A MIXED FINITE ELEMENT APPROXIMATION OF STOKES-BRINKMAN AND NS-BRINKMAN EQUATION FOR NON-DARCIAN FLOWS

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Abstract.
We propose a finite element discretization of the Brinkman equation for modeling non-Darcian fluid flow by allowing the Brinkman viscosity $\tilde{\nu} \to \infty$ and permeability $K \to 0$ in solid obstacles, and $K \to \infty$ in fluid domain. In this context, the Brinkman parameters are generally highly discontinuous. Furthermore, we consider non-generic constraints: non-homogeneous Dirichlet boundary conditions $u|_{\partial\Omega} = \phi \neq 0$ and non-solenoidal velocity $\nabla \cdot u = g \neq 0$ (to model sources/sinks). Coupling between these two conditions makes even existence of solutions subtle. We establish well-posedness of the continuous and discrete problem, a priori stability estimates, and convergence as $\tilde{\nu} \to \infty$ and $K \to 0$ in solid obstacles, as $K \to \infty$ in fluid region, and as the mesh width $h \to 0$. For non-solenoidal Brinkman flows, we include a small data condition to ensure existence of solutions (idea applies directly to the steady Navier-Stokes equations). In addition, we propose a pseudo-skew-symmetrization of the discrete convective term $\int_\Omega u \cdot \nabla v \cdot w$ required for analysis of discrete non-solenoidal Brinkman problem.

1. Introduction. This report considers the approximation of high velocity flow through complex geometries involving pores. Motivating examples include the flow through closely placed turbines on a windfarm [12], [24] and the high velocity flow of helium gas through a packed bed of (tennis-ball sized, uranium fuel) spheres in a pebble bed nuclear reactor [28], [27]. In both applications, the fluid velocities are too large to model accurately with Darcy's equation and the pore geometry is too complex to approximate by the Navier-Stokes equations (NSE) in the pore region with no-slip boundary conditions on the solid obstacles. Therefore, appropriate for this setting, we propose a finite element method for the Brinkman model, beginning with the equilibrium case.

Problem 1.1. *(NS-Brinkman model) For incompressible, viscous fluid flow in Ω, find velocity u^δ and pressure p^δ satisfying*

$$\begin{aligned} -2\nabla \cdot \left(\tilde{\nu} D\left(u^\delta\right)\right) + u^\delta \cdot \nabla u^\delta + \nabla p^\delta + \nu K^{-1} u^\delta &= f, && \text{in } \Omega \\ \nabla \cdot u^\delta &= g, && \text{in } \Omega \\ u^\delta &= \phi, && \text{in } \partial\Omega \end{aligned}$$

Here, $\Omega \subset \mathbb{R}^d$ is an open domain for $d = 2$ or 3 consisting of both the pores and solid obstacles, $D\left(u^\delta\right) = 0.5(\nabla u^\delta + \left(\nabla u^\delta\right)^t)$ is the deformation tensor, f represents body forces, g represents sources and/or sinks in Ω, K is the permeability tensor, ν is the kinematical viscosity, and $\tilde{\nu}$ is the Brinkman viscosity. In particular, we fix $0 < \delta << 1$ and set $\tilde{\nu} = 1/\delta$ and $K = 1/\delta$ in the solid obstacles in Ω and $\tilde{\nu} = \nu$ and $K = \delta$ in the purely fluid parts of Ω. See Figure 1.1.

For low Reynolds numbers, the convective term $u^\delta \cdot \nabla u^\delta$ is negligible; thus, we also consider the Stokes-Brinkman model.

Problem 1.2. *(Stokes-Brinkman model) for u^δ and p^δ*

$$\begin{aligned} -2\nabla \cdot \left(\tilde{\nu} D\left(u^\delta\right)\right) + \nabla p^\delta + \nu K^{-1} u^\delta &= f, && \text{in } \Omega \\ \nabla \cdot u^\delta &= g, && \text{in } \Omega \\ u^\delta &= \phi, && \text{in } \partial\Omega \end{aligned}$$

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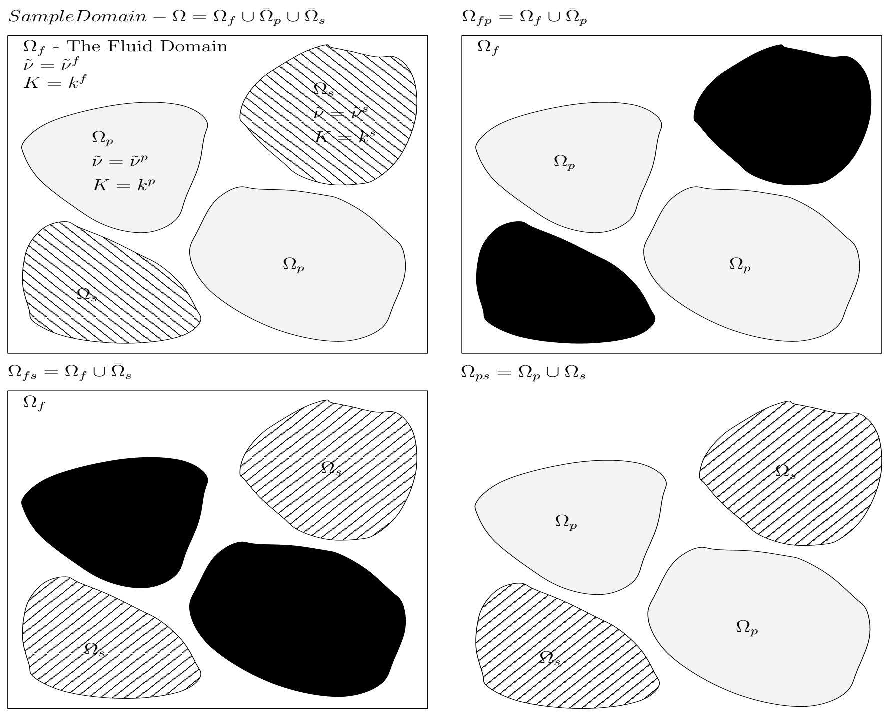

FIG. 1.1. *Sample subdomains $\Omega_* \subset \Omega$ with parameters $\tilde{\nu}^*$, k^*, black regions not part of indicated Ω_*. (top-left) Problem domain $\Omega_{fps} = \Omega$, (top-right) Fluid-Porous domain Ω_{fp}, (bottom-left) Fluid-Solid domain Ω_{fs}, (bottom-right) Porous-Solid domain Ω_{ps}*

For model parameters $\tilde{\nu}$ and K of order $\mathcal{O}(1)$, the numerical analysis of the Brinkman model fits within the framework for the abstract error analysis of the NSE, e.g. [11], [32]. However, the targeted applications of the Brinkman model are often highly non-generic flows involving

- complex geometries, i.e. dense swarm of porous and solid obstacles
- highly discontinuous parameters $\tilde{\nu}$ and K
- non-homogeneous boundary conditions, i.e. $u^\delta|_{\partial\Omega} = \phi \neq 0$
- general divergence conditions on velocity, i.e. $\nabla \cdot u^\delta = g \neq 0$

Thus, we consider herein the numerical analysis associated with the asymptotic limits and rates of convergence as the discretization parameter h δ tend to 0. The last two conditions $u^\delta|_{\partial\Omega} \neq 0$ and $\nabla \cdot u^\delta \neq 0$ in Ω are necessary for many natural and industrial flows in porous media.

We derive a weak formulation of Stokes and NS-Brinkman models in Section 2. Note that Hopf proved in [15] that solutions to the steady NSE exist for general boundary data under certain restrictions on $\partial\Omega$ for the case $\nabla \cdot u = 0$. In Section 2.2, we note coupling between $u^\delta|_{\partial\Omega} = \phi$ and $\nabla \cdot u^\delta = g \neq 0$ preventing a general existence result for nonzero boundary conditions and nonzero divergence. Our analysis is based on the construction of an extension operator $\tilde{u}$ of boundary data ϕ satisfying the constraint $\nabla \cdot \tilde{u} = g$. We show that for $g \in L^2(\Omega)$ and $\phi \in H^{1/2}(\partial\Omega)$ satisfying

- $g \equiv 0$, or
- g has compact support in Ω, $\int_\Omega g = 0$, and g small enough, or
- g has compact support in Ω, and g and ϕ is small enough

there exists a solution $\left(u^{\delta}, p^{\delta}\right) \in H^{1}(\Omega) \times L^{2}(\Omega)$ to the NS-Brinkman Problem 1.1. Furthermore, we show that the continuous (Section 2.3) and discrete (Section 3.1) Stokes- and NS-Brinkman models are well-posed (with small data for the nonlinear problem). We derive a priori estimates for u^{δ} with explicit dependence on ν, $\tilde{\nu}$, and K. For $\delta > 0$, both the continuous and discrete velocities for both Stokes-Brinkman and NSE-Brinkman are of order $\mathcal{O}(\sqrt{\delta}/\nu)$ in H^1 in the solid obstacles embedded in Ω and $\mathcal{O}(1/\nu)$ in H^1 in all Ω; hence, for fixed $\nu > 0$, u^{δ} is uniformly stable with respect to δ with respect to $\delta \to 0$.

For the numerical scheme, we provide a condition for interpolating non-smooth boundary data used in the analysis of the finite element discretization of the Brinkman model. We also propose an innovative (explicitly pseudo-skew symmetrized, defined in Section 3, for general g) discrete form for the convective term $u \cdot \nabla u$. In Section 3.1, we show that the proposed conforming finite element discretization provides a convergent approximation $u^{\delta,h}$ of u^{δ} as $h \to 0$ uniformly with respect to the penalty parameter δ. In Section 3.2, letting u be a solution to the Stokes problem in the purely fluid domain $\Omega_f \subset \Omega$ with no-slip boundary conditions on the solid obstacles, we show that the the discrete Brinkman velocity $u^{\delta,h}$ converges to u as h, $\delta \to 0$ such that

$$\left\|u^{\delta,h} - u\right\|_{H^1(\Omega)} \leq C\left(\frac{\delta}{\nu} + \left\|u^{\delta} - u^{\delta,h}\right\|_{H^1(\Omega)}\right)$$

Finally, we provide numerical validations of our theory in Section 4.

1.1. Overview of Brinkman flow model. Whereas Darcy's law assumes that velocity is proportional to the pressure gradient for a particular porous medium, Brinkman noted that, in general, the viscous effects must also be taken into account to model flow accurately through porous media, see [6], [7]. Heuristic generalizations of Darcy's law have been considered to model non-Darcy flows in porous media (e.g. [14], [18], [26], [5]). Along with heuristic developments, theoretical justifications exist for the Brinkman model as an asymptotic approximation to the NSE, e.g. see [1], [16] and references therein. Straughan presents several of the most popular non-Darcy models for flow in porous media in [30] (a well-cited compilation of his and others' contributions to this theory).

