu^{-1/2}\|\nabla\mathbf{u}_{tt}\|^2_{4,0} + \nu^{-1/2}|||\nabla\mathbf{u}|||^2_{4,0} + \nu^{-1/2}|||\nabla\mathbf{u}_{1/2}|||^2_{4,0}\right)\end{aligned}$$

Proof. First, we note that at time $t_{n+1/2}$, $\mathbf{u}$ given by (3.1)-(3.2) satisfies

$$\begin{aligned}&(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}, \mathbf{v}^h) + b^*(D^h_\mu\overline{\overline{\mathbf{u}}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \mathbf{v}^h) + \nu(\nabla\mathbf{u}_{n+1/2}, \nabla\mathbf{v}^h) - (p_{n+1/2}, \nabla\cdot\mathbf{v}^h) \\ &\quad + \chi(\mathbf{u}_{n+1/2} - D^h_\mu\overline{\overline{\mathbf{u}}_{n+1/2}}^h, \mathbf{v}^h) = (\mathbf{f}_{n+1/2}, \mathbf{v}^h) + Intp(\mathbf{u}^n, p^n; \mathbf{v}^h),\end{aligned} \tag{3.16}$$

for all $\mathbf{v}^h \in X_h$, where $Intp(\mathbf{u}_n, p_n; \mathbf{v}^h)$, representing the interpolating error, denotes

$$\begin{aligned}Intp(\mathbf{u}_n, p_n; \mathbf{v}^h) &= \left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}), \mathbf{v}^h\right) + \nu(\nabla\mathbf{u}_{n+1/2} - \nabla\mathbf{u}(t_{n+1/2}), \nabla\mathbf{v}^h) \\ &\quad + b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v}^h) - b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}(t_{n+1/2}), \mathbf{v}^h) \\ &\quad - b^*(\mathbf{u}_{n+1/2} - D^h_\mu\overline{\overline{\mathbf{u}}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \mathbf{v}^h) \\ &\quad + \chi(\mathbf{u}_{n+1/2} - D^h_\mu\overline{\overline{\mathbf{u}}_{n+1/2}}^h, \mathbf{v}^h) - (p_{n+1/2} - p(t_{n+1/2}), \nabla\cdot\mathbf{v}^h) \\ &\quad + (\mathbf{f}(t_{n+1/2}) - \mathbf{f}_{n+1/2}, \mathbf{v}^h)\,.\end{aligned} \tag{3.17}$$

Subtracting (3.3) from (3.16) and letting $\mathbf{e}_n = \mathbf{u}_n - \mathbf{w}^h_n$ we have

$$\begin{aligned}&\frac{1}{\Delta t}(\mathbf{e}_{n+1} - \mathbf{e}_n, \mathbf{v}^h) + b^*(D^h_\mu\overline{\overline{\mathbf{u}}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \mathbf{v}^h) - b^*(D^h_\mu\overline{\mathbf{w}^h_{n+1/2}}^h, \mathbf{w}^h_{n+1/2}, \mathbf{v}^h) \\ &\quad + \nu(\nabla\mathbf{e}_{n+1/2}, \nabla\mathbf{v}^h) + \chi(\mathbf{e}_{n+1/2} - D^h_\mu\overline{\mathbf{e}_{n+1/2}}^h, \mathbf{v}^h) \\ &\quad = (p_{n+1/2} - q, \nabla\cdot\mathbf{v}^h) + Intp(\mathbf{u}_n, p_n; \mathbf{v}^h), \qquad \forall\mathbf{v}^h \in X_h.\end{aligned} \tag{3.18}$$

Let $\mathbf{e}_n = (\mathbf{u}_n - U_n) - (\mathbf{w}_n^h - U_n) := \boldsymbol{\eta}_n - \boldsymbol{\phi}_n^h$ where $\boldsymbol{\phi}_n^h \in X_h$ and U represents the L^2 projection of $\mathbf{u}$ in X_h. Setting $\mathbf{v}^h = \boldsymbol{\phi}_{n+1/2}^h$ in (3.18) and using $(q, \nabla \cdot \boldsymbol{\phi}_{n+1/2}) = 0$ for all $q \in Q_h$ we obtain

$$\begin{aligned}(\boldsymbol{\phi}_{n+1}^h - \boldsymbol{\phi}_n^h, \boldsymbol{\phi}_{n+1/2}^h) + \Delta t\nu \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + \Delta t b^*(D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h, \mathbf{e}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\ + \Delta t b^*(D_\mu^h \overline{\mathbf{e}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) + \Delta t \chi \left\| {\boldsymbol{\phi}_{n+1/2}^h}^* \right\|^2 \\ = \Delta t(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n, \boldsymbol{\phi}_{n+1/2}^h) + \Delta t\nu(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}_{n+1/2}^h) \\ + \Delta t\chi(\boldsymbol{\eta}_{n+1/2} - D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h, \boldsymbol{\phi}_{n+1/2}^h) \\ + \Delta t(p_{n+1/2} - q, \nabla \cdot \boldsymbol{\phi}_{n+1/2}^h) - \Delta t \ Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}_{n+1/2}^h)\,. \end{aligned} \tag{3.19}$$

Because of our choice of U, we have $(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n, \boldsymbol{\phi}_{n+1/2}^h) = 0$ and we can rewrite

$$\begin{aligned}\frac{1}{2}(\|\boldsymbol{\phi}_{n+1}^h\|^2 - \|\boldsymbol{\phi}_n^h\|^2) + \Delta t\nu \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + \Delta t\chi \left\| {\boldsymbol{\phi}_{n+1/2}^h}^* \right\|^2 \\ = \Delta t\nu(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}_{n+1/2}^h) - \Delta t b^*(D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\ + \Delta t b^*(D_\mu^h \overline{\boldsymbol{\phi}_{n+1/2}^h}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\ - \Delta t b^*(D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h, \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\ + \Delta t\chi(\boldsymbol{\eta}_{n+1/2} - D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h, \boldsymbol{\phi}_{n+1/2}^h) \\ + \Delta t(p_{n+1/2} - q, \nabla \cdot \boldsymbol{\phi}_{n+1/2}^h) + \Delta t \ Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}_{n+1/2}^h)\,. \end{aligned} \tag{3.20}$$

We now estimate the terms on the right hand side of (3.20) separately.

