Numerical Analysis and Computational Comparisons of the NS-alpha and NS-omega regularizations

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Abstract

We study stability, accuracy and efficiency of algorithms for a new regularization of the NSE, the NS-$\overline{\omega}$ model (in which the vorticity term, $\overline{\omega} = \nabla \times \overline{u}$, is averaged) given by

$$u_t - u \times (\nabla \times \overline{u}) + \nabla q - \nu \Delta u = f, \nabla \cdot u = 0.$$

This is similar to the NS-alpha model (in which the nonlinear term is $\overline{u} \times \nabla \times u$), but the small difference opens attractive algorithmic possibilities. We give tests both confirming the predicted rates of convergence and exhibiting some shared limitations of both models. The experiments also show the discrete NS-$\overline{\omega}$ simulation has greater accuracy at much less cost than a comparable NS-α simulation. The experiments suggest consideration of higher accuracy NS-$\overline{\omega}$-deconvolution models as a next logical step.

Key words. NS-α model, finite element method, large eddy simulation
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1 Introduction

This report presents a numerical analysis and computational testing of a (to our limited knowledge) new regularization of the Navier-Stokes equations (NSE) that we call the NS-omega (NS-$\overline{\omega}$) model of turbulent flow. The regularization (1.1)-(1.3) below is motivated by a desire for efficient, accurate, and reliable under-resolved simulations of flow problems and has evolved from our work on algorithms for, e.g., the NS-α model, the rational model and deconvolution models. The NS-$\overline{\omega}$ model is derived from the rotational form of the NSE by filtering, with the differential filter / rational filter / Helmholtz filter, the second term of the nonlinearity (for reasons explained below)

$$u_t - u \times (\nabla \times \overline{u}) + \nabla q - \nu \Delta u = f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$-\alpha^2 \Delta \overline{u} + \overline{u} = u. \tag{1.3}$$

NS-$\overline{\omega}$ is a complement to the NS-α model (1.4)-(1.6) next. The NS-α / viscous Camassa-Holm model (1.4)-(1.6) is a recently developed regularization of the NSE with desirable

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mathematical properties, e.g., [FHT01], [FHT02], [GOP03]. It is given by

$$u_t - \overline{u} \times (\nabla \times u) + \nabla q - \nu \Delta u = f, \tag{1.4}$$
$$\nabla \cdot \overline{u} = 0, \tag{1.5}$$
$$-\alpha^2 \Delta \overline{u} + \overline{u} = u. \tag{1.6}$$

We consider the NS-$\overline{\omega}$ model (1.1)-(1.3) either for internal flow, i.e. for a bounded, polyhedral domain in $\mathbb{R}^d, d = 2, 3$, subject to no-slip boundary conditions, or for $\Omega = (0, L)^d$ and subject to L periodic boundary conditions with zero mean imposed on the solution and all data.

Comparing (1.1)-(1.3) to (1.4)-(1.6), the NS-$\overline{\omega}$ model averages the vorticity term $\omega = \nabla \times u$ rather than the velocity term in the nonlinearity. (Hence calling it the NS-$\overline{\omega}$ regularizations seems descriptive.) This modification is similar in form to that of the modified Leray model, [ILT05] but it differs in motivation and consequence.

The difference (in the non-periodic case) between the imposition of incompressibility in (1.2) and (1.5) arises from filtering the different terms in the nonlinearity in (1.1) and (1.4). The choices (1.2) and (1.5) are necessary for model stability ($\nu > 0$) and energy conservation ($\nu = 0$).

In this report, we compare variants of Crank-Nicolson (CN) finite element method (FEM) algorithms for NS-$\overline{\omega}$ and NS-α. In Section 3, we study CN algorithms for both models (necessary notation and preliminaries are presented in Section 2). We give a numerical analysis for the CN scheme for NS-$\overline{\omega}$ that includes proofs of unconditional stability and solvability, as well as convergence. A similar analysis for the CN scheme for NS-α was performed in [Con08] and in [Cag03] for a reformulated method. We also study a second order, unconditionally stable, linearly implicit variant (CNLE) of CN for NS-$\overline{\omega}$. In Section 4, we present numerical experiments for the schemes. Here we confirm predicted convergence rates, and compare runtimes and errors for the CN schemes for both NS-α and NS-$\overline{\omega}$. We also consider linearly and quadratically extrapolated CN schemes for NS-$\overline{\omega}$; we discuss in Section 3 why such linearizations lose their strong unconditional stability for NS-α. In tests in Section 4, significantly better runtimes and errors are found for the NS-$\overline{\omega}$ schemes.

Lastly, we present experiments for the well known two dimensional flow around a cylinder problem. It is our view that understanding the limitations of a model or algorithm is often more important than reporting yet another similar success. This test problem is a difficult one since it is near a critical Reynolds number Re for vortex shedding. Often, regularizations produce solutions which resemble lower Re solutions (i.e., without shedding). Although both NS-α and NS-$\overline{\omega}$ are dispersive regularizations, they share this limitation.

1.1 Motivation for NS-$\overline{\omega}$

Our motivation for studying the modification (1.1)-(1.3) from (1.4)-(1.6) is the search for efficient, unconditionally stable and (at least second order) accurate methods for the simulation of under-resolved flows. The classic beginning point for such methods, studied by Caglar [Cag03] and Connors [Con08] for NS-α, is a variational discretization in space plus the Crank-Nicolson method in time. Suppressing spatial discretization and letting

$u^{n+1/2} := (u^{n+1} + u^n)/2$, for NS-$\alpha$ (1.4)-(1.6) it is given by

$$\frac{u^{n+1} - u^n}{\triangle t} - w^{n+\frac{1}{2}} \times \nabla \times u^{n+\frac{1}{2}} - \nu \triangle u^{n+\frac{1}{2}} + \nabla p^{n+\frac{1}{2}} = f^{\,n+\frac{1}{2}}, \tag{1.7}$$

$$\nabla \cdot \overline{u}^{n+\frac{1}{2}} = 0\ , \tag{1.8}$$

$$-\alpha^2 \Delta w^{n+\frac{1}{2}} + w^{n+\frac{1}{2}} = u^{n+\frac{1}{2}}. \tag{1.9}$$

This is second order accurate for NS-α and nonlinearly, unconditionally stable (take the inner product of the discrete equations with $w^{n+\frac{1}{2}}$). However, in finite element spaces with N_V velocity degrees of freedom and N_P pressure degrees of freedom, *the method leads to a large, nonlinear system at each time step with $2N_V + N_P$ total degrees of freedom.* There does not appear to be any method for NS-α which preserves these attractive properties and has significantly less complexity.

On the other hand, because of the structure of the nonlinearity, the NS-$\overline{\omega}$ model (1.1)-(1.3) admits simple methods which are nonlinearly, unconditionally stable, second order accurate, *linearly implicit* (only 1 linear system per time step) and require only $N_V + N_P$ total degrees of freedom. Herein we study both the full Crank-Nicolson method for NS-$\overline{\omega}$ as well as the following attractive variant CNLE (Crank-Nicolson with linear extrapolation, known for the NSE at least since Baker's 1976 paper [B76]). Let $u^{n+1/2} := (u^{n+1} + u^n)/2$ and $U^n := \frac{3}{2}u^n - \frac{1}{2}u^{n-1}$

$$\frac{u^{n+1} - u^n}{\triangle t} - u^{n+\frac{1}{2}} \times \nabla \times \overline{U^n} - \nu \triangle u^{n+\frac{1}{2}} + \nabla p^{n+\frac{1}{2}} = f^{\,n+\frac{1}{2}}, \text{ and } \nabla \cdot u^{n+\frac{1}{2}} = 0\ . \tag{1.10}$$

We prove unconditional stability (take the inner product with $u^{n+\frac{1}{2}}$). It is clearly second order accurate and linearly implicit (since $U^n := \frac{3}{2}u^n - \frac{1}{2}u^{n-1}$ is known from previous time levels). Further, since $U^n := \frac{3}{2}u^n - \frac{1}{2}u^{n-1}$ is known, its average can be directly computed, uncoupled from the linear equations for advancing in time. We also discuss the Crank Nicolson quadratic extrapolation (CNQE) in Section 2 and give experiments for it in Section 4, where U^n is extrapolated quadratically as $U^n := \frac{15}{8}u^n - \frac{10}{8}u^{n-1} + \frac{3}{8}u^{n-2}$.

The NS-$\overline{\omega}$ regularization (1.1)-(1.3) exactly conserves (under periodic boundaries and no viscous or external forces) the two key integral invariants for three dimensional fluid flow (note $\|\cdot\|$ and $(\cdot,\cdot)$ denote the usual L^2 norm and inner product): energy, $\frac{1}{2}\|u\|^2$, and (model) helicity, $(u, \nabla \times \overline{u})$, Proposition 1.1 below. Helicity is a fundamentally important quantity in turbulent fluid flow. It is a scalar quantity describing *rotation* that is conserved by the Euler equations, cascaded through the inertial range in homogeneous, isotropic turbulence, and provides important information on the topology of vortex filaments [MT92]. Helicity density has been used as an indicator of coherent structures, and moreover, as an indicator of low turbulent dissipation. The Pythagorean identity gives

$$\frac{(u \cdot \nabla \times u)^2 + |u \times \nabla \times u|^2}{|u|^2\,|\nabla \times u|^2} = 1, \tag{1.11}$$

which can be interpreted as

$$\frac{\text{helicity}^2 + \text{nonlinearity}^2}{\text{energy} \cdot \text{enstrophy}} = 1. \tag{1.12}$$

Hence at constant energy and enstrophy, high helicity implies small nonlinearity, and conversely. The interaction between model's treatment of energy and helicity impacts its predictions of a flow's rotational structures.

The L^2 norm and inner product are $\|\cdot\|$ and $(\cdot,\cdot)$.

Proposition 1.1. *In the absence of viscosity and external forces ($\nu = f = 0$), and for periodic boundary conditions, the NS-$\overline{\omega}$ model (1.1)-(1.3) conserves energy, $\frac{1}{2}\|u\|^2$, and a model helicity, $(u, \nabla \times \overline{u})$: For $t > 0$,*

$$\frac{1}{2}\|u(t)\|^2 = \frac{1}{2}\|u(0)\|^2, \tag{1.13}$$

$$(u(t), \nabla \times \overline{u(t)}) = (u(0), \nabla \times \overline{u(0)}). \tag{1.14}$$

Proof. The proof for each result begins by setting $\nu = f = 0$ in (1.1). For energy, multiply by u and integrate over Ω. The nonlinearity and pressure term vanish. This leaves only $\frac{1}{2}\frac{d}{dt}\|u\|^2 = 0$. Integrating over time gives the result.

The proof for helicity requires multiplying by $\nabla \times \overline{u}$ and integrate over Ω. This leaves $(u_t, \nabla \times \overline{u}) = 0$. Integrating over time and using the fact that differential operators (including the filter) commute under periodic boundary conditions completes the proof. □

Many models and regularizations do not exactly conserve (any form of) helicity, [R07], and this clearly impacts the reliability of their predictions in highly rotational flows and over longer time intervals. The nonlinearity of NS-$\overline{\omega}$ acts in manner physically consistent with true fluid flow in that it neither creates nor dissipates either energy or helicity. In homogeneous, isotropic turbulence, energy and model helicity conservation suggests that both energy and model helicity in NS-$\overline{\omega}$ cascade from large (input) scales to small scales where they are dissipated strongly by viscosity ([LST08], and the helicity case follows [LMNR08]). It is interesting to note that the three dimensional integral invariants for NS-$\overline{\omega}$ are, in some sense, reversed from those of NS-α. The NS-α model conserves a weaker model energy, $\frac{1}{2}(u, \overline{u})$, than NS-$\overline{\omega}$, and the usual helicity $(u, \nabla \times u)$.

1.2 Differential Filters in NS-$\overline{\omega}$ and Large Eddy Simulation

Large eddy simulation is concerned with approximation of local, spacial averages in under-resolved simulations of mixes of laminar, transitional and turbulent flows. Velocity averages can be defined by a projection into a finite dimensional space (e.g., a finite element velocity space) as in the Variational Multiscale Method (VMM) of Hughes and co-workers, e.g., [HKJ00, HOM01]. Traditionally, velocity averages $\overline{u}$ in LES are defined by convolution with a given filter

$$\overline{u}(x, t; \alpha) := (g_\alpha \star u)(x, t), \alpha = \text{ filter radius, typically } O(h).$$

On the surface, there are many choices of filter kernel $g(\cdot)$. However, if the averages are viewed as containing information of physical meaning, there are already results on acceptable filters beginning with the famous paper of Koenderink [Koe84], see also Sagaut [S01]. Scale space analysis begins with some basic postulates that any physically reasonable filter should satisfy. In [Koe84] (see also [Lin94] for interesting developments and applications) the following is proven.