The Brinkman model has been applied to approximate non-Darcian flows in a variety of contexts; e.g. it is used to model oil filtration flows [17], groundwater flows [8], forced convective flows in metal foam-filled pipes (used in the cooling of electronic equipment) [23], gas diffusion through fuel cell membranes [13], Casson fluid flow in porous media (e.g. blood flow in vessels obstructed by fatty plaques and clots) [9], and interstitial fluid flow through muscle cells [31] with good accuracy. The Brinkman equation is also used to model turbulence in porous media in the macroscopic scales [19] (for a discussion concerning turbulence modeling at the macroscopic versus the microscopic pore level see [25]).

Numerical analysis of a discretization of the Stokes and NS-Brinkman flow model is limited. In [33], Xie et.al. provide an innovative numerical analysis of the Stokes-Brinkman equations with a condition that ensures stable finite element spaces for the discrete Stokes-Brinkman equation in the limiting condition for high Reynold's number. In [2], Angot provides a beautifully detailed error analysis for the continuous Stokes-Brinkman fluid velocity in fluid-porous and fluid-solid domains compared to Darcy-Stokes velocities.

1.2. Approximating Brinkman flow in Ω. Solving the NSE in the pores of Ω with no-slip boundary conditions at solid interfaces or non-stationary and/or complex domain boundaries is cumbersome at best and most often simply not feasible [2], [3]. Furthermore, the coupling condition between Stokes flow domains and Darcy flow domains (used for flow in porous media) is physically unresolved even though the Beavers-Joseph-Saffman (BJS) interface condition is widely accepted and generally used in practice [4], [29], [21]. Furthermore, Layton et.al show in [21] that coupled Stokes-Darcy flow using the BJS interface condition is well-posed, but such a conclusion has not been verified for the nonlinear NS-Darcy coupling. In addition, further complications arise because Stokes velocity has meaning of a "pointwise" velocity and Darcy velocity has meaning of an "averaged" velocity providing an unresolved compatability issue between these two velocities, see e.g. [18], [26], [5].

It is exactly these shortcomings in coupling Stokes or NSE with Darcy's equations that are the strengths of the Brinkman flow model. To this end, we consider the penalized Brinkman problem formulated and described in [20] with convergence analysis in [2]. In particular, when approximating flows in Ω, we want u^δ be as small as possible inside all solid obstacles $\Omega_s \subset \Omega$ and recover the no-slip condition on each solid interface $\partial\Omega_s$. This is attained by imposing a large Brinkman viscosity $\tilde{\nu}$ and small permeability k in Ω_s. In addition, in the purely fluid region $\Omega_f \in \Omega$, there are no medium obstacles impeding the flow; thus, the permeability k in Ω_f should be large. Consider a small, parameter $0 < \delta << 1$ and set

$$\tilde{\nu}|_{\Omega_s} = \frac{\nu}{\delta}, \qquad k|_{\Omega_s} = \delta, \qquad k|_{\Omega_f} = \frac{1}{\delta}$$

We are interested in the asymptotic behavior of solutions u^δ to Problems 1.1 and 1.2 as $\delta \to 0$ and the double asymptotic of approximate solutions $u^{\delta,h}$ as $\delta \to 0$, $h \to 0$. This fictitious domain approach has been analyzed in various contexts for the continuous Brinkman velocity u^δ, see e.g. [2], [3], [19], [22]. The Brinkman approach eliminates the mathematical and physical problems with the interface couplings. Moreover, it is simple in implementation and easily adapted to existing computing platforms.

2. Problem formulation. We are interested in fluid flow through a porous medium Ω, an open and connected domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, refer to Figure 1.1 for an illustration. Decompose Ω into a purely fluid domain Ω_f (no flow obstruction), porous domain Ω_p (some flow obstruction) and purely solid domain Ω_s (complete flow obstruction)

$$\Omega = \Omega_f \cup \bar{\Omega}_p \cup \bar{\Omega}_s$$

where $\partial\Omega_f$, $\partial\Omega_p$, and $\partial\Omega_s$ represent the corresponding boundaries of the indicated subdomains. We allow $\partial\Omega$ to be the union of distinct, connected segments. We assume that $\partial\Omega_p$ and $\partial\Omega_s$ do not intersect with the problem domain boundary $\partial\Omega$, thatΩ_p and Ω_s consist of open and connected subsets of Ω, and Ω_p and Ω_s are disjoint and bounded away from $\partial\Omega$

$$\bar{\Omega}_p \cap \bar{\Omega}_s = \emptyset, \qquad \left(\bar{\Omega}_p \cup \bar{\Omega}_s\right) \cap \partial\Omega = \emptyset$$

Lastly, we require that Ω_f is necessarily connected such that

$$\partial\Omega_f = \partial\Omega \cup \partial\Omega_p \cup \partial\Omega_s$$

We write Ω_* for $* = f,\, p,\, s,\, fp,\, fs,\, ps$, and fps such that

$$\begin{array}{ll} \Omega_{fp} & := \Omega_f \cup \bar{\Omega}_p \\ \Omega_{fs} & := \Omega_f \cup \bar{\Omega}_s \\ \Omega_{ps} & := \Omega_p \cup \Omega_s \\ \Omega_{fps} & := \Omega \end{array}$$

See Figure 1.1 for an illustration.

We assume that $\nu > 0$ is constant in Ω. Also, $\tilde{\nu} > 0$ is piecewise constant and constant in each subdomain Ω_f, Ω_p, and Ω_s such that $\tilde{\nu}|_{\Omega_f} = \nu$ and $\tilde{\nu}|_{\Omega_p} = \nu$. We write

$$\tilde{\nu}^f := \tilde{\nu}|_{\Omega_f} = \nu, \qquad \tilde{\nu}^p := \tilde{\nu}|_{\Omega_p} = \nu, \qquad \tilde{\nu}^s := \tilde{\nu}|_{\Omega_s}$$

See Figure 1.1 for an illustration. Moreover, we write

$$\tilde{\nu}^{fp}(x) := \begin{cases} \tilde{\nu}^f, \; x \in \Omega_f \\ \tilde{\nu}^p, \; x \in \Omega_p \end{cases}, \; \tilde{\nu}^{fs}(x) := \begin{cases} \tilde{\nu}^f, \; x \in \Omega_f \\ \tilde{\nu}^s, \; x \in \Omega_s \end{cases}, \; \tilde{\nu}^{ps}(x) := \begin{cases} \tilde{\nu}^p, \; x \in \Omega_p \\ \tilde{\nu}^s, \; x \in \Omega_s \end{cases}$$

and identify $\tilde{\nu} = \tilde{\nu}^{fps}$ (recall, $\Omega_{fps} = \Omega$). The permeability tensor $K \in L^\infty(\Omega)^{d \times d}$ is generally symmetric and positive definite. We assume that K is a constant scalar on each subdomain Ω_f, Ω_p, and Ω_s and write

$$k^* := K|_{\Omega_*}, \qquad * = f,\, p,\, s,\, fp,\, fs,\, ps, \text{ and } fps$$

and identify $K = k^{fps}$. Lastly, for $* = f,\, p,\, s,\, fp,\, fs,\, ps$, and fps write

$$\begin{array}{llll} \tilde{\nu}^*_{max} & := \max\limits_{x \in \Omega_*} \tilde{\nu}(x), & \tilde{\nu}^*_{min} & := \min\limits_{x \in \Omega_*} \tilde{\nu}(x) \\ k^*_{max} & := \max\limits_{x \in \Omega_*} K(x), & k^*_{min} & := \min\limits_{x \in \Omega_*} K(x) \end{array}$$

and identify $\tilde{\nu}_{max} := \tilde{\nu}^{fps}_{max}$, $\tilde{\nu}_{min} := \tilde{\nu}^{fps}_{min}$, $k_{max} := k^{fps}_{max}$, and $k_{max} := k^{fps}_{max}$.

In porous regions Ω_p, the Brinkman viscosity k^p and $\tilde{\nu}^p$ should have moderate values. We suppose that k^p depends on the domain geometry (e.g. see [5]). It is not well understood how to select the Brinkman viscosity $\tilde{\nu}$ in Ω_p. We set $\tilde{\nu}^{fp} = \nu$ which is a common choice in both engineering practice and analytical theory. See, e.g., [5] and [18] for more on this subject.

Lastly, assume that $f \in H^{-1}(\Omega)^d$ (the dual space of $H^1_0(\Omega)^d$ consisting of H^1-functions vanishing on $\partial\Omega$), $\phi \in H^{1/2}(\partial\Omega)^d$, and $g \in L^2(\Omega)$. We suppose that g is localized satisfying

$$g \equiv 0 \text{ in } \bar{\Omega}_f \cup \bar{\Omega}_s$$

and compatible with the boundary data so that $\nabla \cdot u^\delta = g$ and thus

$$\int_\Omega g = \int_{\partial\Omega} \phi \cdot \hat{n} \tag{2.1}$$

In this context, we consider $\left(u^\delta, p^\delta\right)$ satisfying the nonlinear or linear system of equations, Problem 1.1, 1.2.

2.1. Weak formulation. Let V' denote the dual of any linear space V. Let $* = f$, p, s, fp, fs, ps, or fps. We write $(\cdot,\cdot)_*$ to represent the $L^2(\Omega_*)$ inner product and $(\cdot,\cdot)$ when $* = fps$ (recall $\Omega_{fps} = \Omega$). Let $\|\cdot\|_{1,*}$ represent standard norm for $H^1(\Omega_*)$ and write $\|\cdot\|_*$ for the standard $L^2(\Omega_*)$-norm, $\|\cdot\|_1$ for the standard $H^1(\Omega)$-norm ($* = fps$), and $\|\cdot\|$ for the standard $L^2(\Omega)$-norm. Lastly, let $\langle\cdot,\cdot\rangle_{V'\times V}$ represent the duality pairing for linear space V.

DEFINITION 2.1. *Let $\Omega_* \subset \mathbb{R}^d$ be an open set.*

$$L_0^2(\Omega_*) := \left\{q \in L^2(\Omega_*) : \int_{\Omega_*} q = 0\right\}, \qquad H_0^1(\Omega_*) := \left\{v \in H^1(\Omega_*) : v|_{\partial\Omega_*} = 0\right\}$$

Write $Q := L_0^2(\Omega)$ and $X := H_0^1(\Omega)$. For $g \in L^2(\Omega)$, $\phi \in H^{1/2}(\partial\Omega)$

$$\begin{array}{lcl} X_\phi & := & \left\{v \in H^1(\Omega) : v|_{\partial\Omega} = \phi\right\} \\ V_\phi(g) & := & \left\{v \in X_\phi : \int_\Omega q(\nabla\cdot v) = \int_\Omega gq,\ \forall q \in L_0^2(\Omega)\right\} \\ V^* & := & \left\{f \in X' : \langle f, v\rangle_{X'\times H_0^1} = 0,\ \forall v \in V_0(0)\right\} \\ V^\perp & := & \left\{v^\perp \in X : \int_\Omega v^\perp \cdot v = 0,\ \forall v \in V_0(0)\right\} \end{array}$$

and We write $V = V_0(0)$, $V_\phi = V_\phi(0)$, and $V(g) = V_0(g)$. Moreover, $X' = H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)'$ is equipped with the following norm

$$\|f\|_{-1} = \sup_{v\in X, v\neq 0} \frac{\langle f, v\rangle_{X'\times X}}{\|v\|_1}$$

Lastly, we note that $Q' = Q$.