Using Cauchy-Schwarz and Young's inequalities we have

$$\begin{aligned}\nu \triangle t(\nabla \boldsymbol{\eta}_{n+1/2}, \nabla \boldsymbol{\phi}_{n+1/2}^h) &\leq \nu \triangle t \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\ &\leq \frac{\nu \Delta t}{12} \left\| \nabla \boldsymbol{\phi}_{n+1/2}^h \right\|^2 + C\nu \Delta t \left\| \nabla \boldsymbol{\eta}_{n+1/2} \right\|^2. \end{aligned} \tag{3.21}$$

$$\begin{aligned}\triangle t(p_{n+1/2} - q, \nabla \cdot \boldsymbol{\phi}_{n+1/2}^h) &\leq C \triangle t \|p_{n+1/2} - q\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\ &\leq \frac{\nu \Delta t}{12} \left\| \nabla \boldsymbol{\phi}_{n+1/2}^h \right\|^2 + C \triangle t \nu^{-1} \|p_{n+1/2} - q\|^2\,. \end{aligned} \tag{3.22}$$

Lemmas 2.1 and 2.5 and standard inequalities give

$$\begin{aligned}&\triangle t \, b^*(D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\ \leq\ & C \Delta t \|D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h\|^{1/2} \, \|\nabla D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h\|^{1/2} \, \|\nabla \mathbf{u}_{n+1/2}\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\ \leq\ & \frac{\nu \triangle t}{12} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + C \triangle t \, \nu^{-1} \|\boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \|\nabla \mathbf{u}_{n+1/2}\|^2\,. \end{aligned} \tag{3.23}$$

$$\begin{aligned}&\triangle t \, b^*(D_\mu^h \overline{\boldsymbol{\phi}_{n+1/2}^h}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\ \leq\ & C \triangle t \|D_\mu^h \overline{\boldsymbol{\phi}_{n+1/2}^h}^h\|^{1/2} \, \|\nabla D_\mu^h \overline{\boldsymbol{\phi}_{n+1/2}^h}^h\|^{1/2} \, \|\nabla \mathbf{u}_{n+1/2}\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\ \leq\ & C \triangle t \|\boldsymbol{\phi}_{n+1/2}^h\|^{1/2} \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^{3/2} \, \|\nabla \mathbf{u}_{n+1/2}\| \\ \leq\ & \frac{\nu \triangle t}{12} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 \, + \, C \triangle t \, \nu^{-3} \|\boldsymbol{\phi}_{n+1/2}^h\|^2 \, \|\nabla \mathbf{u}_{n+1/2}\|^4\,. \end{aligned} \tag{3.24}$$

$$
\begin{aligned}
& \triangle t\, b^*(D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h, \boldsymbol{\eta}_{n+1/2}, \boldsymbol{\phi}_{n+1/2}^h) \\
\leq \quad & C\|D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h\|^{1/2} \, \|\nabla D_\mu^h \overline{\mathbf{w}_{n+1/2}^h}^h\|^{1/2} \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\phi}_{n+1/2}^h\| \\
\leq \quad & \frac{\nu \triangle t}{12} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + C \triangle t\, \nu^{-1} \|\mathbf{w}_{n+1/2}^h\| \, \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \, . \qquad (3.25)
\end{aligned}
$$

Lemma 2.6 and Young's inequality give

$$
\Delta t \chi(\boldsymbol{\eta}_{n+1/2} - D_\mu^h \overline{\boldsymbol{\eta}_{n+1/2}}^h, \boldsymbol{\phi}_{n+1/2}^h) \;\leq\; \frac{\nu \triangle t}{12} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + C \triangle t\, \nu^{-1} \; \chi^2 \mu^2 \delta^4 \; \left\| \nabla \boldsymbol{\eta}_{n+1/2} \; \right\|^2 . \qquad (3.26)
$$

Substituting (3.21)-(3.25) into (3.20) and summing from $n = 0$ to $M-1$ (assuming that $\|\boldsymbol{\phi}_0^h\| = 0$) we get

$$
\begin{aligned}
& \|\boldsymbol{\phi}_M^h\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \boldsymbol{\phi}_{n+1/2}^h\|^2 + \chi \triangle t \sum_{n=0}^{M-1} \|{\boldsymbol{\phi}_{n+1/2}^h}^*\|^2 \\
\leq \quad & \triangle t \sum_{n=0}^{M-1} C \nu^{-3} \|\nabla \mathbf{u}_{n+1/2}\|^4 \, \|\boldsymbol{\phi}_{n+1/2}^h\|^2 + \triangle t \sum_{n=0}^{M-1} C \nu \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} C \nu^{-1} \|\boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \|\nabla \mathbf{u}_{n+1/2}\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} C \nu^{-1} \|\mathbf{w}_{n+1/2}^h\| \, \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} C\, \nu^{-1} \; \chi^2 \mu^2 \delta^4 \; \left\| \nabla \boldsymbol{\eta}_{n+1/2} \; \right\|^2 + \triangle t \sum_{n=0}^{M-1} C \nu^{-1} \|p_{n+1/2} - q\|^2 \\
& + \triangle t \sum_{n=0}^{M-1} |Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}_{n+1/2}^h)| \, . \qquad (3.27)
\end{aligned}
$$

We now bound each term in the right hand side of (3.27).