Theorem 1 (Koenderink [Koe84])**.** *The only filter satisfying the folowing four conditions* *is the Gaussian filter:*

$$\textit{linearity: } \overline{\alpha u + \beta v} = \alpha \overline{u} + \beta \overline{v}$$
$$\textit{spacial invariance: } \overline{u(x-a)} = \overline{u}(x-a)$$
$$\textit{isotropy, for all rotations } R : \overline{u}(x) = R^* \overline{u(Rx)}$$
scale invariance or the semigroup property (no preferred size):
$$\textit{if } \overline{u}(x,t) := g_\alpha \star u(x,t) \ (\textit{so } \ \overline{\overline{u}} := g_\alpha \star g_\alpha \star u) \textit{ then } \overline{\overline{u}} = g_{\sqrt{2}\alpha} \star u.$$

Thus, there is really only one mathematical correct filter: the Gaussian. Convolution with the Gaussian is expensive so that in spite of this strong uniqueness result usually other filters are used. Koenderink's result indicates that these other filters must be assessed as approximations to the Gaussian.

Differential filters like (1.3) were proposed for large eddy simulation by Germano [G86]. The close connection of the above differential filter to the Gaussian filter can be seen various ways, [BIL06]. For example, the Gaussian is the heat kernel. Thus, one way to compute the exact Gaussian filtered velocity $\overline{u}(x,t) = g_\alpha \star u(x,t)$ is to solve the following evolution equation:

$$v_s(x,s) = \triangle v(x,s) \text{ for } s > 0, \text{ and } v(x,0) = u(x),$$

then set $\overline{u}(x,t) := v(x,s)|_{s=\alpha^2}$. Since the averaging radius α is small, α^2 is smaller still and we can reasonably approximate $v(x, \alpha^2)$ by one step of backward Euler, leading back to the differential filter (1.3).

In comparison with the Gaussian, the differential filter is only approximately scale invariant. Indeed, if we compute $\overline{\overline{u}}$ it fails the semigroup property by $O(\ \alpha^4)$

$$\overline{\overline{u}} \ = \ (-\alpha^2\triangle + 1)^{-1}(-\alpha^2\triangle + 1)^{-1}u = (-(\sqrt{2}\alpha)^2\triangle + 1)^{-1}u + O(\alpha^4).$$

1.3 LES vs. Regularization

In regularization modeling, for example Leray and NS-α, one solves a system similar to the NSE which has better qualitative properties for numerical simulation than the underlying NSE. As a consequence, the computed solution is simply a regularized approximation of the NSE solution rather than a local, spacial average of the fluid velocity. The many possible NSE regularizations can be judged based on (i) accuracy as an approximation of the NSE, and (ii) fidelity to qualitative properties of the NSE's velocity.

The first regularization is due to the seminal 1934 work of J. Leray

$$u_t + \overline{u} \cdot \nabla u + \nabla q - \nu \Delta u = f, \text{ and } \nabla \cdot u = 0. \tag{1.15}$$

This is $O(\alpha^2)$ consistent with the NSE. Leray proved that a unique solution exists and converges (modulo a subsequence) as the averaging radius $\alpha_j \to 0$ [L34a, L34b]. Importantly, Leray's regularization is performed so as to neither create nor destroy kinetic energy: if $\nu = f = 0$ (and under periodic boundary conditions) Leray's model exactly conserves kinetic energy.

As noted above, the 3d Euler equations have two important integral invariants: energy $\frac{1}{2}\int_\Omega |u|^2 dx$ and helicity $\int_\Omega u \cdot \nabla \times u dx$. The interplay between these two are thought to organize coherent structures in fluid motion. The important step of Camassa and Holm from (1.15) to NS-α is that by using Leray's idea with the NSE nonlinearity in rotational form, a regularization results which exactly conserves (in the appropriate context) both a model energy and helicity. Based on the associated á priori bounds, a Leray theory (and beyond) has been established for (1.4)-(1.6) (and as well as for the NS-$\overline{\omega}$ model in [LST08]).

2 Notation and Preliminaries

This section summarizes the notation, definitions and preliminary lemmas needed. We start by introducing the following notation. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space $W_2^k(\Omega)$, and $\|\cdot\|_k$ denotes the norm in H^k. For functions $v(x,t)$ defined on the entire time interval $(0,T)$, we define $(1 \le m < \infty)$

$$\|v\|_{\infty,k} := \operatorname{ess}\sup_{0<t<T} \|v(t,\cdot)\|_k\,, \qquad \text{and} \qquad \|v\|_{m,k} := \left(\int_0^T \|v(t,\cdot)\|_k^m\,dt\right)^{1/m}.$$

In the discrete case we use the analogous norms:

$$\begin{aligned} |||v|||_{\infty,k} &:= \max_{0\le n\le M} \|v_n\|_k\,, & |||v_{1/2}|||_{\infty,k} &:= \max_{0\le n\le M} \|v_{n+1/2}\|_k\,, \\ |||v|||_{m,k} &:= \left(\sum_{n=0}^{M} \|v_n\|_k^m \triangle t\right)^{1/m}, & |||v_{1/2}|||_{m,k} &:= \left(\sum_{n=0}^{M} \|v_{n+1/2}\|_k^m \triangle t\right)^{1/m}. \end{aligned}$$

We consider both the periodic case and the case of internal flow with no slip boundary conditions. (There is mainly only small notational differences between these two cases in the analysis.)

In the periodic case, $\Omega = (0,L)^d, d = 2,3$ and pressure and velocity spaces are, respectively,

$$\begin{aligned} Q &= L_0^2(\Omega) := \{q \in L^2,\ \int_\Omega q = 0\}, \\ X &= H_\#^1(\Omega) := \{v \in H^1(\Omega) \cap L_0^2(\Omega):\ v \text{ is L periodic }\}, \end{aligned}$$

while in the case of internal flow Ω is a regular, bounded, polyhedral domain in $\mathbb{R}^d$ and

$$\begin{aligned} Q &= L_0^2(\Omega), \\ X &= H_0^1(\Omega) := \{v \in H^1(\Omega):\ v|_{\partial\Omega} = 0\}. \end{aligned}$$

We denote the dual space of X by $X^\star$, with the norm $\|\cdot\|_\star$. The space of divergence free functions is denoted

$$V := \{v \in X, (\nabla\cdot v, q) = 0\ \ \forall q \in Q\}\,.$$

The velocity-pressure finite element spaces $X^h \subset X$, $Q^h \subset Q$ are assumed to be conforming and satisfy the LBB^h condition, e.g. [G89]. The discretely divergence free subspace of X^h is, as usual

$$V^h = \{v^h \in X^h, (\nabla\cdot v^h, q^h) = 0\ \ \forall q^h \in Q^h\}\,.$$

In addition, we make use of the following approximation properties,[BS94]:

$$\begin{aligned} \inf_{v\in X_h} \|u - v\| &\le Ch^{k+1}|u|_{k+1},\ \ u \in H^{k+1}(\Omega)^d, \\ \inf_{v\in X_h} \|u - v\|_1 &\le Ch^{k}|u|_{k+1},\ \ u \in H^{k+1}(\Omega)^d, \\ \inf_{r\in Q_h} \|p - r\| &\le Ch^{s+1}|p|_{s+1},\ \ p \in H^{s+1}(\Omega). \end{aligned}$$

Taylor-Hood elements (see e.g. [BS94, G89]) are one common example of such a choice for (X^h, Q^h), and are also the elements we use in our numerical experiments.

We employ a skew-symmetric trilinear form that ensures stability of the method.

Definition 2.1. *Define* $b : X \times X \times X \to \mathbb{R}$ *, by*

$$b(u, v, w) := ((\nabla \times u) \times v, w).$$

We now list important estimates for the b operator necessary in Section 3.

Lemma 2.2. *For* $u, v, w \in X$*, or* $L_\infty(\Omega)$ *and* $\nabla \times u \in L_\infty(\Omega)$*, when indicated, the trilinear term* $((\nabla \times u) \times v, w)$ *satisfies*

$$|((\nabla \times u) \times v, w)| \leq \|\nabla \times u\| \|v\|_\infty \|w\|, \tag{2.1}$$
$$|((\nabla \times u) \times v, w)| \leq \|\nabla \times u\|_\infty \|v\| \|w\|\,, \tag{2.2}$$
$$|((\nabla \times u) \times v, w)| \leq C_0(\Omega) \|\nabla \times u\| \|\nabla v\| \|\nabla w\|\,, \tag{2.3}$$
$$|((\nabla \times u) \times v, w)| \leq C_0(\Omega) \|v\|^{1/2} \|\nabla v\|^{1/2} \|\nabla \times u\| \|\nabla w\|\,. \tag{2.4}$$

Proof. The first two estimates follow immediately from the definition of b. The proof of the other two bounds are easily adapted from the usual bounds of the nonlinearity in non-rotational form. □

Since we study discretizations of the three models, we must deal with discrete differential filters. Continuous differential filters were introduced into turbulence modeling by Germano [G86, G86b] and used various models and regularizations [CTV05, CHOT05, ILT05, GH03, GH05], [GL00].

Definition 2.3 (Continuous differential filter)**.** *For* $\phi \in L^2(\Omega)$ *and* $\alpha > 0$ *fixed, denote the filtering operation on* ϕ *by* $\overline{\phi}$*, where* $\overline{\phi}$ *is the unique solution (in* X*) of*

$$-\alpha^2 \Delta \overline{\phi} + \overline{\phi} = \phi. \tag{2.5}$$

We denote by $A := (-\alpha^2 \Delta + I)$, so $A^{-1} v = \overline{v}$. We define next the discrete differential filter, following Manica and Kaya-Merdan [MKM06].

Definition 2.4 (Discrete differential filter)**.** *Given* $v \in L^2(\Omega)$*, for a given filtering radius* $\alpha > 0$*,* $\overline{v}^h = A_h^{-1} v$ *is the unique solution in* X^h *of*

$$\alpha^2 (\nabla \overline{v}^h, \nabla \chi) + (\overline{v}^h, \chi) = (v, \chi) \quad \forall \chi \in X^h. \tag{2.6}$$

Definition 2.5. *Define the* L^2 *projection* $\Pi^h : L^2(\Omega) \to X^h$ *and discrete Laplacian operator* $\Delta_h : X \to X^h$ *in the usual way by*

$$(\Pi^h v - v, \chi) = 0\,, \qquad (\Delta_h v, \chi) = -(\nabla v, \nabla \chi) \quad \forall \chi \in X^h. \tag{2.7}$$

With Δ_h, we can write $\overline{v}^h = (-\alpha^2 \Delta_h + \Pi^h)^{-1} v$ and $A_h = (-\alpha^2 \Delta_h + \Pi^h)$.

Remark 2.6. *If* Π^h *is the* L^2 *projection on* X^h *and* Π^h_1 *the projection on* X^h *with respect to to the* H^1 *semi-inner product* (grad , grad) *then* $\Delta_h = \Pi^h \Delta_h = \Delta_h \Pi^h_1$*. Further,* Δ_h *is extended from* X^h *to* X *by zero on the orthogonal complement of* X^h *with respect to* (grad , grad).

Remark 2.7. *For other (non-rotational) formulations of the nonlinearity, an attractive alternative is to define the differential filter by a discrete Stokes problem so as to preserve incompressibility approximately. In this case, given* $\phi \in V$, $\overline{\phi}^h \in V^h$ *would be defined by*

$$\alpha^2(\nabla\overline{\phi}^h, \nabla\chi) + (\overline{\phi}^h, \chi) = (\phi, \chi) \text{ for all } \chi \in V^h .$$

Herein we study (2.6) which seems to be perfectly acceptable when working with the rotational form of the nonlinearity.

We begin by recalling from [BIL06, MKM06] some basic facts about discrete differential filters.