If $f \in L^2(\Omega_*)$ and $v \in H^1(\Omega_*)$, we abuse notation and identify $\langle f, v\rangle_* = (f, v)_*$ even though $H^{-1}(\Omega_*)$ is not the dual of $H^1(\Omega_*)$ (rather, it is the dual of $H_0^1(\Omega_*)$).

Next, we define several (bi/tri)linear functionals:

DEFINITION 2.2. *Let $* = f$, p, s, fp, fs, or fsp. Let u, v, $w \in H^1(\Omega)$, and $q \in L^2(\Omega)$. Define the bilinear and linear forms $a_*(\cdot,\cdot)$, $b_*(\cdot,\cdot)$, $l_2(\cdot)$ by*

$$\begin{array}{llll} a_*(\cdot,\cdot) & : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, & a_*(u,v) & := \int_{\Omega_*} \tilde{\nu}\nabla u : \nabla v + \int_{\Omega_*} \nu K^{-1} u \cdot v \\ b_*(\cdot,\cdot) & : H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}, & b_*(v,q) & := -\int_{\Omega_*} q(\nabla\cdot v) \\ l_{2,*}(\cdot) & : L_0^2(\Omega) \to \mathbb{R}, & l_{2,*}(q) & := -\int_{\Omega_*} gq \end{array}$$

Further, define the linear form $l_{1,*}(\cdot) : H_0^1(\Omega) \to \mathbb{R}$,

$$l_{1,*}(v) := \langle f, v\rangle_{H^{-1}(\Omega_*)\times H_0^1(\Omega_*)} - \int_{\Omega*} \tilde{\nu} g \nabla\cdot v$$

and trilinear form $c_*(\cdot,\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$,

$$c_*(u,v,w) := \int_{\Omega_*} u \cdot \nabla v \cdot w$$

To derive the variational formulation of Problems 1.1 and 1.2, we notice note that $\nabla\cdot\nabla u^t = \nabla(\nabla\cdot u)$. Consequently,

$$\nabla\cdot\left(D\left(u^\delta\right)\right) = \frac{1}{2}\left(\Delta u^\delta + \nabla\nabla\cdot u^\delta\right) = \frac{1}{2}\Delta u^\delta + \frac{1}{2}\nabla g$$

The term ∇g is data and included in $l_1(\cdot)$). Thus, we have the following weak formulation of the NS-Brinkman equations, Problem 1.1.

Problem 2.3. *(Weak NS-Brinkman) Given $f \in X'$ and $g \in L^2(\Omega)$. Find $u^\delta \in X_\phi$, $p^\delta \in Q$ such that*

$$\begin{array}{ll} \forall v \in X, & a\left(u^\delta, v\right) + b\left(v, p^\delta\right) + c\left(u^\delta, u^\delta, v\right) = l_1(v) \\ \forall q \in Q, & b\left(u^\delta, q\right) = l_2(q) \end{array}$$

Similarly, the following variational formulation of Stokes-Brinkman equations, Problem 1.2, follows by setting the convective term $c(\cdot,\cdot,\cdot) = 0$ in Problem 2.3.

Problem 2.4. *(Weak Stokes-Brinkman) Given $f \in X'$ and $g \in L^2(\Omega)$. Find $u^\delta \in X_\phi$, $p^\delta \in Q$ such that*

$$\begin{array}{ll} \forall v \in X, & a\left(u^\delta, v\right) + b\left(v, p^\delta\right) = l_1(v) \\ \forall q \in Q, & b\left(u^\delta, q\right) = l_2(q) \end{array}$$

Alternatively, we can formulate the variational Brinkman problem in operator notation.

Definition 2.5. *For any u, $v \in H^1(\Omega)$, $w \in X$, and $q \in Q$, define $A \in \mathcal{L}(X; X')$, $B \in \mathcal{L}(X; Q)$, and $C(u) \in \mathcal{L}(X; X')$ such that*

$$\begin{array}{lcll} \langle Au, w\rangle_{X'\times X} & := & a(u,v) & \\ \langle Bw, q\rangle_{Q\times Q} & := & b(w,q) & =: \langle B'q, w\rangle_{X'\times X} \\ \langle C(u)v, w\rangle_{X'\times X} & := & c(u,v,w) & \end{array}$$

Thus, we can rewrite Problem (2.3): find $u \in X_\phi$, $p \in Q$ satisfying

$$Au + B'p + C(u)u = f \text{ (in } X'), \qquad Bu = g \text{ (in } Q)$$

We can also rewrite Problem (2.4): find $u \in X_\phi$and $p \in Q$ satisfying

$$Au + B'p = f \text{ (in } X'), \qquad Bu = g \text{ (in } Q)$$

2.2. Calculus on subdomains Ω_*. We often decompose Ω into its physical components the purely fluid region Ω_f, the porous region Ω_p, and solid region Ω_s. In doing so, we must be careful in applying Poincaré's Inequality (since we require that functions vanish on a set of positive measure on the domain boundary) and duality pairing (since $H^{-1}(\Omega_*)$ is dual to $H_0^1(\Omega_*)$ and *not* to $H^1(\Omega_*)$). We state these results in the context of our problem.

Theorem 2.6. *(Poincaré's Inequality) For any $w \in H_0^1(\Omega)$, there exists $C_p^* > 0$ such that*

$$\|w\|_{\Omega_*} \le C_p^* \|\nabla w\|_{\Omega_*}, \text{ for } * = f, fp, fs, fps$$

We generically write $C_p = C_p^*$ for all $*$.

Note that this result is *not* applicable in Ω_s or Ω_p since boundary data is generally not provided on $\partial\Omega_p$ or $\partial\Omega_s$. Additionally, the functional $f \in H^{-1}(\Omega_*)$ acts on elements from its dual space $v \in H_0^1(\Omega_*)$. Again, since boundary data is generally not provided on $\partial\Omega_p$ or $\partial\Omega_s$ and

$$\|f\|_{-1,*} = \sup_{v \in X(\Omega_*), v \neq 0} \frac{\langle f, v\rangle_{X'\times X}}{\|v\|_{1,*}}$$

then $\langle f, v\rangle_{H^{-1}(\Omega_*)\times H_0^1(\Omega_*)} \le \|f\|_{-1,*}\|v\|_{1,*}$ only applies when $* = fps$.

Next, we collect some basic properties of the divergence operator.

LEMMA 2.7. *For $\Omega \subset \mathbb{R}^d$ open and connected the divergence operator is an isomorphism between $V^{\perp}$ and $L_0^2(\Omega)$. Therefore, there exists $\beta > 0$ such that the inf-sup condition holds:*

$$\inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)} \frac{\langle \nabla q, v\rangle}{\|q\| \, \|v\|_1} \geq \beta > 0. \tag{2.2}$$

As a consequence, for any $q \in L_0^2(\Omega)$ there exists a unique $v \in V^{\perp} \subset H_0^1(\Omega)$ satisfying

$$\|v\|_1 \leq \beta^{-1} \|q\| \tag{2.3}$$

Proof. See, for example, page 24, Corollary 2.4 in [11] . ☐

2.3. Decoupling $\nabla \cdot u = g$ and $u|_{\Omega} = \phi$. It is necessary to rewrite the general divergence, nonhomogeneous Brinkman problems 1.2 and 1.1 in terms of a divergence-free velocity vanishing on the boundary $\partial\Omega$. We compile several well-known results, e.g. see [11]. Recall that $\phi \in H^{1/2}(\Omega)$, $f \in X'$, and $g \in L^2(\Omega)$ satisfies the compatibility condition Eq. (2.1). From from the trace theorem, there exists an extension

$$\exists u_\phi \in H^1(\Omega) \text{ satisfying } u_\phi|_{\partial\Omega} = \phi \tag{2.4}$$

$$\text{and } \exists \gamma > 0 \text{ such that } \|u_\phi\|_1 \leq \gamma \|\phi\|_{1/2,\partial\Omega} \tag{2.5}$$

Furthermore, from the compatibility condition Eq. (2.1) along with application of the divergence theorem, we have that $\int_\Omega (g - \nabla \cdot u_\phi) = 0$. Hence, $g - \nabla \cdot u_\phi \in Q$. Lemma 2.6 ensures that $\nabla \cdot : V^{\perp} \to Q$ defines an isomorphism so that

$$\exists! u_0 \in V^{\perp} \subset X \text{ satisfying } \nabla \cdot u_0 = g - \nabla \cdot u_\phi \tag{2.6}$$

Hence, we consider looking for $w \in V$ rather than $u^\delta \in V_\phi(g)$ solving Problems 2.3 or 2.4.

PROPOSITION 2.8. *Given $\phi \in H^{1/2}(\partial\Omega)$ and $g \in L^2(\Omega)$ satisfying the compatibility condition (2.1), suppose that u_ϕ and u_0 are defined as above in Eq.'s (2.4) and (2.6) respectively. Then writing $\tilde{u} := u_\phi + u_0$,*

$$\|\tilde{u}\|_1 = \|u_\phi + u_0\|_1 \leq \left(\gamma + \sqrt{d}\beta^{-1}\right) \|\phi\|_{1/2,\partial\Omega} + \beta^{-1} \|g\| \tag{2.7}$$

Moreover, solving Problems 2.3 and 2.4 for $u^\delta \in V_\phi(g)$ is equivalent to solving the same equations for $w \in V$ where

$$u^\delta = w + u_\phi + u_0, \qquad \nabla \cdot w = 0 \text{ in } \Omega, \qquad w|_{\partial\Omega} = 0 \tag{2.8}$$

Proof. The proof of the first bound follows by applying bound (2.3) and Lemma 2.6. The problem equivalency is obvious. ☐

Unfortunately, this bound is unsatisfying . In particular, similar to the NSE, we must control the problematic term $\int_\Omega w \cdot \nabla \tilde{u} \cdot \, w$, where $w \in V$ is as in Eq. (2.8) and $\tilde{u}$ satisfies $\tilde{u}|_{\partial\Omega} = \phi$ and $\nabla \cdot \tilde{u} = g$ in Ω as in Eq. (2.4). Noting that

$$\int_\Omega w \cdot \nabla \tilde{u} \cdot \, w = -\int_\Omega w \cdot \nabla w \cdot \, \tilde{u} \leq C \|\nabla w\|^2 \|\tilde{u}\|_{L^4(\Omega)} \tag{2.9}$$

must be absorbed from the right-hand side to left-hand side (details follow in the next section), the following proposition is necessary in establishing existence of solutions to NS-Brinkman, Problem 2.3 for arbitrary data $\phi \in H^{1/2}(\partial\Omega)$. This result builds upon the subtle and technical work of Leray rigorously compiled by Hopf in [15] and elegantly presented by Raviart and Girault in [11] and Galdi in [10] and specifically concerns nonhomogeneous boundary data for the steady Navier-Stokes equations with divergence-free constraint $\nabla \cdot u = 0$. Particular care must be taken in a domain with holes when

$$\int_{\partial\Omega^i} u^\delta \cdot \hat{n} = 0 \tag{2.10}$$

is *not* required on each connected component of the boundary $\partial\Omega^i \subset \partial\Omega$, $i = 1, \ldots, l$. We consider another interesting case when sources and sinks are present inside the computational domain itself, i.e. when $\nabla \cdot u^\delta = g \neq 0$.