$$
\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C \nu \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \quad & \leq \quad \triangle t C \nu \sum_{n=0}^{M} \|\nabla \boldsymbol{\eta}_n\|^2 \leq \triangle t C \nu \sum_{n=0}^{M} h^{2k} |\mathbf{u}_n|_{k+1}^2 \\
& \leq \quad C \nu h^{2k} |||\mathbf{u}|||_{2,k+1}^2 \, . \qquad (3.28)
\end{aligned}
$$

Next, we consider the term

$$
\begin{aligned}
& \triangle t \sum_{n=0}^{M-1} C\nu^{-1} \|\boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\| \, \|\nabla \mathbf{u}_{n+1/2}\|^2 \\
\leq \;& C\nu^{-1} \triangle t \sum_{n=0}^{M-1} \big(\|\boldsymbol{\eta}_{n+1}\| \, \|\nabla \boldsymbol{\eta}_{n+1}\| + \|\boldsymbol{\eta}_{n}\| \, \|\nabla \boldsymbol{\eta}_{n}\| \\
& \qquad + \|\boldsymbol{\eta}_{n}\| \, \|\nabla \boldsymbol{\eta}_{n+1}\| + \|\boldsymbol{\eta}_{n+1}\| \, \|\nabla \boldsymbol{\eta}_{n}\| \big) \|\nabla \mathbf{u}_{n+1/2}\|^2 \\
\leq \;& C\,\nu^{-1}\, h^{2k+1} \left(\triangle t \sum_{n=0}^{M-1} |\mathbf{u}_{n+1}|_{k+1}^2 \, \|\nabla \mathbf{u}_{n+1/2}\|^2 + \triangle t \sum_{n=0}^{M-1} |\mathbf{u}_{n+1}|_{k+1} |\mathbf{u}_{n}|_{k+1} \, \|\nabla \mathbf{u}_{n+1/2}\|^2 \right. \\
& \qquad \left. + \; \triangle t \sum_{n=0}^{M-1} |\mathbf{u}_{n}|_{k+1}^2 \, \|\nabla \mathbf{u}_{n+1/2}\|^2 \right) \\
\leq \;& C\nu^{-1}\, h^{2k+1} \left(\triangle t \sum_{n=0}^{M} |\mathbf{u}_{n}|_{k+1}^4 + \triangle t \sum_{n=0}^{M} \|\nabla \mathbf{u}_{n}\|^4 \right) \\
= \;& C\nu^{-1}\, h^{2k+1} \left(|||\mathbf{u}|||_{4,k+1}^4 + |||\nabla \mathbf{u}|||_{4,0}^4 \right) . \qquad (3.29)
\end{aligned}
$$

Using (3.11), we have

$$
\begin{aligned}
& \triangle t \sum_{n=0}^{M-1} C\nu^{-1} \left(\|\mathbf{w}_{n+1/2}^h\| \, \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \right) \\
\leq \;& C\nu^{-1} \triangle t \sum_{n=0}^{M-1} \|\nabla \mathbf{w}_{n+1/2}^h\| \, \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 \\
\leq \;& C\nu^{-1} \triangle t \sum_{n=0}^{M-1} \left(\|\nabla \boldsymbol{\eta}_{n+1}\|^2 + \|\nabla \boldsymbol{\eta}_{n}\|^2 \right) \|\nabla \mathbf{w}_{n+1/2}^h\| \\
\leq \;& C\nu^{-1} h^{2k} \triangle t \sum_{n=0}^{M-1} \left(|\mathbf{u}_{n+1}|_{k+1}^2 + |\mathbf{u}_{n}|_{k+1}^2 \right) \|\nabla \mathbf{w}_{n+1/2}^h\| \\
\leq \;& C\nu^{-1} h^{2k} \left(\triangle t \sum_{n=0}^{M} |\mathbf{u}^n|_{k+1}^4 + \triangle t \sum_{n=0}^{M} \|\nabla \mathbf{w}_{n+1/2}^h\|^2 \right) \\
\leq \;& C\nu^{-1} h^{2k} \left(\; |||\mathbf{u}|||_{4,k+1}^4 + \nu^{-1} (\|\mathbf{w}_0^h\|^2 + \nu^{-1} |||\mathbf{f}|||_{2,\star}^2) \; \right) . \qquad (3.30)
\end{aligned}
$$

Lemma 3.1 and (2.5) give

$$
\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C\nu^{-1} \|p_{n+1/2} - q\|^2 \;\leq\; & \triangle t C\nu^{-1} \sum_{n=0}^{M-1} \left(\; \|p(t_{n+1/2}) - q\|^2 + \|p_{n+1/2} - p(t_{n+1/2})\|^2 \; \right) \\
\leq\; & C\nu^{-1} \; (h^{2s+2} \triangle t \sum_{n=0}^{M-1} \|p(t_{n+1/2})\|_{s+1}^2 \\
& + \triangle t \sum_{n=0}^{M-1} \frac{1}{48} (\triangle t)^3 \int_{t_n}^{t_{n+1}} \|p_{tt}\|^2 \; dt \\
\leq\; & C\nu^{-1} (h^{2s+2} |||p_{1/2}|||_{2,s+1}^2 + (\triangle t)^4 \|p_{tt}\|_{2,0}^2). \qquad (3.31)
\end{aligned}
$$

Next, we bound the time relaxation term

$$
\triangle t \sum_{n=0}^{M-1} \nu^{-1} \; \chi^2 \mu^2 \delta^4 \quad \left\| \nabla \boldsymbol{\eta}_{n+1/2}^h \; \right\|^2 \;\leq\; \nu^{-1} \; \chi^2 \mu^2 \delta^4 \; h^{2k} |||\mathbf{u}|||_{2,k+1}^2 \, . \qquad (3.32)
$$