Lemma 2.8. *For* $v \in X$, *we have the following bounds for the discretely filtered and approximately deconvolved* v

$$\|\overline{v}^h\| \le \|v\| , \quad \|\nabla\overline{v}^h\| \le \|\nabla v\| \quad \text{and} \quad \|\nabla \times \overline{v}^h\| \le \|\nabla v\| . \tag{2.8}$$

Proof. The proof of the first part of (2.8) follows from choosing $\chi = \overline{v}^h$ in (2.6), and applying Young's inequality.

For the second inequality in (2.8), we note that the filter definition can be rewritten using Δ_h as

$$-\alpha^2(\Delta_h\overline{v}^h, \chi) + (\overline{v}^h, \chi) = (v, \chi) \quad \forall\chi \in X^h.$$

Choosing $\chi = \Delta_h\overline{v}^h$ and using the definition of Δ_h gives

$$\alpha^2\|\Delta_h\overline{v}^h\|^2 + \|\nabla\overline{v}^h\|^2 = -(\nabla v, \nabla\overline{v}^h).$$

Young's inequality completes the proof.

The last inequality follows from $\|\nabla \times \overline{v}^h\| \le \|\nabla\overline{v}^h\|$ and the second inequality. □

Lemma 2.9. *For* $\phi \in X$ *and* $\Delta\phi \in L^2(\Omega)$

$$\alpha^2||\nabla(\phi - \overline{\phi}^h)||^2 + ||\phi - \overline{\phi}^h||^2 \le C \inf_{v^h \in X^h} \{\alpha^2||\nabla(\phi - v^h)||^2 + ||\phi - v^h||^2\} + C\alpha^4||\Delta\phi||^2.$$

Proof. The functions ϕ and $\overline{\phi}^h$ satisfy respectively for any $v^h \in X^h$

$$\begin{aligned} \alpha^2(\nabla\overline{\phi}^h, \nabla v^h) + (\overline{\phi}^h, v^h) &= (\phi, v^h), \\ \alpha^2(\nabla\phi, \nabla v^h) + (\phi, v^h) &= -\alpha^2(\Delta\phi, v^h) + (\phi, v^h). \end{aligned}$$

If the error is denoted $e := \phi - \overline{\phi}^h$, subtraction gives for any $v^h \in X^h$

$$\alpha^2(\nabla e, \nabla v^h) + (e, v^h) = -\alpha^2(\Delta\phi, v^h).$$

The rest of the proof follows standard error analysis of finite element method. Let $\widetilde{\phi} \in X^h$ be arbitrary and split the error as $e = (\phi - \widetilde{\phi}) - (\overline{\phi}^h - \widetilde{\phi})$. Rearranging the above error equation following that splitting, setting $v^h = \overline{\phi}^h - \widetilde{\phi}$ and using the Cauchy-Schwarz-Young inequality and the triangle inequality completes the proof. □

To compress the analysis of both cases, it will be convenient to define two related trilinear forms that correspond to the regularized models.

Definition 2.10 (b_α and b_ω). *Define b_α and $b_\omega : X \times X \times X \to \mathbb{R}$, as*

$$b_\alpha(u, v, w) := ((\nabla \times u) \times \overline{v}^h, w), \text{ and } b_\omega(u, v, w) := ((\nabla \times \overline{u}^h) \times v, w).$$

We shall assume that the solution to the NSE that is approximated by the model is a strong solution and in particular satisfies $u \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;X)$, $p \in L^2(0,T;Q)$, $u_t \in L^2(0,T;X^*)$ and

$$(u_t, v) + (u \cdot \nabla u, v) - (p, \nabla \cdot v) + \nu(\nabla u, \nabla v) = (f, v) \ \ \forall v \in X, \tag{2.9}$$

$$(q, \nabla \cdot u) = 0 \ \ \forall q \in Q. \tag{2.10}$$

For notational clarity let $v(t_{n+1/2}) = v((t_n + t_{n+1})/2)$ for the continuous variable and $v_{n+1/2} = (v_n + v_{n+1})/2$ for both, continuous and discrete variables.

Algorithm 2.11. *[Full CN-FEM for NS-α or NS-$\overline{\omega}$]*
For $b = b_\alpha$ or b_ω .
Let $\Delta t > 0$, $(u_0, p_0) \in (X^h, Q^h)$, $f \in X^*$ and $T := M\,\Delta t$ for M an integer.
For $n = 0, 1, 2, \cdots, M-1$, find $(u^h_{n+1}, p^h_{n+1}) \in (X^h, Q^h)$ satisfying

$$\frac{1}{\Delta t}(u^h_{n+1} - u^h_n, v^h) + b(u^h_{n+1/2}, u^h_{n+1/2}, v^h) - (P^h_{n+1/2}, \nabla \cdot v^h) + \nu(\nabla u^h_{n+1/2}, \nabla v^h) = (f_{n+1/2}, v^h) \ \ \forall\, v^h \in X^h, \tag{2.11}$$

and

$$\text{for the alpha model} \ : \ (\nabla \cdot \overline{u^h_{n+1}}^h, q^h) = 0 \ \forall q^h \in Q^h, \tag{2.12}$$

$$\text{for the omega model} \ : \ (\nabla \cdot u^h_{n+1}, q^h) = 0 \ \forall q^h \in Q^h. \tag{2.13}$$

Remark 1. *Consider the NS-α model. Since the averaged term $\overline{u^h_{n+1}}^h$ in b_α is nonlocal, this must be implemented as a coupled system for $(u^h_{n+1}, w^h_{n+1}, p^h_{n+1})$, where $w^h_{n+1} = \overline{u^h_{n+1}}^h$, of the following form*

$$\frac{1}{\Delta t}(u^h_{n+1} - u^h_n, v^h) + b_\alpha(u^h_{n+1/2}, \frac{1}{2}w^h_{n+1} + \frac{1}{2}\overline{u^h_n}^h, v^h) - (P^h_{n+1/2}, \nabla \cdot v^h) + \nu(\nabla u^h_{n+1/2}, \nabla v^h) = (f_{n+1/2}, v^h) \ \forall v^h \in X^h, \tag{2.14}$$

$$\alpha^2(\nabla w^h_{n+1}, \nabla \chi^h) + (w^h_{n+1}, \chi^h) = (u^h_{n+1}, \chi^h) \ \forall \chi^h \in X^h, \tag{2.15}$$

$$(\nabla \cdot w^h_{n+1}, q^h) = 0 \ \forall q^h \in Q^h. \tag{2.16}$$

Since (X^h, Q^h) satisfies the LBB^h condition, this is equivalent to

$$\frac{1}{\Delta t}(u^h_{n+1} - u^h_n, v^h) + b_\alpha(u^h_{n+1/2}, \frac{1}{2}w^h_{n+1} + \frac{1}{2}\overline{u^h_n}^h, v^h) + \nu(\nabla u^h_{n+1/2}, \nabla v^h) = (f_{n+1/2}, v^h) \ \ \forall v^h \in X^h, \tag{2.17}$$

$$\alpha^2(\nabla w^h_{n+1}, \nabla \chi^h) + (w^h_{n+1}, \chi^h) = (u^h_{n+1}, \chi^h) \ \ \forall \chi^h \in V^h. \tag{2.18}$$

At each time-step, the full CN method requires the solution of a large, coupled, nonlinear system whose 1-1 block's linearization is potentially highly non-symmetric and indefinite. To simplify the computations, we also consider an extrapolated version of the above, CNLE. Note that for the CNLE implementation of Algorithm 2.11, a choice must be made of which nonlinear term is to be extrapolated. Our choice is made based on the following two considerations:

- unconditional energy stability, which implies the $\nabla \times u^h_{n+1}$ term must be the one to be extrapolated, and
- efficiency, which implies the nonlocal, filtered term should be the one to be extrapolated.

These two constraints can both be satisfied when the $\nabla \times u$ term is filtered, which is the motivation for the precise form of the NS-$\overline{\omega}$ model and its treatment below.

Algorithm 2.12. *[Crank-Nicolson Linear Extrapolation Scheme for NS-$\overline{\omega}$]*
Consider the NS-$\overline{\omega}$ model.
Define $E(\phi^h_n, \phi^h_{n-1}) = \frac{3}{2}\phi^h_n - \frac{1}{2}\phi^h_{n-1}$.
Let $\Delta t > 0$, $(u^h_0, P^h_0) \in (X^h, Q^h)$, $f \in X^*$ *and* $M := \frac{T}{\Delta t}$ *and* $(u^h_{-1}, P^h_{-1}) = (u^h_0, P^h_0)$.
For $n = 0, 1, 2, \cdots, M-1$, *find* $(u^h_{n+1}, P^h_{n+1}) \in (X^h, Q^h)$ *satisfying*

$$\begin{aligned} \frac{1}{\Delta t}(u^h_{n+1} - u^h_n, v^h) \quad &+ \quad b_\omega(E(u^h_n, u^h_{n-1}), u^h_{n+1/2}, v^h) - (P^h_{n+1/2}, \nabla \cdot v^h) \\ &+ \quad \nu(\nabla u^h_{n+1/2}, \nabla v^h) = (f_{n+1/2}, v^h) \ , \forall\, v^h \in X^h, \qquad (2.19) \\ &\quad (\nabla \cdot u^h_{n+1}, q^h) = 0 \ , \forall q^h \in Q^h. \qquad (2.20) \end{aligned}$$

Note that the non-local, filtered term, $\nabla \times \overline{E(u^h_n, u^h_{n-1})}^h$, is an explicit and inexpensive computation on a known function. Each step of the above requires the solution of one linear (rather than nonlinear for CN) (Oseen like) system in which the 1-1 block has positive definite symmetric part (rather than possibly both non-symmetric and indefinite for CN). For CNLE for the NS-$\overline{\omega}$ model, backward Euler suffices for the first time-step. For $n = 0$, this is equivalent to (u_{-1}, P_{-1}) constant extrapolation.

Remark 2.13. *It is also interesting to consider an analogous algorithm with quadratic extrapolation which has better stability and accuracy properties. It can be implemented just as Algorithm 2.12 by replacing the linear extrapolation* $E(\phi^h_n, \phi^h_{n-1}) = \frac{3}{2}\phi^h_n - \frac{1}{2}\phi^h_{n-1}$ *with* $E(\phi^h_n, \phi^h_{n-1}, \phi^h_{n-2}) = \frac{15}{8}\phi^h_n - \frac{10}{8}\phi^h_{n-1} + \frac{3}{8}\phi^h_{n-2}$. *The error in quadratic extrapolation is higher order than that of the base CN and thus might be preferable for problems involving delicate solution behavior.*

Another possible advantage of NS-$\overline{\omega}$ is that it is a rotational form model. Rotational form solvers preconditioners have been studied extensively for the Navier-Stokes equations [LO02],[OR02], and thus the ability for NS-$\overline{\omega}$ to be implemented for efficient large scale computations could be a feasible extension of such work.

CNLE for the Navier-Stokes equations was first investigated in [B76] by G. Baker. It is second order in time, unconditionally stable and linearly implicit. The convergence analysis of CNLE follows closely but it is technically longer than the full CN method that we perform in Section 3.

Lemma 2.14. *Assume* $u \in C^0(t_n, t_{n+1}; L^2(\Omega))$. *If* u *is twice differentiable in time and* $u_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$ *then*

$$\|u_{n+1/2} - u(t_{n+1/2})\|^2 \ \leq \ \frac{1}{48}(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|u_{tt}\|^2 \, dt \,. \qquad (2.21)$$

If $u_t \in C^0(t_n, t_{n+1}; L^2(\Omega))$ *and* $u_{ttt} \in L^2((t_n, t_{n+1}) \times \Omega)$ *then*

$$\|\frac{u_{n+1} - u_n}{\Delta t} - u_t(t_{n+1/2})\|^2 \;\le\; \frac{1}{1280}(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|u_{ttt}\|^2 \, dt \;\; \textit{and} \tag{2.22}$$

if $\nabla u \in C^0(t_n, t_{n+1}; L^2(\Omega))$ *and* $\nabla u_{tt} \in L^2((t_n, t_{n+1}) \times \Omega)$ *then*

$$\|\nabla(u_{n+1/2} - u(t_{n+1/2}))\|^2 \;\le\; \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla u_{tt}\|^2 \, dt\,. \tag{2.23}$$

The proof of Lemma 2.14 is based on the Taylor expansion with remainder. It is more of technical nature and therefore omitted herein. The error analysis uses a discrete Gronwall inequality, recalled from [HR90], for example.