PROPOSITION 2.9. *Let Ω be an open, connected domain with piecewise Lipschitz boundary $\partial\Omega$ and suppose that $\partial\Omega$ satisfies the Hopf condition (2.10). Suppose that $\phi \in H^{1/2}(\partial\Omega)$ and $g \in L^2(\Omega)$ satisfy the compatibility condition Eq. (2.1) and that $g = 0$ in Ω_s and Ω_f and $g \neq 0$ in Ω_p. Fix $\epsilon > 0$. (1) There an extension $u_\phi^\epsilon \in X_\phi$ such that $u_\phi^\epsilon = 0$ in $\bar{\Omega}_p$ and $\bar{\Omega}_s$. There also exists $u_0 \in H_0^1(\Omega_{fp})$ with extension $u_0 = 0$ in $\bar{\Omega}_s$ such that $\nabla \cdot u_0 = g - \nabla \cdot u_\phi^\epsilon$ in Ω. In particular,*

$$\tilde{u} = u_\phi^\epsilon + u_0 = \begin{cases} u_\phi^\epsilon + u_0 & \text{in } \Omega_f \\ u_0 & \text{in } \Omega_p \\ 0 & \text{in } \Omega_s \end{cases}$$

and

$$\|u_0\|_1 \leq \|g\|_p + \left\|\nabla \cdot u_\phi^\epsilon\right\|_f$$

(2)If in addition $\int_\Omega g = 0$, then there exists an extension $u_\phi^\epsilon \in V_\phi$ such that $u_\phi^\epsilon = 0$ in $\bar{\Omega}_p$ and $\bar{\Omega}_s$. There also exists $u_0 \in H_0^1(\Omega_{fp})$ with extension $u_0 = 0$ in $\bar{\Omega}_s$ such that $\nabla \cdot u_0 = g$. In particular,

$$\tilde{u} = u_\phi^\epsilon + u_0 = \begin{cases} u_\phi^\epsilon & \text{in } \Omega_f \\ u_0 & \text{in } \Omega_p \\ 0 & \text{in } \Omega_s \end{cases}$$

and

$$\|u_0\|_1 \leq \|g\|_p$$

In both cases, for any $\epsilon_0 > 0$, there exists an $\epsilon > 0$ such that

$$\left\|u_\phi^\epsilon\right\|_{L^4(\Omega)} < \epsilon_0$$

Proof. Our proof follows the work of Raviart and Girault for divergence-free velocities and non-homogeneous Dirichlet boundary conditions in [11], page 287, Lemma 2.3 (also see [10]). From Eq. (2.4), there exists an extension u_ϕ of $\phi \in H^{1/2}(\partial\Omega)$ satisfying $\|u_\phi\|_1 \leq \gamma \|\phi\|_{1/2,\partial\Omega}$. In order to localize contributions of problem data, we consider the *cut-off function* $\psi_\epsilon \in C^\infty(\bar{\Omega})$ described in [11]. Briefly, ψ_ϵ is identically zero for all points in Ω a certain ϵ-dependent distance away from $\partial\Omega$ and identically 1 a certain ϵ-dependent distance close to $\partial\Omega$ and smoothly connected in between. Moreover, $\|\psi_\epsilon v\|_1 < \infty$ for all $v \in H^1(\Omega)$. For (1), take $u_\phi^\epsilon := \gamma_\epsilon u_\phi$. We can take $\epsilon > 0$ small enough to ensure that $u_\phi^\epsilon = 0$ in Ω_p and Ω_s. In addition

$$\int_{\Omega_{fp}} \nabla \cdot u_\phi^\epsilon = \int_{\partial\Omega} \phi \cdot \hat{n} = \int_{\Omega_{fp}} g$$

and hence $g - \nabla \cdot u_\phi^\epsilon \in L_0^2(\Omega_{fp})$. Thus, by Lemma 2.6, there exists a unique $u_0 \in V^\perp(\Omega_{fp}) \subset H_0^1(\Omega_{fp})$ such that $\nabla \cdot u_0 = g - \nabla \cdot u_\phi^\epsilon$. We extend $u_0 = 0$ on Ω_s. The bound $\|u_0\|_1 \leq \beta^{-1}||g - \nabla \cdot u_\phi^\epsilon||_{fp}$ follows from Lemma 2.6. For (2) refer again to [11] for the existence of a *particular* extension u_ϕ^*, such that for any $\epsilon > 0$,

$$u_\phi^\epsilon = \nabla \times \left(\psi_\epsilon u_\phi^*\right)$$

defines another extension of ϕ satisfying $\nabla \cdot u_\phi^\epsilon = 0$. Now since $g \in L_0^2(\Omega_p)$, as a consequence of Lemma 2.6, there exists a unique $u_0 \in V^\perp(\Omega_p)$ satisfying $\nabla \cdot u_0 = g$. Extend $u_0 = 0$ on Ω_f and Ω_s. We can take $\epsilon > 0$ small enough to ensure that $u_\phi^\epsilon = 0$ on Ω_p and Ω_s. Let $\tilde{u} = u_\phi^\epsilon + u_0$. The bound for u_0 follows from Lemma 2.6. The final result follows from the specially constructed property of γ_ϵ. ☐

Note that for $g \equiv 0$ in Ω, the L^4-norm of $\tilde{u}$ can be made arbitrarily small. However, for general $g \in L^2(\Omega)$, then g and ϕ are coupled via the necessary compatibility condition Eq. (2.1); hence, there is no obvious way to control the size of $\|\tilde{u}\|_{L^4(\Omega)}$ which is required to control the size of the bound in (2.9).

Note that for the Hopf extension, $\|\tilde{u}\|_{L^4(\Omega)} \to 0$ as $\epsilon \to 0$, but the bound on $\|\tilde{u}\|_1$ grows exponentially as $\epsilon \to 0$.

2.4. Continuity and coercivity. We now proceed with some important bounds on the previously defined functionals required in our proceeding stability and error analysis.

LEMMA 2.10. *The linear functionals $l_1(\cdot)$ and $l_2(\cdot)$ are continuous. In particular, for any $v \in H^1(\Omega)$ and $q \in L^2(\Omega)$,*

$$\begin{aligned} l_1(v) &\leq \|f\|_{-1}\|v\|_1 + \tilde{\nu}_{max}\sqrt{d}\,\|g\|\,\|\nabla v\|, \ \text{if } v \in X \\ l_{2,*}(q) &\leq \|g\|_*\|q\|_*, \ \text{for } * = f,\ p,\ s,\ fp,\ fs,\ ps, \text{ or } fps \end{aligned}$$

Moreover, for $* = f,\ p,\ s,\ fp,\ fs,\ ps,$ or fps, if $f \in L^2(\Omega_*)$, and $v = 0$ on $\Gamma_* \subset \partial\Omega_*$ that has positive measure with respect to boundary, then

$$l_{1,*}(v) \leq \|f\|_*\|v\|_* + \tilde{\nu}_{max}^*\sqrt{d}\,\|g\|_*\|\nabla v\|_* \leq \left(C_p\|f\|_* + \tilde{\nu}_{max}^*\sqrt{d}\,\|g\|_*\right)\|\nabla v\|_*$$

Proof. Linearity for the functionals is obvious. Continuity follows by a direct application of the duality for $\|\cdot\|_{-1}$-norm result and Cauchy-Schwarz for the others along with Poincaré and the fact that $\|\nabla \cdot v\| \leq \sqrt{d}\,\|\nabla v\|$ to obtain $\|\cdot\|_1$. ☐

LEMMA 2.11. *The bilinear functional $b(\cdot,\cdot)$ is continuous. In particular, for $* = f,\ p,\ s,\ fp,\ fs,\ ps,$ or fps and for any $v \in H^1(\Omega)$, $q \in L^2(\Omega)$*

$$b_*(v,q) \leq \sqrt{d}\,\|\nabla v\|_*\|q\|_* \qquad (2.11)$$

Proof. Bilinearity is obvious. Continuity follows by a direct application of Cauchy-Schwarz inequality and the fact that $\|\nabla \cdot v\| \leq \sqrt{d}\,\|\nabla v\|$. ∎

LEMMA 2.12. *The bilinear functional $a(\cdot,\cdot)$ is continuous and coercive. In particular, for $* = f,\ p,\ s,\ fp,\ fs,\ ps,$ or fps and for any $u,\ v \in H^1(\Omega)$*

$$a_*(u,v) \leq \tilde{\nu}^*_{max}\,\|\nabla u\|_*\,\|\nabla v\|_* + \nu\,(k^*_{min})^{-1}\,\|u\|_*\,\|v\|_* \leq \alpha^*_1\,\|u\|_{1,*}\,\|v\|_{1,*} \tag{2.12}$$

and

$$\begin{array}{lll} a_*(v,v) & \geq & \tilde{\nu}^*_{min}\int_* |\nabla v|^2 + \nu_{min}\,(k^*_{max})^{-1}\int_* |v|^2 \quad \geq \alpha^*_0\,\|v\|^2_{1,*}\,, \ \ or \\ a_*(v,v) & \geq & \frac{\tilde{\nu}^*_{min}}{2C_p}\,\|v\|^2_{1,*}\,, \ \ if\ v=0\ on\ \Gamma_* \end{array} \tag{2.13}$$

where $\Gamma_* \subset \partial\Omega_*$ has positive measure with respect to boundary and

$$\alpha^*_1 = max\{\tilde{\nu}^*_{max}, \nu_{max}/k^*_{min}\}\,, \qquad \alpha^*_0 = min\{\tilde{\nu}^*_{min}, \nu/k^*_{max}\} \tag{2.14}$$

Proof. Bilinearity is obvious. Continuity follows by bounding problem parameters ν, $\tilde{\nu}$, and K and applying the Cauchy-Schwarz inequality. The coercivity condition follows once again by bounding the problem parameters and, for the second form, applying the Poincaré inequality. ∎

LEMMA 2.13. *The trilinear functional $c(\cdot,\cdot,\cdot)$ is continuous. In particular, for $* = f,\ p,\ s,\ fp,\ fs,\ ps,$ or fps and for any $u,\ v,\ w$ in $H^1(\Omega)$*

$$c_*(u,v,w) \leq C^*_L\,\|u\|_{1,*}\,\|\nabla v\|_*\,\|w\|_{1,*} \tag{2.15}$$

where $C^*_L > 0$ only depends on Ω_*. Moreover,

$$c_*(u,v,w) \leq C^*_L\,\|\nabla u\|_*\,\|\nabla v\|_*\,\|\nabla w\|_*\,, \ \ if\ u|_{\Gamma^*_1}, w|_{\Gamma^*_2} = 0 \tag{2.16}$$

where $\Gamma^*_1,\ \Gamma^*_2 \subset \partial\Omega_*$ have positive measure with respect to the boundary. We write $C_L = C^*_L$ for all $*$.