We now bound the terms in $Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}^h_{n+1/2})$. Using Cauchy-Schwarz and Young's inequalities, Taylor's theorem, and Lemma 2.7,

$$\begin{aligned}
&\left(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2}\right)\\
\leq\ & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1/2}\|^2 + \frac{1}{2}\|\frac{\mathbf{u}^{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\|^2\\
\leq\ & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1}\|^2 + \frac{1}{2}\|\boldsymbol{\phi}^h_{n}\|^2 + \frac{1}{2}\frac{(\Delta t)^3}{1280}\int_{t_n}^{t_{n+1}} \|\mathbf{u}_{ttt}\|^2\, dt\,,
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
&(p_{n+1/2} - p(t_{n+1/2}), \nabla\cdot\boldsymbol{\phi}^h_{n+1/2})\\
\leq\ & \varepsilon_1\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\,\nu^{-1}\|p_{n+1/2} - p(t_{n+1/2})\|^2\\
\leq\ & \varepsilon_1\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\,\nu^{-1}\frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|p_{tt}\|^2\, dt\,,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&(\mathbf{f}(t_{n+1/2}) - \mathbf{f}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2})\\
\leq\ & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1/2}\|^2 + \frac{1}{2}\|\mathbf{f}(t_{n+1/2}) - \mathbf{f}_{n+1/2}\|^2\\
\leq\ & \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1}\|^2 + \frac{1}{2}\|\boldsymbol{\phi}^h_{n}\|^2 + \frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|\mathbf{f}_{tt}\|^2\, dt\,,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&(\nabla\mathbf{u}_{n+1/2} - \nabla\mathbf{u}(t_{n+1/2}), \nabla\boldsymbol{\phi}^h_{n+1/2})\\
\leq\ & \varepsilon_2\nu\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\,\nu\|\nabla\mathbf{u}_{n+1/2} - \nabla\mathbf{u}(t_{n+1/2})\|^2\\
\leq\ & \varepsilon_2\nu\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2 + C\,\nu\frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|\nabla\mathbf{u}_{tt}\|^2\, dt\,,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
&b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) - b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2})\\
=\ & b^*(\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}), \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) + b^*(\mathbf{u}(t_{n+1/2}), \mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}), \boldsymbol{\phi}^h_{n+1/2})\\
\leq\ & C\,\|\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(t_{n+1/2}))\|\,\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|\,\left(\|\nabla\mathbf{u}_{n+1/2}\| + \|\nabla\mathbf{u}(t_{n+1/2})\|\right)\\
\leq\ & C\,\nu^{-1}\left(\|\nabla\mathbf{u}_{n+1/2}\|^2 + \|\nabla\mathbf{u}(t_{n+1/2})\|^2\right)\frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|\nabla\mathbf{u}_{tt}\|^2\, dt + \varepsilon_3\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2\\
\leq\ & C\,\nu^{-1}\frac{(\Delta t)^3}{48}\left(\int_{t_n}^{t_{n+1}} 2(\|\nabla\mathbf{u}_{n+1/2}\|^4 + \|\nabla\mathbf{u}(t_{n+1/2})\|^4)\, dt\right.\\
&\quad\left. + \int_{t_n}^{t_{n+1}} \|\nabla\mathbf{u}_{tt}\|^4\, dt\right) + \varepsilon_3\nu\|\nabla\phi^h_{n+1/2}\|^2\\
\leq\ & C\,\nu^{-1}\,(\Delta t)^4(\|\nabla\mathbf{u}_{n+1/2}\|^4 + \|\nabla\mathbf{u}(t_{n+1/2})\|^4)\\
&\quad + C\,\nu^{-1}\,(\Delta t)^3\int_{t_n}^{t_{n+1}} \|\nabla\mathbf{u}_{tt}\|^4\, dt + \varepsilon_3\nu\|\nabla\boldsymbol{\phi}^h_{n+1/2}\|^2\,.
\end{aligned} \tag{3.37}$$

$$
\begin{aligned}
& b^*(\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h, \mathbf{u}_{n+1/2}, \boldsymbol{\phi}^h_{n+1/2}) \\
\leq\ & \frac{1}{2}\Big(\|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \, \|\nabla \mathbf{u}_{n+1/2}\|_\infty \, \|\boldsymbol{\phi}^h_{n+1/2}\| \\
& + \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \, \|\mathbf{u}_{n+1/2}\|_\infty \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \Big) \\
\leq\ & C \, \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \, \|\nabla \boldsymbol{\phi}^h_{n+1/2}\| \\
\leq\ & \varepsilon_4 \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1} \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\|^2 \\
\leq\ & \varepsilon_4 \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + C\nu^{-1} \Big(\ (\mu\delta^2 h^{2k} + h^{2k+2}) \left\| D_\mu \overline{\mathbf{u}_{n+1/2}} \right\|^2_{k+1} \\
& + \ (\delta^2 h^{2k} + h^{2k+2}) \left\| \mathbf{u}_{n+1/2} \right\|^2_{k+1} + \mu^2 \delta^4 \left\| \mathbf{u}_{n+1/2} \right\|^2_2 \ \Big) .
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
& \chi(\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h, \boldsymbol{\phi}^h_{n+1/2}) \leq \chi \left\| \boldsymbol{\phi}^h_{n+1/2} \right\| \|\mathbf{u}_{n+1/2} - D^h_\mu \overline{\mathbf{u}_{n+1/2}}^h\| \\
& \leq \frac{1}{2}\|\boldsymbol{\phi}^h_{n+1}\|^2 \ + \ \frac{1}{2}\|\boldsymbol{\phi}^h_{n}\|^2 \ + \chi^2 C \Big(\ (\mu\delta^2 h^{2k} + h^{2k+2}) \left\| D_\mu \overline{\mathbf{u}_{n+1/2}} \right\|^2_{k+1} \\
& \quad + \ (\delta^2 h^{2k} + h^{2k+2}) \left\| \mathbf{u}_{n+1/2} \right\|^2_{k+1} + \mu^2 \delta^4 \left\| \mathbf{u}_{n+1/2} \right\|^2_2 \ \Big) .
\end{aligned} \tag{3.39}
$$