Lemma 2.15 (Discrete Gronwall Lemma)**.** *Let* Δt*,* H*, and* a_n, b_n, c_n, d_n *(for integers* $n \ge 0$*) be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \le \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \quad for\ l \ge 0. \tag{2.24}$$

Suppose that $d_n \Delta t < 1\ \forall n$*. Then,*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \le exp\left(\Delta t \sum_{n=0}^{l} \frac{d_n}{1 - \Delta t d_n}\right)\left(\Delta t \sum_{n=0}^{l} c_n + H\right) \quad for\ l \ge 0. \tag{2.25}$$

3 Analysis of Full Crank-Nicolson Scheme for the NS-$\overline{\omega}$ Model

In this section, we show that solutions of the CN schemes for both regularizations are unconditionally stable and well defined. We prove that the CN-FEM is optimally convergent to solutions of the NSE. (The case of the NS-α model is considerably more delicate, see Connors [Con08].) This error analysis, already technical, can be extended to the CNLE (or CNQE) time stepping method.

Lemma 3.1. *Consider the NS-*$\overline{\omega}$ *model and the schemes in algorithms 2.11 and 2.12 (CN and CNLE, respectively). A solution* u_h^l*,* $l = 1, \dots M$*, exists at each time-step. Both schemes are unconditionally stable: their solutions satisfy the following á priori bound:*

$$\|u_M^h\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \|\nabla u_{n+1/2}^h\|^2 \le \|u_0^h\|^2 + \frac{\Delta t}{\nu} \sum_{n=0}^{M-1} \|f_{n+1/2}\|_*^2. \tag{3.1}$$

Proof. The existence of a solution u_h^n to the schemes in Algorithms 2.11 and 2.12 follows from the Leray-Schauder Principle [Zei951]. The main step is deriving an á priori estimate, which can be obtained by setting $v^h = u_{n+1/2}^h$ both in (2.11) and (2.19). The nonlinear term in the two schemes considered vanishes with this choice. Thus,

$$\frac{1}{2\Delta t}(\|u_{n+1}^h\|^2 - \|u_n^h\|^2) + \nu \|\nabla u_{n+1/2}^h\|^2 \le \frac{1}{2\nu}\|f_{n+1/2}\|_*^2 + \frac{\nu}{2}\|\nabla u_{n+1/2}^h\|^2 \quad \text{for every } n,$$

i.e.,

$$\frac{1}{\Delta t}(\|u_{n+1}^h\|^2 - \|u_n^h\|^2) + \nu \|\nabla u_{n+1/2}^h\|^2 \le \frac{1}{\nu}\|f_{n+1/2}\|_*^2, \quad \text{for every } n.$$

Summing from $n = 0 \dots M-1$ gives the desired result. □

Remark 3.2. *Since the kinetic energy and energy dissipation of NS-$\overline{\omega}$, KE_ω and ε_ω, take the usual form, Lemma 3.1 implies*

$$KE_\omega(u^h_M) + \nu\frac{\Delta t}{2}\sum_{n=0}^{M-1}\varepsilon_\omega(u^h_{n+1/2}) \le KE_\omega(u^h_0) + \frac{\Delta t}{2\nu}\sum_{n=0}^{M-1}\|f_{n+1/2}\|^2_*. \tag{3.2}$$

Thus if $\nu = f = 0$, $KE_\omega(u^h_M) = KE_\omega(u^h_0)$. Hence Algorithms 2.11 and 2.12 are energy conserving.

Next we review stability of the CN method for NS-α. For NS-α it is stable with respect to a modified kinetic energy with a modified energy dissipation, given by

$$KE_\alpha(u) := \frac{1}{2}\|\overline{u}^h\|^2 + \frac{\alpha^2}{2}\|\nabla\overline{u}^h\|^2, \quad \varepsilon_\alpha(u) := \qquad \nu||\nabla\overline{u}^h||^2 + \nu\alpha^2||\triangle^h\overline{u}^h||^2.$$

Discrete versions of these two are also exactly conserved by the fully discrete schemes. We begin by considering the case of NS-α.

Lemma 3.3. *Consider Algorithm 2.11. The scheme (2.11)-(2.12) is well defined and unconditionally stable. A solution u^l_h, $l = 1, \dots M$, exists at each time-step that satisfies the following á priori bound:*

$$KE_\alpha(u^h_M) + \frac{\Delta t}{2}\sum_{n=0}^{M-1}\varepsilon_\alpha(u^h_{n+1/2}) \le KE_\alpha(u^h_0) + \frac{\Delta t}{2\nu}\sum_{n=0}^{M-1}\|f_{n+1/2}\|^2_*, \tag{3.3}$$

If $\nu = f = 0$ then,

$$KE_\alpha(u^h_M) = KE_\alpha(u^h_0). \tag{3.4}$$

Proof. The key for existence and stability is the á priori estimate and the key for the á priori estimate is the exact choice of test function which makes the nonlinear term vanish. Thus we first note that choosing $v^h = \overline{u^h_{n+1/2}}^h$ does this since

$$b^*_\alpha(u^h_{n+1/2}, u^h_{n+1/2}, \overline{u^h_{n+1/2}}^h) = 0 \quad .$$

With this choice, and $q^h = P^h_{n+1/2}$, we have for NS-alpha:

$$\begin{aligned}\frac{1}{\Delta t}(u^h_{n+1} - u^h_n, \overline{u^h_{n+1/2}}^h) - (P^h_{n+1/2}, \nabla\cdot\overline{u^h_{n+1/2}}^h) + \nu(\nabla u^h_{n+1/2}, \nabla\overline{u^h_{n+1/2}}^h) &= (f_{n+1/2}, \overline{u^h_{n+1/2}}^h)\ , \\ (\nabla\cdot\overline{u^h_{n+1/2}}^h, P^h_{n+1/2}) &= 0,\end{aligned}$$

Adding the equations the Bernoulli pressure term drops out as well. We thus obtain

$$\frac{1}{\Delta t}(u^h_{n+1} - u^h_n, \overline{u^h_{n+1/2}}^h) + \nu(\nabla u^h_{n+1/2}, \nabla\overline{u^h_{n+1/2}}^h) = (f_{n+1/2}, \overline{u^h_{n+1/2}}^h).$$

Consider the viscous term. This can be written as (using the fact that the operators involved commute on X^h)

$$\begin{aligned}\nu(\nabla u^h_{n+1/2}, \nabla\overline{u^h_{n+1/2}}^h) &= -\nu(\triangle^h u^h_{n+1/2}, \overline{u^h_{n+1/2}}^h) = -\nu(\triangle^h A_h\overline{u^h_{n+1/2}}^h, \overline{u^h_{n+1/2}}^h) \\ &= \nu(\nabla\overline{u^h_{n+1/2}}^h, \nabla\overline{u^h_{n+1/2}}^h) + \nu\alpha^2(\triangle^h\overline{u^h_{n+1/2}}^h, \triangle^h\overline{u^h_{n+1/2}}^h) \\ &= \nu||\nabla\overline{u^h_{n+1/2}}^h||^2 + \nu\alpha^2||\triangle^h\overline{u^h_{n+1/2}}^h||^2 \\ &= \varepsilon_\alpha(u^h_{n+1/2}),\end{aligned}$$

Similarly, the time difference term can be rewritten as

$$\begin{aligned}
\frac{1}{\Delta t}(u^h_{n+1} - u^h_n, \overline{u^h_{n+1/2}}^h) &= \frac{1}{\Delta t}(A_h \overline{u^h_{n+1}}^h - A_h \overline{u^h_n}^h, \overline{u^h_{n+1/2}}^h) \\
&= \frac{1}{\Delta t}(\overline{u^h_{n+1}}^h - \overline{u^h_n}^h, \overline{u^h_{n+1/2}}^h) + \frac{\alpha^2}{\Delta t}(\nabla \overline{u^h_{n+1}}^h - \nabla \overline{u^h_n}^h, \nabla \overline{u^h_{n+1/2}}^h) \\
&= \frac{1}{2\Delta t}(\|\overline{u^h_{n+1}}^h\|^2 - \|\overline{u^h_n}^h\|^2) + \frac{\alpha^2}{2\Delta t}(\|\nabla \overline{u^h_{n+1}}^h\|^2 - \|\nabla \overline{u^h_{n+1}}^h\|^2) \\
&= \frac{1}{\Delta t}(KE_\alpha(u^h_{n+1}) - KE_\alpha(u^h_n)).
\end{aligned}$$

Thus,

$$\frac{1}{\Delta t}(KE_\alpha(u^h_{n+1}) - KE_\alpha(u^h_n)) + \varepsilon_\alpha(u^h_{n+1/2}) \le \frac{1}{2\nu}\|f_{n+1/2}\|^2_* + \frac{\nu}{2}\|\nabla \overline{u^h_{n+1/2}}^h\|^2 \quad \text{, for every } n,$$

i.e.,

$$\frac{1}{\Delta t}(KE_\alpha(u^h_{n+1}) - KE_\alpha(u^h_n)) + \frac{1}{2}\varepsilon_\alpha(u^h_{n+1/2}) \le \frac{1}{2\nu}\|f_{n+1/2}\|^2_*.$$

Summing from $n = 0 \dots M-1$ gives the desired result. The á priori bound and an argument using the Leray-Schauder fixed point theorem yields existence of the approximate solution at each time level as well. □

Our main convergence result for the discrete NS-$\overline{\omega}$ model is given next.

Theorem 3.4 (Convergence for discrete NS-omega regularization)**.** *Consider the discrete NS-$\overline{\omega}$ model. Let $(u(t), P(t))$ be a smooth, strong solution of the NSE satisfying either no-slip or periodic with zero-mean boundary conditions such that the norms of $(u(t), P(t))$ on the right hand side of (3.5)-(3.6) are finite. Suppose (u^h_0, P^h_0) are the V_h and Q_h interpolants of $(u(0), P(0))$, respectively. Suppose (u^h, P^h) is the CN approximation (2.11)-(2.13) of the NS-$\overline{\omega}$ model. Then for $\triangle t$ small enough there is a constant $C = C(u, P)$ such that*

$$|||u - u^h|||_{\infty,0} \le F(\triangle t, h, \alpha) + Ch^{k+1}|||u|||_{\infty,k+1}\,, \tag{3.5}$$

$$\begin{aligned}
\left(\nu\triangle t\sum_{n=0}^{M-1}\|\nabla(u_{n+1/2} - (u^h_{n+1} + u^h_n)/2)\|^2\right)^{1/2} &\le F(\triangle t, h, \alpha) + C\nu^{1/2}(\triangle t)^2\|\nabla u_{tt}\|_{2,0} \\
&\quad + C\nu^{1/2}h^k|||u|||_{2,k+1}\,,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
F(\triangle t, h, \alpha) := C^* &\Big\{\nu^{-1/2}\, h^{k+1/2}\left(|||u|||^2_{4,k+1} + |||\nabla u|||^2_{4,0}\right) + \nu^{1/2}h^k|||u|||_{2,k+1} \\
&+\nu^{-1/2}h^k\left(|||u|||^2_{4,k+1} + \nu^{-1/2}(\|u^h_0\| + \nu^{-1/2}|||f|||_{2,\star})\right) + \nu^{-1/2}h^{s+1}|||P_{1/2}|||_{2,s+1} \\
&+\nu^{-1/2}\alpha^2\|\,|u|\,\|_{4,1}\|\,|u|\,\|_{4,3} + (\Delta t)^2\Big(\|u_{ttt}\|_{2,0} + \nu^{-1/2}\|P_{tt}\|_{2,0} + \|f_{tt}\|_{2,0} \\
&\qquad +\nu^{1/2}\|\nabla u_{tt}\|_{2,0} + \nu^{-1/2}\|\nabla u_{tt}\|^2_{4,0} \\
&\qquad +\nu^{-1/2}|||\nabla u|||^2_{4,0} + \nu^{-1/2}|||\nabla u_{1/2}|||^2_{4,0}\Big)\Big\}\,.
\end{aligned} \tag{3.7}$$

Remark 3.5. *The constant C^* arising from Gronwall's lemma depends on ν like $exp(\nu^{-3}T)$ and the smallness assumption on the time-step is $\triangle t < C(\nu^{-3}|||\nabla u|||^4_{\infty,0} + 1)^{-1}$. We believe that this last condition on $\triangle t$ is improvable.*