Separately, if $\nabla \cdot u = g$ in Ω_* and $(u \cdot \hat{n}\,(v \cdot w))\,|_{\partial\Omega_*} = 0$, then

$$c_*(u,v,w) = -c_*(u,w,v) - \int_{\Omega_*} g\,(v \cdot w)\,, \ \ and\ hence \tag{2.17}$$

$$c_*(u,v,v) = -\frac{1}{2}\int_{\Omega_*} g\,|v|^2 \tag{2.18}$$

We call $c_*(\cdot,\cdot,\cdot)$ pseudo-skew symmetric. Moreover, if $\nabla \cdot u = 0$ in Ω_*, then $c_*(\cdot,\cdot,\cdot)$ is actually skew-symmetric.

Proof. Trilinearity is obvious. The two continuity bounds are classical results. To prove the first identity, we make use of Einstein's tensor notation for indices and vector/tensor operations: $u \cdot \nabla v \cdot w = u_i v_{j,i} w_j$. Apply the divergence theorem and the fact that $(u \cdot \hat{n}\,(v \cdot w))\,|_{\partial\Omega_*}$ vanishes on $\partial\Omega_*$. Skew-symmetry for div-free functions then follows easily. ∎

We call $c_*(\cdot,\cdot,\cdot)$ pseudo-skew symmetric because the function

$$(u,v,w) \mapsto c_*(u,v,w) + \frac{1}{2}\int_{\Omega_*} g\,(v \cdot w)$$

is skew-symmetric.

REMARK 2.14. *Note that, $b(\cdot,\cdot)$ is uniformly continuous; $a(\cdot,\cdot)$ is uniformly continuous when, for some $C > 0$ we restrict ν, $\tilde{\nu}$ and $\|K\|_{L^\infty(\Omega)} < C_1 < \infty$; $a(\cdot,\cdot)$ is uniformly coercive when, for some $C_0 > 0$ we restrict ν, $\tilde{\nu} \geq C_0 > 0$. Hence, for turbulent flows there is a question concerning stability as $\nu \to 0$. Xie et.al. in [33] discuss the numerical aspects of properly selecting stable finite element spaces.*

2.5. Well-posedness of Stokes-Brinkman. First, we establish existence and uniqueness along with a priori estimates for the weak Stokes-Brinkman equation before proceeding with the NS-Brinkman equation. We suppose that $f \in L^2(\Omega)$, $\phi \in H^{1/2}(\Omega)$, and that $g \in L^2(\Omega)$ ($g \neq 0$ in Ω_p and $g = 0$ in Ω_f, Ω_s) satisfies the compatability condition Eq. (2.1) throughout. We are particularly interested in rigorously tracking the dependence of solutions u^δ to Stokes-Brinkman on $\tilde{\nu}$, K and ν. In the next theorem we prove that

$$\left\|\nabla u^\delta\right\| \leq \mathcal{O}\left(\tfrac{1}{\nu}\right), \qquad \left\|u^\delta\right\|_{1,s} \leq \mathcal{O}\left(\tfrac{\sqrt{\delta}}{\nu}\right)$$

THEOREM 2.15. *(Well-posedness Stokes-Brinkman) There exists a unique* $\left(u^\delta, p^\delta\right) \in (X_\phi, Q)$ *satisfying Stokes-Brinkman, Problem 2.4. Let* $\tilde{u}$ *be an extension of boundary data* $\phi \in H^{1/2}(\Omega)$ *satisfying* $\nabla \cdot \tilde{u} = g \in L^2(\Omega)$*, e.g. see Proposition 2.8. Then* $\left(u^\delta, p^\delta\right)$ *satisfy*

$$\left\|\nabla u^\delta\right\| \leq \frac{1}{\nu}\sqrt{C_\delta}, \qquad \left\|u^\delta\right\|_{1,s} \leq \frac{\sqrt{\delta}}{\nu}\sqrt{C_\delta}$$

where

$$C_\delta := C_\delta(f, \tilde{u}) := C\left(\|f\|_{fp}^2 + \delta\|f\|_s^2 + \frac{\nu^2}{k^p}\|\tilde{u}\|_{1,fp}^2\right)$$

where the constant C *is independent of parameters* $\tilde{\nu}$*,* ν*, and* K*.*

Proof. Restrict $v \in V$. Then $l_1(v) = \langle f, v\rangle$ and we have $a(w, v) = \langle f, v\rangle - a(\tilde{u}, v)$. By the (bi)linearity and continuity of $l_1(\cdot)$, $a(\cdot,\cdot)$ along with coercivity of $a(\cdot,\cdot)$ on Ω established in Lemmas 2.9 and 2.11 we apply Lax-Milgram theorem to establish existence and uniqueness of $w \in V \subset X$. For such a w, we now note that $a(w, v) + a(\tilde{u}, v) - \langle f, v\rangle = 0$ for any $v \in V$. Thus, $Aw + A\tilde{u} - f \in V^*$. Eq. (2.6) implies that $B = \nabla : Q \to V^*$ defines an isomorphism. This establishes existences of a unique $p^\delta \in Q$ such that $Bp^\delta = Aw + A\tilde{u} - L_1$. Finally, to show that $u^\delta = w + \tilde{u}$ is unique solution is an easily follows from the coercivity of $a(\cdot,\cdot)$ on Ω. For the estimates, since $w \in V$, apply bounds from Lemmas 2.9 and 2.11 and bound on $\tilde{u}$ in Eq. (2.7) and (2.3)

$$\begin{aligned}\alpha_0\|w\|_1^2 &\leq a(w, w) = \langle f, w\rangle - a(\tilde{u}, w)\\ &\leq \|f\|_{-1}\|w\|_1 + \alpha_1\|\tilde{u}\|_1\|w\|_1\end{aligned}$$

Divide by $\|w\|_1$ and α_0. To obtain estimate for u^δ, apply triangle inequality $\left\|u^\delta\right\|_1 \leq \|w\|_1 + \|\tilde{u}\|_1$. Finally, to bound p^δ, take $v \in X$ and solve for p^δ

$$b\left(p^\delta, v\right) = l_1(v) - a\left(u^\delta, v\right) \leq \left(\|f\|_{-1} + \sqrt{d}\tilde{\nu}_{max}\|g\|\right)\|v\|_1 + \alpha_1\left\|u^\delta\right\|_1\|v\|_1$$

To finish, apply the inf-sup condition, Eq. (2.2), to obtain

$$\left\|p^\delta\right\| \leq \beta^{-1}\sup\left(b\left(p^\delta, v\right)/\|v\|_1\right).$$

Fix small $\delta > 0$ and set $\nu/\tilde{\nu}^s$, $1/k^f$, and $k^s = \delta$. Then to summarize, we have preliminary bounds on u^δ and p^δ

$$\begin{aligned}\left\|u^\delta\right\|_1 &\leq \tfrac{1}{\nu}\|f\|_{-1} + \left(\tfrac{1}{\nu\delta} + 1\right)\|\tilde{u}\|_1\\ \left\|p^\delta\right\| &\leq \tfrac{1}{\beta}\left(\|f\|_{-1} + \sqrt{d}\nu\|g\|_p\right) + \tfrac{\nu}{\beta}\left\|u^\delta\right\|_1\end{aligned}$$

The bound for u^δ is unsatisfying since it is not uniform as $\delta \to 0$. For sharper bounds, require $f \in L^2(\Omega)$ and decompose Ω to its components Ω_*. Recall that $\|w\|_* \le C_p \|\nabla w\|_*$ for $* = f,\ fp$ and $\tilde{u} = 0$ in Ω_s. Applications of Cauchy-Schwarz and Young's inequalities provide,

$$\begin{aligned}
&\nu \|\nabla w\|_{fp}^2 + \tilde{\nu}^s \|\nabla w\|_s^2 + \sum_{*=f,p,s} \frac{\nu}{k^*} \|w\|_*^2 = a(w,w) = \langle f, w\rangle - a(\tilde{u}, w) \\
&\le \sum_{*=f,p,s} \left(\|f\|_* \|w\|_*\right) + \sum_{*=f,p} \left(\tilde{\nu}^* \|\nabla \tilde{u}\|_* \|\nabla w\|_* + \frac{\nu}{k^*} \|\tilde{u}\|_* \|w\|_{\Omega_*}\right) \\
&\le \frac{C_P^2}{\nu} \|f\|_{fp}^2 + \frac{\nu}{4} \|\nabla w\|_{fp}^2 + \frac{k^s}{\nu} \|f\|_s^2 + \frac{\nu}{4k^s} \|w\|_s^2 \\
&\quad + \nu \|\nabla \tilde{u}\|_{fp}^2 + \frac{\nu}{4} \|\nabla w\|_{fp}^2 + \sum_{*=f,p} \frac{\nu}{k^*} \left(\|\tilde{u}\|_*^2 + \frac{1}{4} \|w\|_*^2\right)
\end{aligned}$$

Combining terms, and simplifying we have

$$\nu \|\nabla w\|_{fp}^2 + \alpha_0^s \|w\|_{1,s}^2 \le \frac{2C_p^2}{\nu} \|f\|_{fp}^2 + \frac{2k^s}{\nu} \|f\|_s^2 + 4\alpha_1^{fp} \|\tilde{u}\|_{1,fp}^2$$

Recall $\nu/\tilde{\nu}^s$, $1/k^f$, and $k^s = \delta$. Also, recall that $\nu < \alpha_0^s = \nu/\delta$ and $k^p < k^f = 1/\delta$ such that

$$\alpha_1^{fp} = \max_{x \in \Omega_{fp}} \{\nu, \nu/k(x)\} \le \nu/k^p$$

Then

$$\begin{aligned}
\|w\|_{1,s}^2 &\le C\left(\tfrac{\delta}{\nu^2} \|f\|_{fp}^2 + \tfrac{\delta^2}{\nu^2} \|f\|_s^2 + \tfrac{\delta}{k^p} \|\tilde{u}\|_{1,fp}^2\right) \\
\|w\|_1^2 &\le C\left(\tfrac{1}{\nu^2} \|f\|_{fp}^2 + \tfrac{\delta}{\nu^2} \|f\|_s^2 + \tfrac{1}{k^p} \|\tilde{u}\|_{1,fp}^2\right)
\end{aligned}$$

Applying the triangle inequality

$$\|u\|_{1,s} \le \|w\|_{1,s} + \|\tilde{u}\|_{1,s}$$

recalling that $\tilde{u} \equiv 0$ in Ω_s (for the bound in Ω_s) and assuming that $k^p \le 1$ (for the bound in Ω) proves the claim. ◻

2.6. Well-posedness of NS-Brinkman. We shall prove existence using the Leray-Schauder fixed point theorem. This requires some preliminary notation and estimates for NS-Brinkman (2.3).