Combine (3.33)-(3.39) to obtain

$$
\begin{aligned}
\sum_{n=0}^{M-1} \ \Delta t |Intp(\mathbf{u}_n, p_n; \boldsymbol{\phi}^h_{n+1/2})| \ \ \leq \ \ & \sum_{n=0}^{M-1} \ [\ \Delta t \, C \|\boldsymbol{\phi}^h_{n+1}\|^2 \\
& + (\varepsilon_1 \ + \ \varepsilon_2 \ + \ \varepsilon_3 \ + \ \varepsilon_4) \Delta t \, \nu \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \\
& + C\nu^{-1} \Big(\ (\mu\delta^2 h^{2k} + h^{2k+2}) \left\| D_\mu \overline{\mathbf{u}_{n+1/2}} \right\|^2_{k+1} \\
& \quad + \ (\delta^2 h^{2k} + h^{2k+2}) \left\| \mathbf{u}_{n+1/2} \right\|^2_{k+1} + \mu^2 \delta^4 \left\| \overline{\mathbf{u}_{n+1/2}} \right\|^2_2 \ \Big) \\
& + C\chi^2 \Big(\ (\mu\delta^2 h^{2k} + h^{2k+2}) \left\| D_\mu \overline{\mathbf{u}_{n+1/2}} \right\|^2_{k+1} \\
& \quad + \ (\delta^2 h^{2k} + h^{2k+2}) \left\| \mathbf{u}_{n+1/2} \right\|^2_{k+1} + \mu^2 \delta^4 \left\| \overline{\mathbf{u}_{n+1/2}} \right\|^2_2 \ \Big) \ ] \\
& + C(\Delta t)^4 \big(\|\mathbf{u}_{ttt}\|^2_{2,0} + \nu^{-1} \|p_{tt}\|^2_{2,0} + \|\mathbf{f}_{tt}\|^2_{2,0} \\
& \qquad + \nu \|\nabla \mathbf{u}_{tt}\|^2_{2,0} \ + \nu^{-1} \|\nabla \mathbf{u}_{tt}\|^4_{4,0} \\
& \qquad + \nu^{-1} \||\nabla \mathbf{u}|\|^4_{4,0} \ + \nu^{-1} \||\nabla \mathbf{u}_{1/2}|\|^4_{4,0} \big) \ .
\end{aligned} \tag{3.40}
$$

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1/12$ and putting everything together, from (3.27) we obtain

$$
\begin{aligned}
&\|\boldsymbol{\phi}^h_M\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \chi \triangle t \sum_{n=0}^{M-1} \|{\boldsymbol{\phi}^h_{n+1/2}}^*\|^2 \\
\leq \quad & \triangle t \sum_{n=0}^{M-1} C\,(\nu^{-3} \left\|\nabla \mathbf{u}_{n+1/2}\right\|^4 + 1) \left\|\boldsymbol{\phi}^h_{n+1/2}\right\|^2 \\
& + C\ \nu\, h^{2k}\, |||\mathbf{u}|||^2_{2,k+1} + C\,\nu^{-1} h^{2k+1} \left(|||\mathbf{u}|||^4_{4,k+1} + |||\nabla \mathbf{u}|||^4_{4,0} \right) \\
& + C\ \nu^{-1} h^{2k} \left(|||\mathbf{u}|||^4_{4,k+1} + \nu^{-1}(\|\mathbf{w}^h_0\|^2 + \nu^{-1}|||\mathbf{f}|||^2_{2,\star}) \right) \\
& + C\ \nu^{-1} \left(h^{2s+2}|||p_{1/2}|||^2_{2,s+1} + (\triangle t)^4 \|p_{tt}\|^2_{2,0} \right) + C\ \nu^{-1}\chi^2\mu^2\delta^4\, h^{2k}|||\mathbf{u}|||^2_{2,k+1} \\
& + C\,(\nu^{-1} + \chi^2) \Big((\mu\delta^2 h^{2k} + h^{2k+2}) \|D_\mu \overline{\mathbf{u}}\|^2_{2,k+1} \\
& \qquad + (\delta^2 h^{2k} + h^{2k+2}) \|\mathbf{u}\|^2_{2,k+1} + \mu^2\delta^4 \|\mathbf{u}\|^2_{2,2} \Big) \\
& + C(\Delta t)^4 \big(\|\mathbf{u}_{ttt}\|^2_{2,0} + \|\mathbf{f}_{tt}\|^2_{2,0} + \nu\|\nabla \mathbf{u}_{tt}\|^2_{2,0} \\
& \qquad + \nu^{-1}\|\nabla \mathbf{u}_{tt}\|^4_{4,0} + \nu^{-1}|||\nabla \mathbf{u}|||^4_{4,0} + \nu^{-1}|||\nabla \mathbf{u}_{1/2}|||^4_{4,0} \big)\ . \qquad (3.41)
\end{aligned}
$$

Hence, with $\triangle t$ sufficiently small, i.e. $\triangle t < C(\nu^{-3}|||\nabla \mathbf{u}|||^4_{\infty,0} + 1)^{-1}$, from Gronwall's Lemma (see Lemma 3.2), we have