Corollary 3.6. *Suppose that in addition to the assumptions made in Theorem 3.4, the finite element spaces X^h and Q^h are composed of Taylor-Hood elements. Suppose u is smooth in the sense that the indicated norms on the right hand side of (3.5)-(3.6) are finite for $k = 2$ and $s = 1$. Then the error in the CN scheme for the model (2.11)-(2.13), is of the order*

$$|||u - u^h|||_{\infty,0} + \left(\nu \Delta t \sum_{n=1}^{M} \|\nabla(u_{n+1/2} - u^h_{n+1/2})\|^2 \right)^{1/2} = O(h^2 + \Delta t^2 + \alpha^2). \tag{3.8}$$

Proof of Theorem 3.4. Note that for $u,\ v,\ w,\ \in X$, by adding and subtracting terms, we can write

$$b_\omega(u, v, w) \ = \ ((\nabla \times u) \times v, w) \ - \ \text{filtering error},$$

specifically

$$b_\omega(u, v, w) \ = \ ((\nabla \times u) \times v, w) \ - ((\nabla \times u - \nabla \times \overline{u}^h) \times v, w),$$

Define the filtering error accordingly:

$$FE = FE_\omega(u, v, w) := ((\nabla \times u - \nabla \times \overline{u}^h) \times v, w).$$

At time $t_{n+1/2}$, the solution of the NSE (u, P) satisfies

$$\begin{aligned} (\frac{u_{n+1} - u_n}{\Delta t}, v^h) \ &+ \ b_\omega(u_{n+1/2}, u_{n+1/2}, v^h) + \nu(\nabla u_{n+1/2}, \nabla v^h) - (P_{n+1/2}, \nabla \cdot v^h) \\ &= \ (f_{n+1/2}, v^h) + Intp(u^n, P^n; v^h), \end{aligned} \tag{3.9}$$

for all $v^h \in V^h$. The term $Intp(u_n, P_n; v^h)$ collects the interpolation error, the above filtering error and the consistency error. It is given by

$$\begin{aligned} Intp(u_n, p_n; v^h) \ = \ & \left(\frac{u_{n+1} - u_n}{\Delta t} - u_t(t_{n+1/2}), v^h \right) + \nu(\nabla u_{n+1/2} \ - \ \nabla u(t_{n+1/2}), \nabla v^h) \\ & + b_\omega(u_{n+1/2}, u_{n+1/2}, v^h) - b_\omega(u(t_{n+1/2}), u(t_{n+1/2}), v^h) \\ & - FE(u_{n+1/2}, u_{n+1/2}, v^h) \\ & - (P_{n+1/2} - P(t_{n+1/2}), \nabla \cdot v^h) + (f(t_{n+1/2}) - f_{n+1/2}, v^h) \, . \end{aligned} \tag{3.10}$$

Subtracting (3.9) from (2.11) and letting $e_n = u_n - u^h_n$ we have

$$\begin{aligned} & \frac{1}{\Delta t}(e_{n+1} - e_n, v^h) + b_\omega(u_{n+1/2}, u_{n+1/2}, v^h) - b_\omega(u^h_{n+1/2}, u^h_{n+1/2}, v^h) \\ & \qquad + \nu(\nabla e_{n+1/2}, \nabla v^h) = (P_{n+1/2}, \nabla \cdot v^h) + Intp(u_n, p_n; v^h) \quad , \quad \forall v^h \in V^h. \end{aligned} \tag{3.11}$$

Decompose the error as $e_n = (u_n - U_n) - (u^h_n - U_n) := \eta_n - \phi^h_n$ where $\phi^h_n \in V^h$, and U is the L^2 projection of u in V^h. Setting $v^h = \phi^h_{n+1/2}$ in (3.11) and using $(q, \nabla \cdot \phi_{n+1/2}) = 0$ for all $q \in Q^h$ we obtain

$$\begin{aligned} & (\phi^h_{n+1} - \phi^h_n, \phi^h_{n+1/2}) + \nu \triangle t \|\nabla \phi^h_{n+1/2}\| + \triangle t\, b_\omega(u^h_{n+1/2}, e_{n+1/2}, \phi^h_{n+1/2}) \\ & \quad + \triangle t\, b_\omega(e_{n+1/2}, u_{n+1/2}, \phi^h_{n+1/2}) = (\eta_{n+1} - \eta_n, \phi^h_{n+1/2}) + \triangle t \nu(\nabla \eta_{n+1/2}, \nabla \phi^h_{n+1/2}) \\ & \qquad + \triangle t(P_{n+1/2} - q, \nabla \cdot \phi^h_{n+1/2}) + \triangle t\, Intp(u_n, p_n; v^h) \, , \end{aligned} \tag{3.12}$$

i.e.

$$\frac{1}{2}(\|\phi^h_{n+1}\| - \|\phi^h_n\|) + \nu\triangle t\|\nabla\phi^h_{n+1/2}\| = (\eta_{n+1} - \eta_n, \phi^h_{n+1/2}) + \triangle t\nu(\nabla\eta_{n+1/2}, \nabla\phi^h_{n+1/2})$$
$$- \triangle t\, b_\omega(\eta_{n+1/2}, u_{n+1/2}, \phi^h_{n+1/2}) + \triangle t\, b_\omega(\phi^h_{n+1/2}, u_{n+1/2}, \phi^h_{n+1/2})$$
$$- \triangle t\, b_\omega(u^h_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) + \triangle t(P_{n+1/2} - q, \nabla\cdot\phi^h_{n+1/2})$$
$$+ \triangle t\, Intp(u_n, p_n; \phi^h_{n+1/2})\,. \qquad (3.13)$$

We now bound the terms in the RHS of (3.13) individually. According to the choice of U, $(\eta_{n+1} - \eta_n, \phi^h_{n+1/2}) = 0$. The Cauchy-Schwarz and Young's inequalities give

$$\begin{aligned} \nu\triangle t(\nabla\eta_{n+1/2}, \nabla\phi^h_{n+1/2}) &\le \nu\triangle t\|\nabla\eta_{n+1/2}\|\,\|\nabla\phi^h_{n+1/2}\| \\ &\le \frac{\nu\Delta t}{12}\|\nabla\phi^h_{n+1/2}\|^2 + C\nu\Delta t\|\nabla\eta_{n+1/2}\|^2. \qquad (3.14) \end{aligned}$$

$$\begin{aligned} \triangle t(P_{n+1/2} - q, \nabla\cdot\phi^h_{n+1/2}) &\le C\triangle t\|P_{n+1/2} - q\|\,\|\nabla\phi^h_{n+1/2}\| \\ &\le \frac{\nu\Delta t}{12}\|\nabla\phi^h_{n+1/2}\|^2 + C\triangle t\nu^{-1}\|P_{n+1/2} - q\|^2\,. \qquad (3.15) \end{aligned}$$

Lemmas 2.2, 2.8 and standard inequalities give

$$\begin{aligned} &\triangle t\, b_\omega(\eta_{n+1/2}, u_{n+1/2}, \phi^h_{n+1/2}) \\ &\quad\le C\Delta t\|\nabla\overline{\eta_{n+1/2}}^h\|\,\|\nabla u_{n+1/2}\|\,\|\nabla\phi^h_{n+1/2}\| \\ &\quad\le C\Delta t\|\nabla\eta_{n+1/2}\|\,\|\nabla u_{n+1/2}\|\,\|\nabla\phi^h_{n+1/2}\| \\ &\quad\le \frac{\nu\Delta t}{12}\|\nabla\phi^h_{n+1/2}\|^2 + C\triangle t\,\nu^{-1}\,\|\nabla\eta_{n+1/2}\|^2\|\nabla u_{n+1/2}\|^2\,. \qquad (3.16) \end{aligned}$$

$$\begin{aligned} \triangle t\, b_\omega(\phi^h_{n+1/2}, u_{n+1/2}, \phi^h_{n+1/2}) &= \triangle t\,((\nabla\times\overline{\phi^h_{n+1/2}}^h) \times u_{n+1/2}, \phi^h_{n+1/2}) \\ &\le C\triangle t\|\nabla\times\overline{\phi^h_{n+1/2}}^h\|\,\|\nabla u_{n+1/2}\|\|\phi^h_{n+1/2}\|^{1/2}\,\|\nabla\phi^h_{n+1/2}\|^{1/2} \\ &\le C\triangle t\|\phi^h_{n+1/2}\|^{1/2}\,\|\nabla\phi^h_{n+1/2}\|^{3/2}\,\|\nabla u_{n+1/2}\| \\ &\le \frac{\nu\triangle t}{24}\|\nabla\phi^h_{n+1/2}\|^2 \,+\, C\triangle t\,\nu^{-3}\|\phi^h_{n+1/2}\|^2\,\|\nabla u_{n+1/2}\|^4\,. \qquad (3.17) \end{aligned}$$

The final trilinear term requires a bit more effort. Begin by splitting the first entry of this term by adding and subtracting $u_{n+1/2}$, followed by rewriting the resulting error term as pieces inside and out of the finite element space.

$$\begin{aligned} \triangle t b_\omega(u^h_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) &= \triangle t b_\omega(\eta_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) \\ &\quad + \triangle t b_\omega(\phi^h_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) + \triangle t b_\omega(u_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}). \qquad (3.18) \end{aligned}$$

We bound each of the terms on the right hand side of (3.18) using the same inequalities and lemmas as above:

$$\begin{aligned} \triangle t b_\omega(\eta_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) &\le \triangle t\|\nabla\times\overline{\eta_{n+1/2}}^h\|\|\nabla\eta_{n+1/2}\|\|\nabla\phi^h_{n=1/2}\| \\ &\le \triangle t\|\nabla\eta_{n+1/2}\|^2\|\nabla\phi^h_{n+1/2}\| \\ &\le \frac{\nu\triangle t}{24}\|\nabla\phi^h_{n+1/2}\|^2 + C\triangle t\nu^{-1}\|\nabla\eta_{n+1/2}\|^4 \qquad (3.19) \end{aligned}$$

$$\begin{aligned}
&\triangle t b_\omega(\phi^h_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) \\
&\quad \leq \; C\triangle t \|\nabla \phi^h_{n+1/2}\|^{3/2} \|\nabla \eta_{n+1/2}\| \|\phi^h_{n+1/2}\| \\
&\quad \leq \; \frac{\nu \triangle t}{24} \|\nabla \phi^h_{n+1/2}\|^2 + C\triangle t \nu^{-3} \|\nabla \eta_{n+1/2}\|^4 \|\phi^h_{n+1/2}\|^2
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\triangle t b_\omega(u_{n+1/2}, \eta_{n+1/2}, \phi^h_{n+1/2}) \\
&\quad \leq \; \triangle t \|\nabla \times \overline{u_{n+1/2}}^h\| \|\nabla \eta_{n+1/2}\| \|\nabla \phi^h_{n+1/2}\| \\
&\quad \leq \; \frac{\nu \triangle t}{24} \|\nabla \phi^h_{n+1/2}\|^2 + C\triangle t \nu^{-1} \|\nabla \eta_{n+1/2}\|^2 \|\nabla u_{n+1/2}\|^2
\end{aligned} \tag{3.21}$$

Combining (3.14)-(3.21) and summing from $n = 0$ to $M-1$ (assuming that $\|\phi^h_0\| = 0$) reduces (3.13) to

$$\begin{aligned}
&\|\phi^h_M\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \phi^h_{n+1/2}\|^2 \\
&\quad \leq \; C\triangle t \{ \sum_{n=0}^{M-1} C\nu^{-3} (\|\nabla u_{n+1/2}\|^4 + \|\nabla \eta_{n+1/2}\|^4) \|\phi^h_{n+1/2}\|^2 \qquad (3.22) \\
&\qquad + \sum_{n=0}^{M-1} \Big((\nu + \nu^{-1}) \|\nabla \eta_{n+1/2}\|^2 + \nu^{-1} \|\nabla \eta_{n+1/2}\|^4 \Big) \\
&\qquad + \sum_{n=0}^{M-1} \nu^{-1} \|\eta_{n+1/2}\| \, \|\nabla \eta_{n+1/2}\| \|\nabla u_{n+1/2}\|^2 \\
&\qquad + \sum_{n=0}^{M-1} \|\nabla \times \overline{u^h_{n+1/2}}^h\| \, \|\nabla \eta_{n+1/2}\| \, \|\phi^h_{n+1/2}\|^{\frac{1}{2}} \|\nabla \phi^h_{n+1/2}\|^{\frac{1}{2}} \\
&\qquad + \sum_{n=0}^{M-1} \nu^{-1} \|P_{n+1/2} - q\|^2 + \sum_{n=0}^{M-1} |Intp(u_n, p_n; \phi^h_{n+1/2})| \}.
\end{aligned} \tag{3.23}$$