DEFINITION 2.16. *Let $\tilde{u} \in H^1(\Omega)$ be an extension of boundary data $\phi \in H^{1/2}(\Omega)$ preserving $\nabla \cdot \tilde{u} = g \in L^2(\Omega)$; e.g. consider (1) or (2), Proposition 2.8. Then, we define*
(1) $T : X' \to V$ such that $T(f) := w$ where $w \in V$ solves

$$\forall v \in V, \qquad a(w,v) = l_1(v) - a(\tilde{u}, v)$$

(2) $N : V \to X'$ such that $N(w) := f + \nabla(\tilde{\nu} g) - \tilde{u} \cdot \nabla \tilde{u} - \tilde{u} \cdot \nabla w - w \cdot \nabla \tilde{u} - w \cdot \nabla w$
(3) $F : V \to V$ such that $F := T \circ N$.

In order to apply Leray Schauder's fixed point theorem, we show that F is a compact linear operator with a fixed point that satisfies NS-Brinkman, Problem 2.3. We prove this through the proceeding lemmas.

LEMMA 2.17. *T is a well-defined linear, continuous operator.*

Proof. T is clearly linear. Well-posedness and boundedness of T follows from Theorem 2.14. Since T is a linear and bounded operator, it follows that T is continuous. $\square$

LEMMA 2.18. *For any $w \in V$, then $N(w) \in H^{-d/4}(\Omega)$ and N maps $V \to X'$ continuously.*

Proof. The Ladyzhenskaya inequalities imply that there exists $\sqrt{C_L} > 0$ such that $\|w\|_{L^4} \le \sqrt{C_L}\,\|w\|_1$. By the Sobolev embedding theorem, $H^{d/4}(\Omega) \hookrightarrow L^4(\Omega)$; hence, there exists $C_{4,d} > 0$ satisfying $\|v\|_{L^4} \le C_{4,d}\,\|v\|_{H^{d/4}}$ for any $v \in H^{d/4}(\Omega)$. Now fixing $v \in H^{d/4}(\Omega)$ it is straightforward to bound $\int_\Omega w \cdot \nabla w \cdot v$, $\int_\Omega \tilde{u} \cdot \nabla w \cdot v$, $\int_\Omega w \cdot \nabla \tilde{u} \cdot v$, $\int_\Omega \tilde{u} \cdot \nabla \tilde{u} \cdot v$ by $\left(\sqrt{C_L} C_{4,d}\right) \|w\|_1 \|w\|_1 \|v\|_{d/4}$. Dividing each of the resulting inequalities by $\|v\|_{d/4}$ and taking the supremum over all $v \in H^{d/4}(\Omega)$, we get the desired conclusion. Continuity follows by expanding $\|N(u_1) - N(u_2)\|_{-d/4}$ and successively applying Cauchy-Schwarz, the definition of the negative, fractional Sobolev norm, and the fact that N is bounded. $\square$

PROPOSITION 2.19. *F is a compact operator.*

Proof. By the Rellich Lemma, $H^{-d/4}(\Omega)$ is compactly imbedded in $H^{-1}(\Omega)$. Hence, we summarize,

$$H^1 \underset{N,\text{ cont.}}{\longmapsto} H^{-d/4} \underset{compact}{\hookrightarrow} H^{-1} \underset{T,\text{ cont.}}{\longmapsto} H^1$$

Hence, F is compact as a continuous composition of a compact operator. $\square$

Before concluding existence, we require the following technical result necessary to control the size of of the troublesome term arising in proving existence and derivation of the a priori estimate for u^δ.

PROPOSITION 2.20. *F is a compact operator.*

Proof. By the Rellich Lemma, $H^{-d/4}(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$. Hence, we summarize,

$$H^1 \underset{N,\text{ cont.}}{\longmapsto} H^{-d/4} \underset{compact}{\hookrightarrow} H^{-1} \underset{T,\text{ cont.}}{\longmapsto} H^1$$

Hence, F is compact as a continuous composition of a compact operator. $\square$

Before concluding existence, we require the following technical result necessary to control the size of of the troublesome term arising in proving existence and derivation of the a priori estimate for u^δ.

LEMMA 2.21. *Fix $\epsilon_0 > 0$. Consider boundary data $\phi \in H^{1/2}(\partial\Omega)$ and $g \in L^2(\Omega)$ satisfying the compatibility condition Eq. (2.1). For $\tilde{u}$ defined by (1), Proposition 2.8, we have that for $\epsilon > 0$ small enough*

$$\|\tilde{u}\|_{L^4(\Omega_*)} \le \sqrt{C_L}\left(\|g\|_p + 2\sqrt{d}\,\|\phi\|_{H^{1/2}(\partial\Omega)}\right), \qquad * = f,\, p \tag{2.19}$$

Suppose further that $\int_{\partial\Omega} \phi \cdot \hat{n} = 0$ and hence, $g \in L^2_0(\Omega)$. Then choosing $\tilde{u}$ as in (2), Proposition 2.8, we have that for $\epsilon > 0$ small enough

$$\|\tilde{u}\|_{L^4(\Omega_f)} \le \epsilon_0 + \sqrt{C_L}\,\|g\|_p\,, \qquad \|\tilde{u}\|_{L^4(\Omega_p)} \le \sqrt{C_L}\,\|g\|_p$$

Proof. Apply Proposition 2.8 and Lemma 2.6. ☐

Based on the definitions and lemmas above, we can conclude the following theorem.

THEOREM 2.22. *(Well-posedness NS-Brinkman) Suppose that the small data condition is satisfied*

$$2\sqrt{C_L}\,\|\tilde{u}\|_{L^4(\Omega_{fp})} \le \frac{\nu}{2} \tag{2.20}$$

where $\tilde{u}$ is an extension of boundary data $\phi \in H^{1/2}(\Omega)$ satisfying $\nabla\cdot\tilde{u} = g \in L^2(\Omega)$; e.g. consider (1) or (2), Proposition 2.8 with bounds provided in Lemma 2.20. Then there is at least one pair $\left(u^\delta, p^\delta\right) \in (X_\phi, Q)$ satisfying NS-Brinkman, Problem 2.3 and

$$\begin{array}{llll} \left\|u^\delta\right\|_{1,s}^2 & \le \nu^{-2}\delta C_{\delta,NSE}, & \text{or } \left\|u^\delta\right\|_{1,s} & \le \mathcal{O}(\nu^{-1}\sqrt{\delta}) \\ \left\|\nabla u^\delta\right\|^2 & \le \nu^{-2}C_{\delta,NSE}, & \text{or } \left\|\nabla u^\delta\right\| & \le \mathcal{O}(\nu^{-1}) \end{array}$$

where

$$C_{\delta,NSE} := C_{\delta,NSE}(\tilde{u}, f) := C\left(\delta\|f\|_s^2 + \|f\|_{fp}^2 + \left(\nu^2 + \frac{\nu^2}{k^p} + \|\tilde{u}\|_{1,fp}^2\right)\|\tilde{u}\|_{1,fp}^2\right)$$

and the constant C is independent of parameters $\tilde{\nu}$, ν, and K. Furthermore, there is at most one such solution $\left(u^\delta, p^\delta\right)$ when the additional small data condition is satisfied:

$$\begin{array}{rcl} \frac{1}{2}C_L\|g\|_p + \sqrt{C_L}\,\|\tilde{u}\|_{L^4(\Omega_p)} + \nu^{-1}C_L\sqrt{C_{\delta,NSE}} & \le & \frac{1}{2}\alpha_0^p \\ \sqrt{C_L}\,\|\tilde{u}\|_{L^4(\Omega_f)} + \nu^{-1}C_L\sqrt{C_{\delta,NSE}} & \le & \frac{1}{2}\nu \\ \nu^{-1}C_L\sqrt{C_{\delta,NSE}} & \le & \frac{1}{2}\nu\delta^{-3/2} \end{array} \tag{2.21}$$

Proof. We prove existence via the Leray-Schauder fixed point theorem. Fix $v \in V$. Then $l_1(v) = \langle f, v\rangle$ and Problem 2.3 becomes

$$a(w,v) + c(w,w,v) + c(w,\tilde{u},v) + c(\tilde{u},w,v) = \langle f, v\rangle - a(\tilde{u},v) + c(\tilde{u},\tilde{u},v)$$

It is easy to see that a fixed point of the nonlinear, compact operator F is a solution of this variational problem. Thus, consider the family of fixed point problems: for any $0 < \lambda \le 1$, find $u_\lambda \in X_0$ satisfying $u_\lambda = \lambda F(u_\lambda)$. Noting that $\tilde{u} \equiv 0$ in Ω_s and the pseudo-skew symmetry of $c_*(\cdot,\cdot,\cdot)$ (Eq. (2.18)), applications of Hölder's and Young's inequalities provide

$$
\begin{aligned}
&\nu \left\|\nabla u_\lambda\right\|_{fp}^2+\tilde{\nu}^s\left\|\nabla u_\lambda\right\|_s^2+\sum_{*=f,p,s}\frac{\nu}{k^*}\left\|u_\lambda\right\|_*^2\\
&=\lambda\left[\sum_{*=f,p,s}\left(l_{1,*}\left(u_\lambda\right)-\underbrace{c_*\left(u_\lambda,u_\lambda,u_\lambda\right)}_{=0}\right)\right.\\
&\qquad\left.+\sum_{*=f,p}\left(-a_*\left(\tilde{u},u_\lambda\right)-c_*\left(\tilde{u},\tilde{u},u_\lambda\right)-c_*\left(u_\lambda,\tilde{u},u_\lambda\right)-c_*\left(\tilde{u},u_\lambda,u_\lambda\right)\right)\right]\\
&\leq\sum_{*=f,p,s}\|f\|_*\left\|u_\lambda\right\|_*+\sum_{*=f,p}\left(\tilde{\nu}^*\|\nabla\tilde{u}\|_*\left\|\nabla u_\lambda\right\|_*+\frac{\nu}{k^*}\|\tilde{u}\|_*\left\|u_\lambda\right\|_*\right)\\
&\qquad+C_L\|\tilde{u}\|_{1,fp}^2\left\|\nabla u_\lambda\right\|_{fp}+2\sqrt{C_L}\|\tilde{u}\|_{L^4(\Omega_{fp})}\left\|\nabla u_\lambda\right\|_{fp}^2\\
&\leq\frac{k^s}{2\nu}\|f\|_s^2+\frac{\nu}{2k^s}\left\|u_\lambda\right\|_s+\frac{3C_p^2}{\nu}\|f\|_{fp}^2+\frac{\nu}{12}\left\|\nabla u_\lambda\right\|_{fp}^2\\
&\qquad+3\nu\|\nabla\tilde{u}\|_{fp}^2+\frac{\nu}{12}\left\|\nabla u_\lambda\right\|_{fp}^2+\sum_{*=f,p}\frac{\nu}{2k^*}\left(\|\tilde{u}\|_*^2+\left\|u_\lambda\right\|_*^2\right)\\
&\qquad+\frac{3C_L^2}{\nu}\|\tilde{u}\|_{1,fp}^4+\frac{\nu}{12}\left\|\nabla u_\lambda\right\|_{fp}^2+2\sqrt{C_L}\|\tilde{u}\|_{L^4(\Omega_{fp})}\left\|\nabla u_\lambda\right\|_{fp}^2
\end{aligned}
$$