$$
\begin{aligned}
&\|\boldsymbol{\phi}^h_M\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 + \chi \triangle t \sum_{n=0}^{M-1} \|{\boldsymbol{\phi}^h_{n+1/2}}^*\|^2 \\
&\leq C^* \,\{\ h^{2k+2}\,(\nu^{-1} + \chi^2)(\ \|D_\mu \overline{\mathbf{u}}\|^2_{k+1} + \|\mathbf{u}\|^2_{k+1}\ ) \\
&\quad + \nu^{-1}\, h^{2k+1} \left(|||\mathbf{u}|||^4_{4,k+1} + |||\nabla \mathbf{u}|||^4_{4,0} \right) + h^{2k}(\nu + \nu^{-1}\ \chi^2\mu^2\delta^4\ )|||\mathbf{u}|||^2_{2,k+1} \\
&\quad + \nu^{-1} h^{2k} \left(|||\mathbf{u}|||^4_{4,k+1} + \nu^{-1}(\|\mathbf{w}^h_0\|^2 + \nu^{-1}|||\mathbf{f}|||^2_{2,\star}) \right) \\
&\quad + \delta^2 h^{2k} (\nu^{-1} + \chi^2) \left(\mu \|D_\mu \overline{\mathbf{u}}\|^2_{2,k+1} + \|\mathbf{u}\|^2_{2,k+1} \right) \\
&\quad \nu^{-1} h^{2s+2} |||p_{1/2}|||^2_{2,s+1} + (\nu^{-1} + \chi^2)\mu^2\delta^4 \|\mathbf{u}\|^2_{2,2} \\
&\quad + (\Delta t)^4 \left(\|\mathbf{u}_{ttt}\|^2_{2,0} + \nu^{-1}\|p_{tt}\|^2_{2,0} + \|\mathbf{f}_{tt}\|^2_{2,0} + \nu\|\nabla \mathbf{u}_{tt}\|^2_{2,0} \right. \\
&\quad \left. + \nu^{-1}\|\nabla \mathbf{u}_{tt}\|^4_{4,0} + \nu^{-1}|||\nabla \mathbf{u}|||^4_{4,0} + \nu^{-1}|||\nabla \mathbf{u}_{1/2}|||^4_{4,0} \right)\ \} \qquad (3.42)
\end{aligned}
$$

where $C^* = C \exp(C\nu^{-3}T)$.

Estimate (3.14) then follows from the triangle inequality and (3.42).

To obtain (3.15), we use (3.42) and

$$
\begin{aligned}
& \|\nabla \left(\mathbf{u}(t_{n+1/2}) - (\mathbf{w}^h_{n+1} + \mathbf{w}^h_n)/2 \right) \|^2 \\
\leq \quad & \|\nabla(\mathbf{u}(t_{n+1/2}) - \mathbf{u}_{n+1/2})\|^2 + \|\nabla \boldsymbol{\eta}_{n+1/2}\|^2 + \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2 \\
\leq \quad & \frac{(\triangle t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla \mathbf{u}_{tt}\|^2\, dt + Ch^{2k}||\mathbf{u}_{n+1}||^2_{k+1} + Ch^{2k}||\mathbf{u}_n||^2_{k+1} + \|\nabla \boldsymbol{\phi}^h_{n+1/2}\|^2\,.
\end{aligned}
$$

□

4 Numerical Illustrations

In this section, we present two numerical experiments. Our first test confirms the predicted rates of convergence. In our second experiment, we study a simple, under-resolved flow with recirculation: the flow across a step. The computations were performed with the software FreeFem++, see [14].

4.1 Convergence Rate Verification

To test the predicted convergence rates, we consider the Chorin vortex decay problem, [7, 26, 38]. The prescribed solution in $\Omega = (0,1) \times (0,1)$ is

$$\begin{aligned} u_1(x,y,t) &= -\cos(n\pi x)\sin(n\pi y)e^{-2n^2\pi^2 t/\tau} \\ u_2(x,y,t) &= \sin(n\pi x)\cos(n\pi y)e^{-2n^2\pi^2 t/\tau} \\ p &= -\frac{1}{4}(\cos(n\pi x)+\cos(n\pi y))e^{-2n^2\pi^2 t/\tau}. \end{aligned}$$

When the relaxation time $\tau = Re$, the pair $(\mathbf{u}, p)$ defined above is a solution of the NSE with $\mathbf{f} = 0$. This solution consists of an $n \times n$ array of oppositely signed vortices that decay as $t \to \infty$.

The model was discretized in time with the implicit second order Crank-Nicolson scheme and in space with the Taylor-Hood finite element method, i.e. the velocity was approximated by continuous piecewise quadratics and the pressure by continuous piecewise linears. Theorem 3.2 shows that under sufficient regularity of the solution of the variational problem, the error $||\nabla(\mathbf{u}-\mathbf{w}^h)||_{2,0}$ is of $O(h^2)$. This experiment suggests that it might be possible to establish an upper bound for the error $||\mathbf{u}-\mathbf{w}^h||_{2,0}$ as follows

$$||\mathbf{u}-\mathbf{w}^h||_{2,0} \le C\ h\ ||\nabla(\mathbf{u}-\mathbf{w}^h)||_{2,0},$$

i.e., $||\mathbf{u}-\mathbf{w}^h||_{2,0} \simeq O(h^3)$. Generally, Nitsche's duality trick is employed to derive error estimates for L^2 norms. This is often referred to as the L^2 lift, [4]. In our test, we choose $n = 1$, $dt = 0.005$, $T = 0.5$, $\mu = 1/m$, $\chi = 0.1$, $\delta = \sqrt{1/m}$ and $h = 1/m$, where m is the number of subdivisions of the interval $(0, 1)$. We performed the same test for different Reynolds numbers. The results are in Tables 1 and 2 below. In both cases, $Re = 1$ and $Re = 10^4$, the convergence rate approaches the second order predicted for $||\nabla(\mathbf{u}-\mathbf{w}^h)||_{2,0}$. For $Re = 10^4$ we also see what appears to be an L^2 lift for $\left\|\mathbf{u}-\mathbf{w}^h\right\|_{2,0}$.