Now, we continue to bound the terms on the RHS of (3.23). We have that

$$\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C\nu \|\nabla \eta_{n+1/2}\|^2 \; &\leq \; \triangle t C(\nu + \nu^{-1}) \sum_{n=0}^{M} \|\nabla \eta_n\|^2 \leq \triangle t C(\nu + \nu^{-1}) \sum_{n=0}^{M} h^{2k} |u^n|^2_{k+1} \\
&\leq \; C(\nu + \nu^{-1}) h^{2k} \||u|\|^2_{2,k+1},
\end{aligned} \tag{3.24}$$

and similarly,

$$\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C\nu^{-1} \|\nabla \eta_{n+1/2}\|^4 \; &\leq \; \triangle t C\nu^{-1} \sum_{n=0}^{M} \|\nabla \eta_n\|^4 \leq \triangle t C\nu^{-1} \sum_{n=0}^{M} h^{4k} |u^n|^4_{k+1} \\
&\leq \; C\nu^{-1} h^{4k} \||u|\|^4_{2,k+1}.
\end{aligned} \tag{3.25}$$

For the term

$$
\begin{aligned}
&\triangle t \sum_{n=0}^{M-1} C\nu^{-1}\|\nabla \eta_{n+1/2}\|^2 \|\nabla u_{n+1/2}\|^2 \\
&\qquad \leq \; C\nu^{-1}\triangle t h^{2k} \sum_{n=0}^{M-1} (|u_{n+1}|^2_{k+1} + |u_n|^2_{k+1})\|\nabla u_{n+1/2}\|^2 \\
&\qquad \leq \; C\nu^{-1}\triangle t h^{2k}\left(\|u\|^4_{4,k+1} + \|\nabla u\|^4_{4,0}\right). \qquad (3.26)
\end{aligned}
$$

¿From (2.14),

$$
\begin{aligned}
\triangle t \sum_{n=0}^{M-1} C\nu^{-1}\|P_{n+1/2} - q\|^2 &\leq C\nu^{-1}\triangle t \sum_{n=0}^{M-1} \|P(t_{n+1/2}) - q\|^2 + \|P_{n+1/2} - P(t_{n+1/2})\|^2 \\
\leq \; & C\nu^{-1}\left(h^{2s+2}\triangle t \sum_{n=0}^{M-1} \|P(t_{n+1/2})\|^2_{s+1} + \triangle t \sum_{n=0}^{M-1} \frac{1}{48}(\triangle t)^3 \int_{t_n}^{t_{n+1}} \|p_{tt}\|^2\, dt \right) \\
\leq \; & C\nu^{-1}\left(h^{2s+2}\||p_{1/2}|\|^2_{2,s+1} + (\triangle t)^4\|p_{tt}\|^2_{2,0}\right) \qquad (3.27)
\end{aligned}
$$

We now bound the terms in $Intp(u_n, p_n; \phi^h_{n+1/2})$. Using Cauchy-Schwarz and Young's inequalities, Taylor's theorem, and Lemmas 2.8 and 2.9,

$$
\begin{aligned}
&\left(\frac{u^{n+1} - u_n}{\Delta t} - u_t(t_{n+1/2}), \phi^h_{n+1/2}\right) \\
&\qquad \leq \; \frac{1}{2}\|\phi^h_{n+1/2}\|^2 + \frac{1}{2}\|\frac{u^{n+1} - u_n}{\Delta t} - u_t(t_{n+1/2})\|^2 \\
&\qquad \leq \; \frac{1}{2}\|\phi^h_{n+1}\|^2 + \frac{1}{2}\|\phi^h_n\|^2 + \frac{1}{2}\frac{(\Delta t)^3}{1280}\int_{t_n}^{t_{n+1}} \|u_{ttt}\|^2\, dt, \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
&(P_{n+1/2} - P(t_{n+1/2}), \nabla \cdot \phi^h_{n+1/2}) \\
&\qquad \leq \; \varepsilon_1 \nu \|\nabla \phi^h_{n+1/2}\|^2 + C\,\nu^{-1}\|P_{n+1/2} - P(t_{n+1/2})\|^2 \\
&\qquad \leq \; \varepsilon_1 \nu \|\nabla \phi^h_{n+1/2}\|^2 + C\,\nu^{-1}\frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|p_{tt}\|^2\, dt, \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
&(f(t_{n+1/2}) - f_{n+1/2}, \phi^h_{n+1/2}) \\
&\qquad \leq \; \frac{1}{2}\|\phi^h_{n+1/2}\|^2 + \frac{1}{2}\|f(t_{n+1/2}) - f_{n+1/2}\|^2 \\
&\qquad \leq \; \frac{1}{2}\|\phi^h_{n+1}\|^2 + \frac{1}{2}\|\phi^h_n\|^2 + \frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1}} \|f_{tt}\|^2\, dt, \qquad (3.30)
\end{aligned}
$$

$$
\begin{aligned}
&\nu(\nabla u_{n+1/2} - \nabla u(t_{n+1/2}), \nabla \phi^h_{n+1/2}) \\
&\qquad \leq \; \varepsilon_2 \nu\, \|\nabla \phi^h_{n+1/2}\|^2 + C\,\nu\|\nabla u_{n+1/2} - \nabla u(t_{n+1/2})\|^2 \\
&\qquad \leq \; \varepsilon_2 \nu\, \|\nabla \phi^h_{n+1/2}\|^2 + C\,\nu\frac{(\Delta t)^3}{48}\int_{t_n}^{t_{n+1/2}} \|\nabla u_{tt}\|^2\, dt, \qquad (3.31)
\end{aligned}
$$

$$
\begin{aligned}
&b_\omega(u_{n+1/2}, u_{n+1/2}, \phi^h_{n+1/2}) - b_\omega(u(t_{n+1/2}), u(t_{n+1/2}), \phi^h_{n+1/2}) \\
&\quad = \; b_\omega(u_{n+1/2} - u(t_{n+1/2}), u_{n+1/2}, \phi^h_{n+1/2}) + b_\omega(u(t_{n+1/2}), u_{n+1/2} - u(t_{n+1/2}), \phi^h_{n+1/2}) \\
&\quad \le \; C \, \|\nabla \times \overline{(u_{n+1/2} - u(t_{n+1/2}))}^h\| \, \|\nabla \phi^h_{n+1/2}\| \, \left(\|\nabla u_{n+1/2}\| + \|\nabla u(t_{n+1/2})\| \right) \\
&\quad \le \; C \, \|\nabla (u_{n+1/2} - u(t_{n+1/2}))\| \, \|\nabla \phi^h_{n+1/2}\| \, \left(\|\nabla u_{n+1/2}\| + \|\nabla u(t_{n+1/2})\| \right) \\
&\quad \le \; C \, \nu^{-1} \left(\|\nabla u_{n+1/2}\|^2 + \|\nabla u(t_{n+1/2})\|^2 \right) \frac{(\Delta t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla u_{tt}\|^2 \, dt + \varepsilon_3 \nu \|\nabla \phi^h_{n+1/2}\|^2 \\
&\quad \le \; C \, \nu^{-1} \frac{(\Delta t)^3}{48} \left(\int_{t_n}^{t_{n+1}} 2(\|\nabla u_{n+1/2}\|^4 + \|\nabla u(t_{n+1/2})\|^4) \, dt \right. \\
&\qquad \left. + \int_{t_n}^{t_{n+1}} \|\nabla u_{tt}\|^4 \, dt \right) + \varepsilon_3 \nu \|\nabla \phi^h_{n+1/2}\|^2 \\
&\quad \le \; C \, \nu^{-1} \, (\Delta t)^4 (\|\nabla u_{n+1/2}\|^4 + \|\nabla u(t_{n+1/2})\|^4) \\
&\qquad + C \, \nu^{-1} \, (\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\nabla u_{tt}\|^4 \, dt + \varepsilon_3 \nu \|\nabla \phi^h_{n+1/2}\|^2 \, . \qquad (3.32)
\end{aligned}
$$

Next we will bound the filtering error using the definition of the discrete filter.

$$
\begin{aligned}
FE &\le \left| (\nabla \times u_{n+1/2} - \nabla \times \overline{u_{n+1/2}}^h \times u_{n+1/2}, \phi^h_{n+1/2}) \right| \\
&\le \; C \|\nabla \times (u_{n+1/2} - \overline{u_{n+1/2}}^h)\| \|\nabla u_{n+1/2}\| \|\nabla \phi^h_{n+1/2}\| \\
&\le \; \varepsilon_4 \nu \|\nabla \phi^h_{n+1/2}\|^2 + C \nu^{-1} \alpha^4 \|\nabla u_{n+1/2}\|^2 \|\nabla \Delta^h \overline{u_{n+1/2}}^h\|^2 \\
&\le \; \varepsilon_4 \nu \|\nabla \phi^h_{n+1/2}\|^2 + C \nu^{-1} \alpha^4 \|\nabla u_{n+1/2}\|^2 \left| u_{n+1/2} \right|_3^2 \qquad (3.33)
\end{aligned}
$$

Combine (3.28)-(3.33) to obtain

$$
\begin{aligned}
\Delta t \sum_{n=0}^{M-1} |Intp(u_n, p_n; \phi^h_{n+1/2})| \; &\le \; \Delta t \, C \|\phi^h_{n+1}\|^2 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \Delta t \, \nu \|\nabla \phi^h_{n+1/2}\|^2 \\
&\quad + C (\Delta t)^4 \left(\|u_{ttt}\|^2_{2,0} + \nu^{-1} \|p_{tt}\|^2_{2,0} + \|f_{tt}\|^2_{2,0} \right. \\
&\qquad + \nu \|\nabla u_{tt}\|^2_{2,0} + \nu^{-1} \|\nabla u_{tt}\|^4_{4,0} \\
&\qquad \left. + \nu^{-1} |||\nabla u|||^4_{4,0} + \nu^{-1} |||\nabla u_{1/2}|||^4_{4,0} \right) . \qquad (3.34)
\end{aligned}
$$

Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1/12$ and with (3.24)-(3.27), (3.34), from (3.23) we obtain

$$
\begin{aligned}
&\|\phi^h_M\|^2 + \nu \triangle t \sum_{n=0}^{M-1} \|\nabla \phi^h_{n+1/2}\|^2 \\
&\quad \le \; \triangle t \sum_{n=0}^{M-1} C (\nu^{-3} \|\nabla u_{n+1/2}\|^4 + 1) \, \|\phi^h_{n+1/2}\|^2 + C \nu h^{2k} |||u|||^2_{2,k+1} \\
&\qquad + C \nu^{-1} \, h^{2k+1} \left(|||u|||^4_{4,k+1} + |||\nabla u|||^4_{4,0} \right) \\
&\qquad + C \nu^{-1} h^{2k} \left(|||u|||^4_{4,k+1} + \nu^{-1} (\|u^h_0\|^2 + \nu^{-1} |||f|||^2_{2,\star}) \right) + C \nu^{-1} h^{2s+2} |||p_{1/2}|||^2_{2,s+1} \\
&\qquad + C \nu^{-1} \alpha^4 \| \, |u| \, \|^2_{4,1} \| \, |u| \, \|^2_{4,3} \\
&\qquad + C (\Delta t)^4 \left(\|u_{ttt}\|^2_{2,0} + \nu^{-1} \|p_{tt}\|^2_{2,0} + \|f_{tt}\|^2_{2,0} \right. \\
&\qquad\qquad + \nu \|\nabla u_{tt}\|^2_{2,0} + \nu^{-1} \|\nabla u_{tt}\|^4_{4,0} \\
&\qquad\qquad \left. + \nu^{-1} |||\nabla u|||^4_{4,0} + \nu^{-1} |||\nabla u_{1/2}|||^4_{4,0} \right) . \qquad (3.35)
\end{aligned}
$$