Let $\tilde{u}$ be one of the extensions in Proposition 2.8 so that we have bounds in Lemma 2.20 and the first small data condition Eq. (2.20) satisfied. Absorbing terms and simplifying, we obtain

$$
\begin{aligned}
&\nu\left\|\nabla u_\lambda\right\|_{fp}^2+\alpha_0^s\left\|u_\lambda\right\|_{1,s}^2\\
&\leq\frac{k^s}{\nu}\|f\|_s^2+\frac{6C_p^2}{\nu}\|f\|_{fp}^2+\left(6\nu+\frac{\nu}{k^p}+\frac{6C_L^2}{\nu}\|\tilde{u}\|_{1,fp}^2\right)\|\tilde{u}\|_{1,fp}^2
\end{aligned}
$$

Recall $\nu/\tilde{\nu}^s$, $1/k^f$, and $k^s=\delta$ and that $\nu<\alpha_0^s=\nu/\delta$. Then,

$$
\begin{aligned}
\left\|\nabla u_\lambda\right\|_{fp}^2 &\leq C\left(\tfrac{\delta}{\nu^2}\|f\|_s^2+\tfrac{1}{\nu^2}\|f\|_{fp}^2+\left(1+\tfrac{1}{k^p}+\tfrac{1}{\nu^2}\|\tilde{u}\|_{1,fp}^2\right)\|\tilde{u}\|_{1,fp}^2\right)\\
\left\|u_\lambda\right\|_{1,s}^2 &\leq C\left(\tfrac{\delta^2}{\nu^2}\|f\|_s^2+\tfrac{\delta}{\nu^2}\|f\|_{fp}^2+\left(\delta+\tfrac{\delta}{k^p}+\tfrac{\delta}{\nu^2}\|\tilde{u}\|_{1,fp}^2\right)\|\tilde{u}\|_{1,fp}^2\right)
\end{aligned}
$$

Thus, we have the necessary bound uniform in λ to conclude existence of $w\in V$ to homogeneous, NS-Brinkman via Leray-Schauder. Hence, there exists $u^\delta=w+\tilde{u}$ satisfying Problem 2.3.

The stability bound for any solution $w\in V$ to homogeneous NS-Brinkman is similar and leads to the same result as for u_λ. We recall that $u^\delta=w+\tilde{u}$. Thus, the stability bound for u^δ follows by application of the triangle inequality, $\left\|u^\delta\right\|_{1,*}\leq\|w\|_{1,*}+\|\tilde{u}\|_{1,*}$. Noting that $\tilde{u}\equiv 0$ in Ω_s we prove the a priori estimate.

To establish uniqueness, suppose w_1, w_2 are two such solutions. Then subtracting the corresponding equations for fixed $v\in V$, we get

$$
\begin{aligned}
0&=a\left(w_1-w_2,v\right)+c\left(w_1,w_1,v\right)-c\left(w_2,w_2,v\right)\\
&=a\left(w_1-w_2,v\right)+c\left(w_1-w_2,\tilde{u},v\right)+c\left(\tilde{u},w_1-w_2,v\right)
\end{aligned}
$$

Write $w = w_1 - w_2$ and take $v = w$ (indeed, $w \in V$). Recall that $\tilde{u} \equiv 0$ in Ω_s and $g \equiv 0$ in Ω_f. By rearranging the above equality, decomposing the domains, noting the pseudo-skew symmetry of $c_*(\cdot,\cdot,\cdot)$ (Eq. (2.18)), and applying Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
&\sum_{*=f,p,s} \tilde{\nu}^* \|\nabla w\|_*^2 + \frac{\nu}{k^*}\|w\|_*^2 \\
&= \sum_{*=f,p,s} -c_*(w,w_2,w) + \sum_{*=f,p} -c_*(w,\tilde{u},w) - c_*(\tilde{u},w,w) \\
&= \sum_{*=f,p,s} (-c_*(w,w_2,w)) + \sum_{*=f,p} (-c_*(w,\tilde{u},w)) - \frac{1}{2}\int_{\Omega_p} g\,|w|^2 \\
&\leq \frac{C_L}{2}\|g\|_p \|w\|_{1,p}^2 + \sum_{*=f,p} \sqrt{C_L}\,\|\tilde{u}\|_{L^4(\Omega_*)}\|w\|_{1,*}^2 + \sum_{*=f,p,s} C_L \|w_2\|_* \|w\|_{1,*}^2
\end{aligned}$$

Thus, requiring

$$\begin{array}{rcl}
\frac{C_L}{2}\|g\|_p + \sqrt{C_L}\,\|\tilde{u}\|_{L^4(\Omega_p)} + \frac{1}{\nu}C_L\sqrt{C_{\delta,NSE}} & < & \alpha_0^p \\
\sqrt{C_L}\,\|\tilde{u}\|_{L^4(\Omega_f)} + \frac{1}{\nu}C_L\sqrt{C_{\delta,NSE}} & < & \nu \\
\frac{\sqrt{\delta}}{\nu}C_L\sqrt{C_{\delta,NSE}} & < & \alpha_0^s
\end{array}$$

is a sufficient condition to ensure $w \equiv 0$. Note that $\alpha_0^s = \nu/\delta$. Now set $u^\delta = w + \tilde{u}$. Then $u^\delta \in V_\phi(g)$ satisfies the nonhomogeneous, NS-Brinkman. Suppose that there are two such solutions u_1 and u_2. Then subtracting the corresponding equations for fixed $v \in V$, we get $a(u_1 - u_2, v) + c(u_1,u_1,v) + c(u_2,u_2,v) = 0$. Add/subtract $c(u_1,u_2,v)$. Set $v = u_1 - u_2$ (indeed, $(u_1-u_2)|_{\partial\Omega} = 0$ and $\nabla\cdot(u_1-u_2) = 0$). Write $w = u_1 - u_2$. Recall again that $g \equiv 0$ in Ω_{fs}. Then rearranging, noting the pseudo-skew symmetry of $c_*(\cdot,\cdot,\cdot)$ (Eq. (2.18)), and applying Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
&\sum_{*=f,p,s} \tilde{\nu}^* \|\nabla w\|_*^2 + \frac{\nu}{k^*}\|w\|_*^2 = \sum_{*=f,p,s} c_*(w,u_1,w) + c_*(u_2,w,w) \\
&= \sum_{*=f,p,s} (c_*(w,u_1,w)) - \int_{\Omega_p} g\,|w|^2 \\
&\leq C_L\|g\|_p \left\|u^\delta\right\|_{1,p}^2 + C_L\|\nabla u_1\|_f \|\nabla w\|_f^2 + \sum_{*=p,s}\left(C_L\|u_1\|_{1,*}\|w\|_{1,*}^2\right)
\end{aligned}$$

Thus, the following is a sufficient condition to ensure $w \equiv 0$

$$\tfrac{1}{\nu}C_L\sqrt{C_{\delta,NSE}} + \|g\|_p < \alpha_0^p, \quad \tfrac{1}{\nu}C_L\sqrt{C_{\delta,NSE}} < \nu, \quad \tfrac{\delta}{\nu}C_L\sqrt{C_{\delta,NSE}} < \alpha_0^s$$

Hence, the second small data condition Eq. (2.21) is sufficient to ensure uniqueness. Establishing existence/uniqueness of $p^\delta \in Q$ follows by applying usual techniques derived from the inf-sup condition. ☐

Note that in the case $g \equiv 0$, $u_0 \equiv 0$ and thus $\|\tilde{u}\|_{L^4(\Omega)} = ||u_\phi^\epsilon||_{L^4(\Omega)}$ can be taken arbitrarily small. Returning to the application of Leray-Schauder fixed point theorem in the previous proof, we can apply this result to conclude existence for any data.

Remark 2.23. *The question of existence for large data $g \in L^2(\Omega)$ is an open problem. The difficulty is that there is an irrevocable coupling between the divergence* *condition $\nabla \cdot u^{\delta} = g$ and the boundary data $u^{\delta}|_{\partial\Omega} = \phi$ via the compatibility condition (2.1). By this compatibility condition and Lemma 2.6, we have that for any extension $\tilde{u}_{\phi} \in X_{\phi}$, there exists a unique $\tilde{u}_0 \in V^{\perp} \in X$ satisfying $\nabla \cdot \tilde{u}_0 = g - \nabla \cdot \tilde{u}_{\phi}$ with only a bound $\|\tilde{u}_0\|_1$ available. In other words, even though through the Hopf extension we can control the size of $\|\tilde{u}_{\phi}\|_{L^4(\Omega)}$, we have no obvious way to control the size of $\|\tilde{u}_0\|_{L^4(\Omega)}$ to an arbitrary degree as required in applying the first small data condition (2.20) in the proof for existence in Theorem 2.21.*

3. Convergence and consistency analysis. To derive a mixed finite element formulation of continuous Brinkman, 2.4 and 2.3, assume that Ω is polygonal with polygonal subdomains Ω_* for $* = p,\ s$. Let $\mathcal{T}_h$ be a triangulation of Ω with $E_h \in \mathcal{T}_h$ triangles for $d = 2$ or tetrahedra for $d = 3$. Moreover, we require that any $E_h \in \mathcal{T}_h$ be such that interior(E_h) is completely contained in Ω_f, Ω_p, or Ω_s.

For non-homogeneous boundary data $\phi \in H^{1/2}(\partial\Omega)$, we consider associated interpolant of this problem data ϕ^h satisfying

$$\int_{\partial\Omega} \phi \cdot \hat{n} = \int_{\partial\Omega} \phi^h \cdot \hat{n} \tag{3.1}$$

which is required in this analysis, explicitly used in proving Lemma 3.7. Accordingly, we define the following general finite element spaces for our analysis

$$\begin{array}{lcl} X^h_{\phi^h} & := & \left\{ v \in C^0(\Omega)^d : \forall E_h,\ v|_{E_h} \in (P_v)^d \in \mathcal{T}_h \text{ and } v|_{\partial E_h \cap \partial\Omega} = \phi^h \right\} \\ Q^h & := & \left\{ q \in L^2_0(\Omega) : \forall E_h \in \mathcal{T}_h,\ q|_{E_h} \in P_q \right\} \end{array}$$

for some finite dimensional spaces polynomial spaces P_v and P_q so that $X^h_{\phi^h} \subset H^1(\Omega)$ and $Q^h \subset Q$ be conforming, finite element subspaces. We write $X^h = X^h_0 \subset X$. For example, let $X^h_{\phi^h}$ and Q^h be spaces of piecewise polynomials on each element of $\mathcal{T}_h$ that satisfy the discrete inf-sup condition

$$\inf_{q \in Q^h} \sup_{v \in X^h} \frac{b(v,q)}{\|v\|_1 \|q\|} \geq \beta_h > 0 \tag{3.2}$$

The well-known Taylor-Hood mixed finite elements are one such example where X^h consists of piecewise quadratic elements and Q^h piecewise linears. We also define the discrete analogue to $V_{\phi}(g)$.

$$V^h_{\phi^h}(g) = \left\{ v \in X^h_{\phi^h} : \int_{\Omega} (\nabla \cdot v)\, q = \int_{\Omega} gq\ \forall q \in Q^h \right\}$$

Write $V^h_{\phi^h}$ for $g \equiv 0$, $V^h(g)$ when $\phi \equiv 0$, and V^h with g and ϕ are 0. We consider the general case $V^h_{\phi^h}(g) \notin V_{\phi}(g)$ (which is true for Taylor-Hood elements). We will also need the following discrete analogue of the convective term.