Note that the error plateau around 10^{-5} already on coarse mesh. This is likely related to the stopping criteria ($\mathbf{u}_{residual} < 10^{-5}$) or the $O(10^{-5})$ stabilization used in the (2,2) block of the linear Stokes system for the solver used.

mesh	$\|\|\mathbf{u}-\mathbf{w}^h\|\|_{2,0}$	ratio	$\|\|\nabla(\mathbf{u}-\mathbf{w}^h)\|\|_{2,0}$	ratio
10×10	0.000316786	-	0.00461599	-
20×20	$6.66405 \cdot 10^{-5}$	2.25	0.00116309	1.98
30×30	$5.31594 \cdot 10^{-5}$	0.6	0.000522003	1.97
40×40	$5.16375 \cdot 10^{-5}$	0.1	0.000300615	1.91

Table 1: Errors and convergence rates for the Leray-Tikhonov model at $Re = 1$

mesh	$\|\|\mathbf{u}-\mathbf{w}^h\|\|_{2,0}$	ratio	$\|\|\nabla(\mathbf{u}-\mathbf{w}^h)\|\|_{2,0}$	ratio
10×10	0.0226085	-	1.35783	-
20×20	0.00428244	2.40	0.502447	1.43
30×30	0.00131237	2.91	0.23989	1.82
40×40	0.000531236	3.14	0.131774	2.08

Table 2: Errors and convergence rates for the Leray-Tikhonov model at $Re = 10^4$

4.2 Step Problem

In our second test, we consider a flow in transition via shedding of eddies behind the step. At a critical Reynolds number, for which the flow should be time dependent, some models are not able to capture the correct (non stationary) physical properties of the flow, e.g., [27]. Herein, we present results for a parabolic inflow profile, which is given by $\mathbf{u} = (u_1, u_2)^T$, with $u_1 = y(10-y)/25$ and $u_2 = 0$. No-slip boundary condition is prescribed on the top and bottom boundary as well as on the step. At the outflow we have "do nothing" boundary condition, an accepted outflow condition in CFD. The model was discretized in time with the Crank-Nicolson and in space with the Taylor Hood finite-element method. Figure 3 shows that expected behavior of the flow: behind the step the flow simulation correctly develops vortices separate from the step. Figure 3 shows the results for a course mesh (4585 degrees of freedom) at $T = 10, 20, 30, 40$, for $\nu^{-1} = 750$, $\chi = 0.01$, $\mu = 0.01$, $dt = 0.0025$, $\delta = 1.5$.

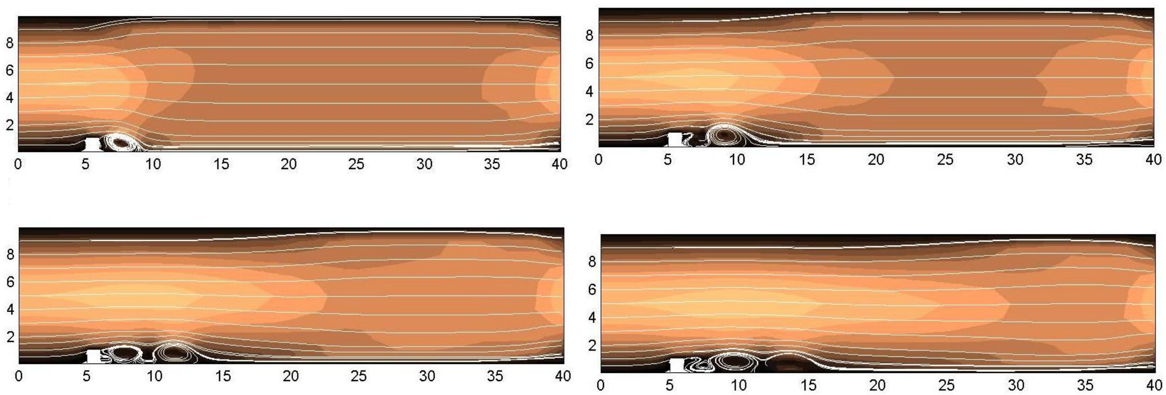

Figure 3: Leray-Tikhonov deconvolution model: Flow field at T = 10, 20, 30, 40.

5 Conclusions

Numerical simulation of complex flows presents many challenges. Often, simulations are based on various regularizations of the NSE rather than the NSE themselves. The resulting models have remarkable and positive effects on computation results: errors are observed to be much better over much larger time intervals and the transition from one type of flow to another is not retarded.

Herein, we developed and studied a new family of such NSE-regularizations, based on a modification of Tikhonov-Lavrentiev regularization process for ill-posed problems. With this method, we obtained an approximation to the unfiltered solution by one filtering step. Using the differential filter (2.7), the error in approximation is $\mathbf{u} - D_\mu \overline{\mathbf{u}} = O(\mu\delta^2)$. We studied a fully discrete algorithm for the model, Crank-Nicolson in time and finite element in space. We have given a numerical analysis for the scheme and included proofs of unconditional stability and solvability. We have also given a convergence analysis which was also verified in our numerical computations.

Acknowledgement

We thank Professor William Layton for the idea, all the support and help. We thank Dr. Myron Sussman for suggesting the modified version of Tikhonov-Lavrentiev operator and for all his constructive comments when writing this paper.

References

[1] N. A. Adams and S. Stolz, *Deconvolution methods for subgrid-scale approximation in large eddy simulation*, Modern Simulation Strategies for Turbulent Flow, R.T. Edwards, 2001.

[2] M. Bertero and B. Boccacci, *Introduction to Inverse Problems in Imaging*, IOP Publishing Ltd., Bristol, 1998.

[3] L. C. Berselli, T. Iliescu, and W. Layton, *Mathematics of Large Eddy Simulation*, Springer, Berlin, 2005.

[4] S. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994.

[5] A. Cheskidov, D.D. Holm, E. Olson and E.S. Titi, *On a Leray- model of turbulence,* Royal Society London, Proceedings, Series A, Mathematical, Physical and Engineering Sciences, 461, 2005, 629-649.

[6] V.V.Chepyzhov, E.S. Titi, and M.I. Vishik, *On the convergence of the Leray-alpha model to the trajectory attractor of the 3d Navier-Stokes system*, to appear: Journal of Discrete and Continuous Dynamical Systems, 2007.