Hence, with $\triangle t$ sufficiently small, i.e. $\triangle t < C(\nu^{-3}|||\nabla u|||_{\infty,0}^4+1)^{-1}$, from Gronwall's Lemma (see Lemma 2.15), we have

$$
\begin{aligned}
&\|\phi_M^h\|^2 + \nu\triangle t \sum_{n=0}^{M-1} \|\nabla\phi_{n+1/2}^h\|^2 \\
&\leq \; C^* \Big\{ \nu^{-1}\, h^{2k+1} \left(|||u|||_{4,k+1}^4 + |||\nabla u|||_{4,0}^4 \right) + \nu h^{2k} |||u|||_{2,k+1}^2 \\
&\quad +\nu^{-1} h^{2k} \Big(|||u|||_{4,k+1}^4 + \nu^{-1}(\|u_0^h\|^2 + \nu^{-1}|||f|||_{2,\star}^2) \Big) + \nu^{-1} h^{2s+2} |||p_{1/2}|||_{2,s+1}^2 \\
&\quad + C\nu^{-1}\alpha^4 \|\,|u|\,\|_{4,1}^2 \|\,|u|\,\|_{4,3}^2 \\
&\quad + (\Delta t)^4 \left(\|u_{ttt}\|_{2,0}^2 + \nu^{-1}\|p_{tt}\|_{2,0}^2 + \|f_{tt}\|_{2,0}^2 \right. \\
&\qquad +\nu\|\nabla u_{tt}\|_{2,0}^2 + \nu^{-1}\|\nabla u_{tt}\|_{4,0}^4 \\
&\qquad \left. +\nu^{-1}|||\nabla u|||_{4,0}^4 + \nu^{-1}|||\nabla u_{1/2}|||_{4,0}^4 \right) \Big\} \,. \qquad (3.36)
\end{aligned}
$$

where $C^* = Cexp(C\nu^{-3}T)$

Estimate (3.5) then follows from the triangle inequality and (3.36).

To obtain (3.6), we use (3.36) and

$$
\begin{aligned}
&\|\nabla \Big(u(t_{n+1/2}) - (u_{n+1}^h + u_n^h)/2 \Big) \|^2 \\
&\leq \; \|\nabla(u(t_{n+1/2}) - u_{n+1/2})\|^2 \; + \; \|\nabla\eta_{n+1/2}\|^2 \; + \; \|\nabla\phi_{n+1/2}^h\|^2 \\
&\leq \; \frac{(\triangle t)^3}{48} \int_{t_n}^{t_{n+1}} \|\nabla u_{tt}\|^2 \, dt \; + \; Ch^{2k}|u_{n+1}|_{k+1}^2 \; + \; Ch^{2k}|u_n|_{k+1}^2 \; + \; \|\nabla\phi_{n+1/2}^h\|^2 \,.
\end{aligned}
$$

□

4 Numerical Experiments

4.1 Convergence Rate Verification

Our first test is designed to test (and does confirm) the predicted rates of convergence. We use the software *FreeFem++* [HePi] to run the numerical tests. To test the predicted convergence rates for CN, the models are discretized with the full CN method, the nonlinear system was solved by a fixed point iteration and discretized in space using the FEM with the Taylor-Hood element (continuous piecewise quadratic polynomials for the velocity and linears for the pressure). Using the same method for both also allows a fair comparison of NS-α and NS-$\overline{\omega}$.

The Chorin vortex decay problem is an interesting test problem in which the true solution is known. It can be found in Chorin [Cho68] and was also used by Tafti [Tafti] and John and Layton [JL02]. The prescribed solution in $\Omega = (0,1) \times (0,1)$ has the form

$$
\begin{aligned}
u_1(x,y,t) &= -\cos(n\pi x)\sin(n\pi y)e^{-2n^2\pi^2 t/\tau} \\
u_2(x,y,t) &= \sin(n\pi x)\cos(n\pi y)e^{-2n^2\pi^2 t/\tau} \\
p(x,y,t) &= -\frac{1}{4}(\cos(2n\pi x) + \cos(2n\pi x))e^{-2n^2\pi^2 t/\tau}
\end{aligned}
$$

When the relaxation time $\tau = Re$, this is a solution of the NSE with $f = 0$, consisting of an $n \times n$ array of oppositely signed vortices that decay as $t \to \infty$.

In our test, we choose $n = 1$, $\triangle t = 0.005$, $T = 1$ and $\alpha = h = 1/m$, where m is the number of subdivisions of the interval $(0, 1)$. The results for the NS-$\overline{\omega}$ model are presented in Table 1. The convergence rate is calculated from the error at two successive values of h in the usual manner by postulating $e(h) = Ch^{\beta}$ and solving for β via $\beta = \ln(e(h_1)/e(h_2))/\ln(h_1/h_2)$.

From the table we see the convergence rate approaches the second order predicted for $|||\nabla u - \nabla u^h|||_{2,0}$. We also see what appears to be an L^2 lift for $|||u - u^h|||_{\infty,0}$.

m	$\|\|\|u - u^h\|\|\|_{\infty,0}$	rate	$\|\|\|\nabla u - \nabla u^h\|\|\|_{2,0}$	rate
8	$1.3974 \cdot 10^{-1}$		5.03788	
16	$4.06505 \cdot 10^{-2}$	1.78	3.36582	0.58
24	$1.70328 \cdot 10^{-2}$	2.15	2.27963	0.96
32	$8.5897 \cdot 10^{-3}$	2.38	1.6152	1.19
40	$4.81706 \cdot 10^{-3}$	2.59	1.18214	1.40
48	$2.90948 \cdot 10^{-3}$	2.76	$8.85739 \cdot 10^{-1}$	1.58
56	$1.85706 \cdot 10^{-3}$	2.91	$6.76137 \cdot 10^{-1}$	1.75
64	$1.23504 \cdot 10^{-3}$	3.05	$5.24409 \cdot 10^{-1}$	1.90
72	$8.48737 \cdot 10^{-4}$	3.18	$4.12513 \cdot 10^{-1}$	2.04

Table 1: Errors and convergence rates for the NS-omega model at $Re = 10^5$.

4.2 Algorithm Runtime Comparison

The main motivation for computing with NS-$\overline{\omega}$ versus NS-α was for more efficiency, but not at the cost of less accuracy. In the next experiment, we compute the CN NS-α scheme (2.11),(2.12) and the CN NS-$\overline{\omega}$ scheme (2.11),(2.13) for the Chorin problem with the same parameters as used in the simulation presented in Table 1, and with $Re = 100$. Shown in Table 2 are runtimes and in Tables 3,4 errors for both of these algorithms, as well as runtimes and errors for the CNLE and CNQE schemes for NS-$\overline{\omega}$ (as discussed above, such schemes are not possible for NS-α).

Table 2 shows, as expected, that the CN scheme for CN NS-$\overline{\omega}$ is significantly faster than that for NS-α. This is no surprise since it generates much smaller matrices and is stable in a stronger norm. It is more interesting that in Tables 3 and 4 we can observe that CN NS-$\overline{\omega}$ and two linearizations of it (CNLE and CNQE) are also much more accurate than the expensive NS-α scheme.

m	NS-Omega	NS-alpha	NS-Omega CNLE	NS-omega CNQE
4	25.45	89.85	13.86	14.06
8	89.06	278.41	49.80	50.687
16	361.27	1117.48	205.75	206.27
32	1509.67	4161.5	830.015	840.78
64	6883.17	22557.7	3787.27	3948.25

Table 2: Run times of models at $Re = 100$.

m	NS-Omega	NS-alpha	NS-Omega CNLE	NS-omega CNQE
4	$5.5263 \cdot 10^{-2}$	$8.54457 \cdot 10^{-1}$	$5.53164 \cdot 10^{-2}$	$5.53156 \cdot 10^{-2}$
8	$7.68681 \cdot 10^{-3}$	$2.17741 \cdot 10^{-1}$	$7.69444 \cdot 10^{-3}$	$7.69427 \cdot 10^{-3}$
16	$1.52416 \cdot 10^{-3}$	$5.36912 \cdot 10^{-2}$	$1.52558 \cdot 10^{-3}$	$1.52553 \cdot 10^{-3}$
32	$2.19401 \cdot 10^{-4}$	$1.30726 \cdot 10^{-2}$	$2.19642 \cdot 10^{-4}$	$2.19637 \cdot 10^{-4}$
64	$2.39093 \cdot 10^{-5}$	$3.2016 \cdot 10^{-3}$	$2.39978 \cdot 10^{-5}$	$2.39979 \cdot 10^{-5}$

Table 3: The $|||u - u^h|||_{\infty,0}$ errors at $Re = 10^2$.

m	NS-Omega	NS-alpha	NS-Omega CNLE	NS-omega CNQE
4	1.10152	7.30353	1.10395	1.1039
8	$2.97759 \cdot 10^{-1}$	2.13637	$2.98372 \cdot 10^{-1}$	$2.98329 \cdot 10^{-1}$
16	$5.32635 \cdot 10^{-2}$	$6.86788 \cdot 10^{-1}$	$5.33942 \cdot 10^{-2}$	$5.33945 \cdot 10^{-2}$
32	$8.18802 \cdot 10^{-3}$	$2.31259 \cdot 10^{-1}$	$8.2195 \cdot 10^{-3}$	$8.24665 \cdot 10^{-3}$
64	$1.22089 \cdot 10^{-3}$	$7.97553 \cdot 10^{-2}$	$1.2308 \cdot 10^{-3}$	$1.24955 \cdot 10^{-3}$

Table 4: The $|||\nabla u - \nabla u^h|||_{2,0}$ errors at $Re = 10^2$.

4.3 Flow around a cylinder

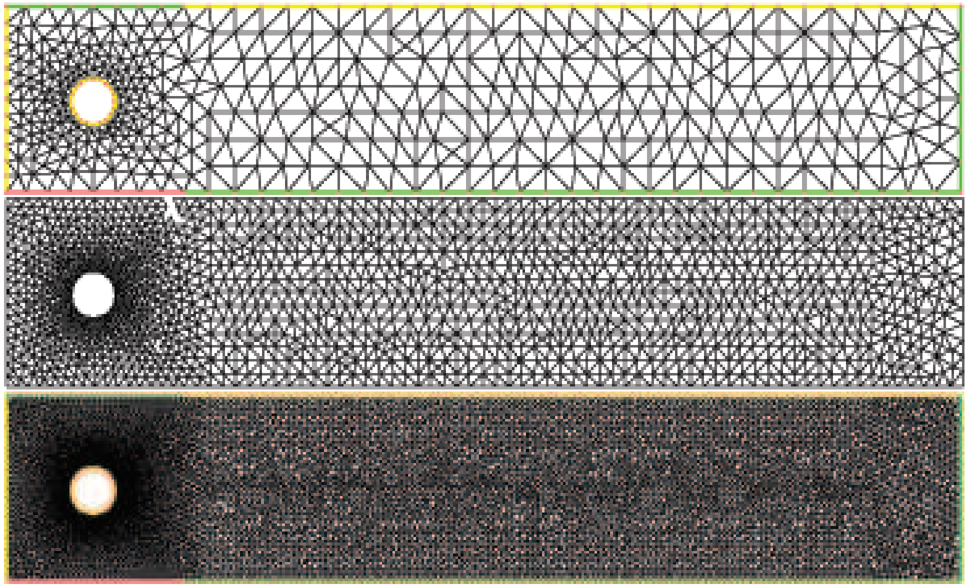
Our final numerical experiment is for two dimensional under-resolved flow around a cylinder. This is a well known benchmark problem taken from Shăfer and Turek [ST96]. It is not turbulent but does have interesting features. The flow patterns are driven by interaction of a fluid with a wall. Such flows are critical if regularizations (and LES models too) are to be useful for real, industrial type flows. It is also interesting since success and failure are clear (vortex street or not) and thus comparison of higher order statistics is not necessary to reach a clear conclusion.

The time dependent inflow and outflow profile are

$$u_1(0,y,t) = u_1(2.2,y,t) = \frac{6}{0.41^2}\sin(\pi\, t/8)y(0.41-y)$$
$$u_2(0,y,t) = u_2(2.2,y,t) = 0.$$

No slip boundary conditions are prescribed along the top and bottom walls and the initial condition is $u(x,y,0) = 0$. The viscosity is $\nu = 10^{-3}$ and the external force $f = 0$. The Reynolds number of the flow, based on the diameter of the cylinder and on the mean velocity inflow is $0 \le Re \le 100$. Freefem provides 3 meshes for testing this problem, the finest of which is able to resolve the problem for the Navier-Stokes equations. These are shown in Figure 1. The filtering radius α is chosen to be the average element length of the respective mesh.

Figure 1: Shown below are three levels of mesh refinement provided by Freefem for computing flow around a cylinder. The meshes provide, respectively, 3,975, 14,455, and 56,702 degrees of freedom for the computations.



For this setting, it is expected that, as the flow increases, from time $t = 2$ to $t = 4$, two vortices start to develop behind the cylinder. Between $t = 4$ and $t = 5$, the vortices should separate from the cylinder, so that a vortex street develops, and they continue to be visible through the final time $t = 8$. This can be seen in Figure 2, which is the solution for resolved Navier-Stokes equations on mesh 3.

However, we were unable to obtain similar results for NS-α or NS-$\overline{\omega}$. Regularizations often give approximate solutions that look like NSE solutions at lower Reynolds numbers (rather than restrictions of the fine mesh solutions to coarser meshes). Some of this behavior is observed for the NS-α and NS-$\overline{\omega}$ on the coarser meshes. For NS-α, we were only able to get results for mesh 1, because we were unable to make the nonlinear iteration converge for the finer meshes. The flow resulting from NS-α on mesh 1 at time $t = 8$ is shown in Figure 4, and does not capture the correct flow dynamics. Mesh 2 results for NS-$\overline{\omega}$ are presented for $t = 8$ in Figure 3, and it can be seen that it was also unable to capture the correct flow behavior.

Our goal in this test has been to make the models fail on the coarser meshes. Since we believe understanding the limitations of a model are just as important (if not more so) as knowing what it does well, we believe presenting these shortcomings are perhaps more important than showing another similar success.

Figure 2: NSE Rotational Form, Formation of a vortex street, times 2,4,5,6,7,8, dt=0.005

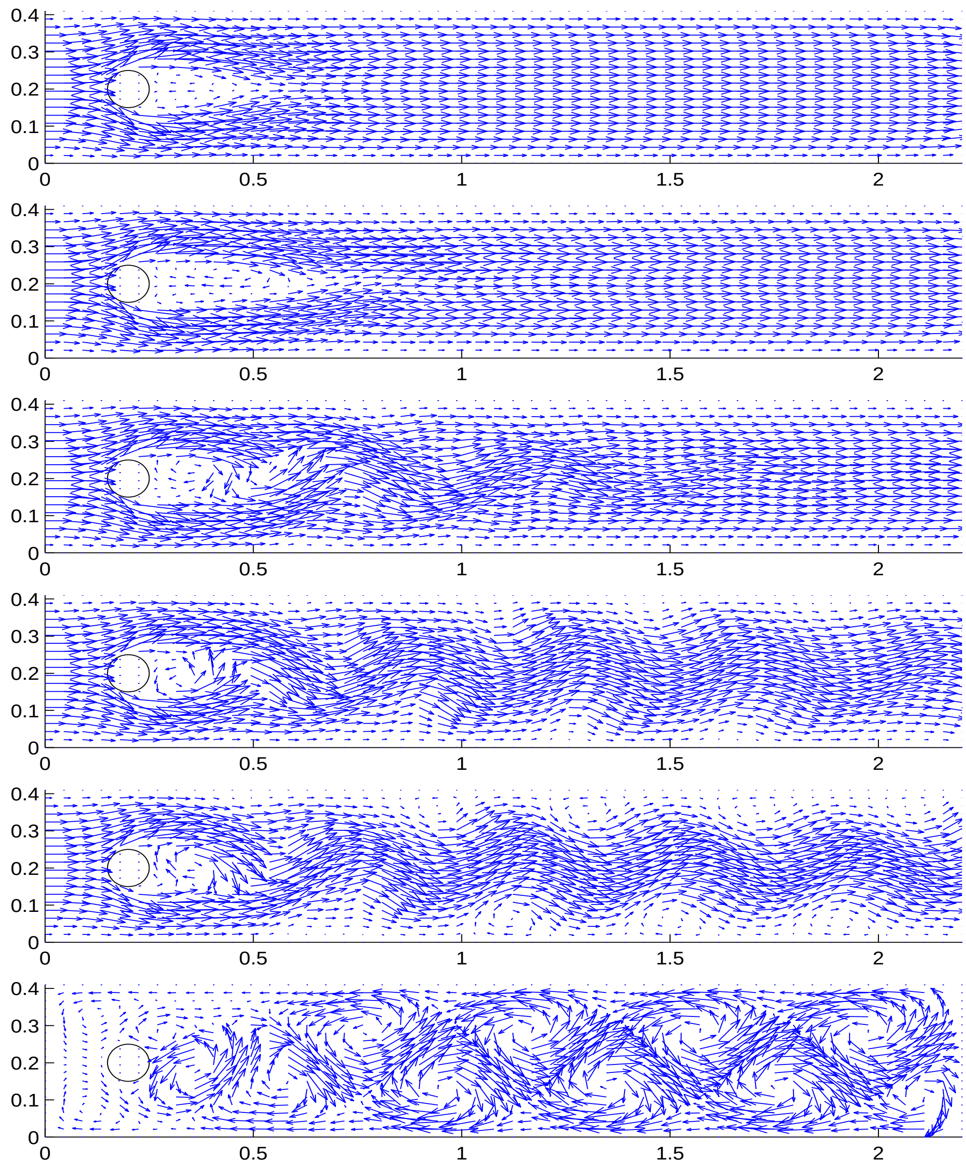


Figure 3: NS-ω model computed on mesh 2 at time t=8, dt=0.005

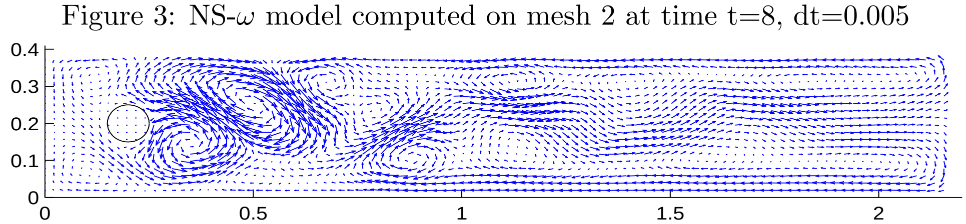
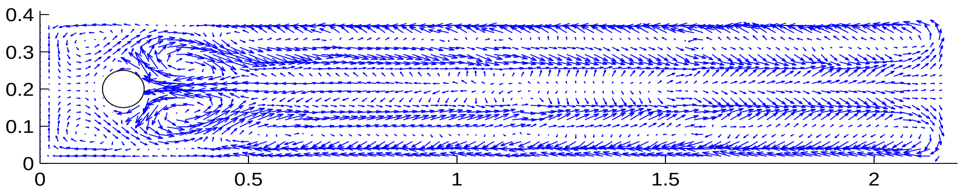


Figure 4: NS-α model computed on mesh 1 at time t=8, dt=0.005



5 Conclusions and Future Directions

Reduced models of fluid motion exist as intermediate steps in under-resolved flow simulations. Thus, within any approach to flow modeling the pairing of efficient algorithms with interesting models must be considered as part of the solution process going from fluid phenomena to numerical simulation. Considering possible combinations leads to interesting developments in continuum models as well as algorithms. We have considered one such development herein. The form of (1.1)-(1.3), its integral invariants, and its development from the deconvolution and NS-α circle of ideas suggest some possibilities for future progress. Three natural ones are synthesis of NS alpha and NS-$\overline{\omega}$ models, higher accuracy in modeling through deconvolution and enhanced scale truncation through combination with VMM / time relaxation ideas.

Synthesis of NS-α and $\overline{\omega}$ models. Although our intuition is the contrary, further study of (1.1)-(1.3) and (1.4)-(1.6) could indicate that they perform well in different flow regions. If this is the case it might be valuable to study combinations and self-adaptive local transition between the models. A simple combination, preserving attractive theoretical properties is given by

$$u_t - \overline{u} \times \nabla \times \overline{u} + \nabla q - \nu \Delta u = f, \tag{5.1}$$

$$\nabla \cdot \overline{u} = 0, \text{ and } -\alpha^2 \Delta \overline{u} + \overline{u} = u. \tag{5.2}$$

An interesting possibility is to include a switching parameter[1], $0 \leq \theta(x,t) \leq 1$, and consider

$$u_t - [\theta\overline{u} + (1-\theta)u] \times \nabla \times [\theta\overline{u} + (1-\theta)u] + \nabla q - \nu \Delta u = f, \tag{5.3}$$

$$\nabla \cdot [\theta\overline{u} + (1-\theta)u] = 0, \text{ and } -\alpha^2 \Delta \overline{u} + \overline{u} = u. \tag{5.4}$$

This possibility must include determination of a method for self-adaptively switching between models locally, i.e., determining θ.

The NS-$\overline{\omega}$-deconvolution models. Computational experience of many years in CFD affirms that accuracy is still important. Two difficulties with (1.1)-(1.3) and (1.4)-(1.6) are that (i) low model accuracy of only $O(\alpha^2)$, and (ii) the differential / Helmholz filter (1.3) does not truncate small scales effectively enough. One solution to both difficulties (following its success in the Leray-deconvolution models [LMNR08b] is to study NS-$\overline{\omega}$-deconvolution models. To avoid extra notation, we give the first nontrivial example.

CNLE uses linear time extrapolation: $u(t_{n+1}) = 2u(t_n) - u(t_{n-1}) + O(\triangle t^2)$. A similar extrapolation can be used in scale space as a deconvolution operator: $u = 2\overline{\overline{u}} - \overline{u} + O(\alpha^4)$. Calling $D_1\overline{u} := 2\overline{\overline{u}} - \overline{u}$ it is a remarkable fact that not only does $D_1\overline{u}$ approximate u to higher order but it also truncates small scales much more effectively than $\overline{u}$, e.g., [LMNR08b]. The

[1]Included in a way to ensure the possibility of a robust mathematical theory.

NS-$\overline{\omega}$-deconvolution models are then (with D denoting the deconvolution operator)

$$u_t - u \times \nabla \times D\overline{u} + \nabla q - \nu \Delta u = f, \tag{5.5}$$

$$\nabla \cdot \overline{u} = 0, \text{ and } -\alpha^2 \Delta \overline{u} + \overline{u} = u, \tag{5.6}$$

and CNLE algorithms preserve their attractive properties for the entire family of deconvolution regularizations: with $U^n := \frac{3}{2}u^n - \frac{1}{2}u^{n-1}$

$$\frac{u^{n+1} - u^n}{\triangle t} - u^{n+\frac{1}{2}} \times \nabla \times (D\overline{U^n}) - \nu \triangle u^{n+\frac{1}{2}} + \nabla p^{n+\frac{1}{2}} = f^{\,n+\frac{1}{2}}, \text{ and } \nabla \cdot u^{n+\frac{1}{2}} = 0 \ . \tag{5.7}$$

The deconvolution operator involves repeated filtering. Efficiency is preserved if, as here, it acts on a velocity known from previous time levels.

Scale truncation, eddy viscosity, VMMs and time relaxation. One basic difficulty shared by (1.1)-(1.3) and (1.4)-(1.6), e.g., [FHT01],[LST08], is that scale truncation is insufficient. While the model's microscales, η_{model}, (as predicted by turbulence phenomenology) is larger that the NSE microscale, η_{NSE}, it is far larger than the filter length scale:

$$\text{for (1.1)-(1.3) and (1.4)-(1.6):} \quad \eta_{model} >> \alpha \simeq O(h).$$

In the usual K41 phenomenology, [F95], microscales can only be altered by increasing energy dissipation. There are three possibilities (the last two have less influence on the resolved scales) which can be added to any model to enhance scale truncation: (i) use algorithms which include extra numerical dissipation (which if not carefully done acts on all scales), (ii) subgrid eddy viscosity / Variational Multiscale small-small Smagorinsky, [HKJ00, HOM01], and (iii) time relaxation, e.g., [LN07, SA99, SAK01a, SAK01b]. We believe that the last two are related and are fundamental tools for effective simulation of under-resolved flows.

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