[7] A. J. Chorin, *Numerical solution for the Navier-Stokes equations,* Math. Comp., 22 (1968) 745-762.

[8] A. Dunca, V.John, W. Layton *The Commutation Error of the Space Averaged Navier-Stokes Equations on a Bounded Domain* , Advances in Mathematical Fluid Mechanics 3, Birkhuser Verlag Basel, (2004), 53 - 78.

[9] A. Dunca, *Optimal design of fluid flow using subproblems reduced by large eddy simulation*, Preprint ANL/MCS-p1117-1003, 2003.

[10] A. Dunca , *Space avereged Navier-Stokes equations in the presence of walls* , Ph.D. Thesis, University of Pittsburgh. 2004.

[11] A. Dunca and Y. Epshteyn, *On the Stolz-Adams Deconvolution Model for the Large-Eddy Simulation of Turbulent Flows*, SIAM J. Math. Anal. (2006), 1890-1902.

[12] C. Foias, *What do the Navier-Stokes equations tell us about turbulence?* Contemporary Mathematics, 208 (1997), 151-180.

[13] C. Foias, D. D. Holm and E. S. Titi, *The Navier-Stokes-alpha model of fluid turbulence*, Physica D, 152-153(2001), 505-519.

[14] F. Hecht, O. Pironneau, and K. Ohtsuka, *FreeFem++*, webpage: http://www.freefem.org.

[15] G. P. Galdi, *An introduction to the Mathematical Theory of the Navier-Stokes equations*, Springer, Berlin, 1994.

[16] M.D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms*, Academic Printers, 1989.

[17] V. Girault and P.A. Raviart, *Finite Element approximation of the Navier-Stokes equations*, Springer-Verlag, 1979.

[18] B.J. Geurts, *Inverse modeling for large-eddy simulation*, Physics of Fluids, 9 (1997), 3585-3587.

[19] G. P. Galdi and W. J. Layton, *Approximation of the large eddies in fluid motion II: A model for space-filtered flow*, Math. Models and Methods in the Appl. Sciences, 10(2000), 343-350.

[20] M. Germano, *Differential filters of elliptic type*, Phys. Fluids, 29(1986), 1757-1758.

[21] B.J. Geurts and D.D. Holm, *Leray and LANS-alpha modeling of turbulent mixing*, J. of Turbulence, 7(2006), 1-33.

[22] B.J. Geurts and D.D. Holm, *Regularization modeling for large eddy simulation*, Physics of Fluids, 15(2003), L13 L16.

[23] B.J. Geurts, *Elements of direct and large eddy simulation*, Edwards Publishing, 2003.

[24] J.G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem., Part IV: Error analysis for the second order time discretization,* SIAM J. Numer. Anal., 2, 1990, 353-384.

[25] A.A. Ilyin, E.M. Lunasin and E.S. Titi, *A modified Leray-α subgrid-scale model of turbulence*, Nonlinearity, 19 (2006), No. 4, 879-897.

[26] V. John, W. Layton, *Analysis of Numerical Errors in Large Eddy Simulation*, SIAM J. Numer. Anal., 40 (2002) 995-1020.

[27] V. John, W. Layton, and N. Sahin, *Derivation and Analysis of Near Wall Models for Channel and Recirculating Flows*, Computers and Mathematics with Applications 48 (2004) 1135-1151.

[28] J.G.M. Kuerten, B.J.Geurts, A.W. Vreman and M.Germano, *Dynamic inverse modelling and its testing in LES of the mixing layer*, Phys. Fluids, 11 (1999), 3778-3785.

[29] J. Leray, *Sur le movement d'un fluide visqueux emplissant l'espace*, Acta Math, 63: 193-248, 1934, Kluwer Academic Publishers, 1997.

[30] W. Layton and R. Lewandowski, *A simple and stable scale similarity model for large eddy simulation: energy balance and existence of weak solutions*, Applied Math. letters 16(2003) 1205-1209.

[31] W. Layton and R. Lewandowski, *Residual stress of approximate deconvolution models of turbulence*, Journal of Turbulence 7 (2006), 1-21.

[32] W. Layton, *Introduction to the Numerical Analysis of Incompressible, Viscous Flows*, SIAM Comp. Sci. and Engr. 6, 2007.

[33] W. Layton and M. Neda, *Truncation of scales by time relaxation*, JMAA 325 (2006), 788-807.

[34] C. Manica and S. Kaya Merdan, *Convergence analysis of the finite element method for a fundamental model in turbulence*, Technical Report, University of Pittsburgh, 2006.

[35] P. Sagaut, *Large eddy simulation for Incompressible flows,* Springer, Berlin, 2001.

[36] I. Stanculescu, *Existence Theory of Abstract Approximate Deconvolution Models of Turbulence,* Annali dell'Universita' di Ferrara, Springer Milan, 54 (2008), 145-168 .

[37] S. Stolz and N.A. Adams, and L.Kleiser, *The approximate deconvolution model for large-eddy simulations of compressible flows and its application to shock-turbulent-boundary-layer interaction* Physics of Fluids, 13 (2001), 997-1015.

[38] D. Tafti, *Comparison of some upwind-biased high-order formulations with a second order central-difference scheme for time integration of the incompressible Navier-Stokes equations*, Comput. & Fluids, 25 (1996) 647-665.

[39] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, SIAM, Philadelphia, 1995.

[40] A.N. Tikhonov and V.Y. Arsenin, *Solutions of ill-posed problem*, Winston, New York, 1977.

[41] M.I. Vishik, E.S. Titi and V.V. Chepyzhov, *Trajectory attractor approximations of the* $3d$ *Navier-Stokes system by the Leray-*α *model*, Russian Math Dokladi, 71(2005), 91-95.

[42] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Springer-Verlag, New York, 1995.