Definition 3.1. *Fix $g \in L^2(\Omega)$ and $u,\ v,\ w \in H^1(\Omega)$ such that $\int_{\Omega} (\nabla \cdot u)\, q^h = \int_{\Omega} gq^h$ for all $q^h \in Q^h$. Let $c^h_* : H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be such that*

$$c^h_*(u,v,w) := \frac{1}{2}\left(c_*(u,v,w) - c_*(u,w,v)\right) - \frac{1}{2}\int_{\Omega_*} g\,(v \cdot w)$$

By construction, $c^h_*(\cdot,\cdot,\cdot)$ is continuous, (explicitly) pseudo-skew symmetric, and consistent with $c_*(\cdot,\cdot,\cdot)$ in the sense stated in the following lemma.

Lemma 3.2. *Fix $g \in L^2(\Omega)$. The trilinear functional $c^h(\cdot,\cdot,\cdot)$ is continuous; in particular, for u, v, w in $H^1(\Omega)$ such that $\int_\Omega (\nabla \cdot u) q^h = \int_\Omega g q^h$ for all $q^h \in Q^h$,*

$$c_*^h(u,v,w) \le C_L^h \|u\|_{1,*} \|v\|_{1,*} \|w\|_{1,*}$$

where $C_L^h = C_L\left(1+\sqrt{d}/2\right)$. Write generically $C_L^h := C_L$. Moreover,

$$c_*^h(u,v,v) = -\frac{1}{2}\int_{\Omega_*} g|v|^2, \qquad \nabla \cdot u = g \Rightarrow c^h(u,v,w) = c(u,v,w)$$

Proof. Trilinearity of $c^h(\cdot,\cdot,\cdot)$ is obvious. For continuity, first note that it is clear that $\|g\| \le \sqrt{d}\|u\|_1$ (indeed, consider $\|g\| = sup_{q\in Q^h}((\nabla\cdot u, q)/\|q\|)$). Then,

$$\begin{aligned}c_*^h(u,v,w) &= \frac{1}{2}(c_*(u,v,w) - c_*(u,w,v)) - \frac{1}{2}\int_{\Omega_*} g(v\cdot w)\\ &\le C_L\|u\|_{1,*}\|v\|_{1,*}\|w\|_{1,*} + \frac{C_L}{2}\left(\sqrt{d}\|u\|_{1,*}\right)\|v\|_{1,*}\|w\|_{1,*}\end{aligned}$$

The other results are obvious applications of the definition of $c_*(\cdot,\cdot,\cdot)$. ∎

We can now state the discrete NS-Brinkman problem:

Problem 3.3. *(Discrete NS-Brinkman) Fix $f \in X'$ and $g \in L^2(\Omega)$ satisfying the compatibility condition Eq. (2.1). Find $u^{\delta,h} \in V_{\phi^h}^h(g)$, $p^{\delta,h} \in Q^h$ satisfying*

$$\begin{aligned}&\forall v \in X^h, && a\left(u^{\delta,h}, v\right) + b\left(v, p^{\delta,h}\right) + c^h\left(u^{\delta,h}, u^{\delta,h}, v\right) = l_1(v)\\ &\forall q \in Q^h, && b\left(u^{\delta,h}, q\right) = l_2(q)\end{aligned}$$

Existence of solutions to the discrete NS-Brinkman, Problem 3.3, closely follows the proof of for the continuous case. We conclude without further proof:

Theorem 3.4. *(Well-posedness of Discrete NS-Brinkman) Under small data conditions of Theorem 2.21, there exists a uniqe solution $(u^{\delta,h}, p^{\delta,h}) \in (V_{\phi^h}^h(g), Q^h)$ of the Discrete NS-Brinkman, Problem 3.3 that satisfies the same stability bound as Theorem 2.14 with u^δ, p^δ, β, and C_L replaced by $u^{\delta,h}$, $p^{\delta,h}$, β_h, and C_L^h respectively. Write generically $\beta_h = \beta$ and $C_L^h = C_L$.*

For low Reynold's numbers, the convective term in NS-Brinkman is negligible. Hence, taking $c^h(\cdot,\cdot,\cdot) = 0$, we consider the discrete analogue of Stokes-Brinkman.

Problem 3.5. *(Discrete Stokes-Brinkman) Fix $f \in X'$ and $g \in L^2(\Omega)$ satisfying the compatibility condition Eq. (2.1). Find $u^{\delta,h} \in V_{\phi^h}^h(g)$, $p^{\delta,h} \in Q^h$ satisfying*

$$\begin{aligned}&\forall v \in X^h, && a\left(u^{\delta,h}, v\right) + b\left(v, p^{\delta,h}\right) = l_1(v)\\ &\forall q \in Q^h, && b\left(u^{\delta,h}, q\right) = l_2(q)\end{aligned}$$

Existence of solutions to discrete Stokes-Brinkman, Problem 3.5, closely follows the proof of for the continuous case. We conclude without further proof:

Theorem 3.6. *(Well-posedness of Discrete NS-Brinkman) There exists a unique $(u^{\delta,h}, p^{\delta,h}) \in (V_{\phi^h}^h(g), Q^h)$ satisfying discrete Stokes-Brinkman, Problem 3.5. Moreover, any such solution satisfies the same stability bound as the continuous problem shown in Theorem 2.14 with u^δ, p^δ, β and C_L replaced by $u^{\delta,h}$, $p^{\delta,h}$, β_h, and C_L^h respectively. Write generically $\beta_h = \beta$ and $C_L^h = C_L$.*

3.1. Convergence analysis of discrete Brinkman. We now derive error estimates $u^{\delta,h}$ obtained from both Problems 3.5 and (3.3). For what follows, let $\left(u^{\delta}, p^{\delta}\right) \in \left(V_{\phi}\left(g\right), Q\right)$ and $\left(u^{\delta,h}, p^{\delta,h}\right) \in \left(V^h_{\phi^h}\left(g\right), Q^h\right)$ represent solutions of the continuous NS-Brinkman (or Stokes-Brinkman) and discrete NS-Brinkman (or Stokes-Brinkman) problems respectively. We show that these error estimates for the discrete Brinkman velocity $u^{\delta,h}$ relative to the continuous Brinkman velocity u^{δ} are uniform with respect to penalty parameter $\delta \to 0$.

The following lemma is a technical result required in proving the error estimate.

LEMMA 3.7.

$$\mathop{inf}_{\tilde{v}^h \in V^h_{\phi^h}(g)} \left\|u^{\delta} - \tilde{v}^h\right\|_1 \leq \left(1 + \frac{\sqrt{d}}{\beta_h}\right) \mathop{inf}_{v^h \in X^h_{\phi^h}} \left\|u^{\delta} - v^h\right\|_1$$

Proof. Fix $v^h \in X^h_{\phi^h}$. By choosing a *good* approximation ϕ^h of boundary data $\phi \in H^{1/2}\left(\Omega\right)$ via Eq. (3.1), we ensure that $\nabla \cdot \left(u^{\delta} - v^h\right) \in L^2_0\left(\Omega\right)$. Hence, as a discrete analogue of Lemma 2.6 implied by the discrete inf-sup condition, there exists $w^h \in \left(V^h\right)^{\perp}$ satisfying $\nabla \cdot w^h = \nabla \cdot \left(u^{\delta} - v^h\right)$. Moreover,

$$\left\|w^h\right\|_1 \leq \frac{1}{\beta_h} \sup_{q \in Q^h} \frac{b\left(w^h, q\right)}{\|q\|} = \frac{1}{\beta_h} \sup_{q \in Q^h} \frac{b\left(\left(u^{\delta} - v^h\right), q\right)}{\|q\|} \leq \frac{\sqrt{d}}{\beta_h} \left\|u^{\delta} - v^h\right\|_1$$

Since $\nabla \cdot \left(w^h + v^h\right) = \nabla \cdot u^{\delta} = g$, it follows that $v^*_h := \left(w^h + v^h\right) \in V^h\left(g\right)$. Hence,

$$\left\|u^{\delta} - v^*_h\right\|_1 \leq \left\|u^{\delta} - v^h\right\|_1 + \left\|w^h\right\|_1 \leq \left(1 + \frac{\sqrt{d}}{\beta_h}\right) \left\|u^{\delta} - v^h\right\|_1$$

This inequality holds for arbitrary $v^h \in X^h_{\phi^h}$, the conclusion follows. $\square$

We state the following error estimate for Stokes-Brinkman velocities.

THEOREM 3.8. *(Stokes-Brinkman error estimate) Suppose that $\left(u^{\delta}, p^{\delta}\right) \in \left(X_{\phi}, Q\right)$ solves Stokes-Brinkman, Problem 2.4, and $u^{\delta,h} \in X^h_{\phi^h}$ solves discrete Stokes-Brinkman, Problem 3.5. Then,*

$$\begin{aligned}
\left\|\nabla\left(u^{\delta} - u^{\delta,h}\right)\right\|^2 \quad &\leq \quad C\Bigg[\mathop{inf}_{q^h \in Q^h} \left(\tfrac{1}{\nu^2}\left\|p^{\delta} - q^h\right\|^2_{fp} + \tfrac{\delta}{\nu}\left\|p^{\delta} - q^h\right\|^2_s\right) \\
&\qquad + \mathop{inf}_{v^h \in X^h_{\phi^h}} \left(\left(1 + \tfrac{1}{k^p}\right)\left\|u^{\delta} - v^h\right\|^2_{1,fp} + \tfrac{1}{\delta}\left\|u^{\delta} - v^h\right\|^2_{1,s}\right)\Bigg] \\
\left\|u^{\delta} - u^{\delta,h}\right\|^2_{1,s} \quad &\leq \quad C\Bigg[\mathop{inf}_{q^h \in Q^h} \left(\left(\tfrac{\delta}{\nu^2}\left\|p^{\delta} - q^h\right\|^2_{fp} + \tfrac{\delta^2}{\nu}\left\|p^{\delta} - q^h\right\|^2_s\right)\right) \\
&\qquad + \mathop{inf}_{v^h \in X^h_{\phi^h}} \delta\left(\left(1 + \tfrac{1}{k^p}\right)\left\|u^{\delta} - v^h\right\|^2_{1,fp} + \left\|u^{\delta} - v^h\right\|^2_{1,s}\right)\Bigg]
\end{aligned}$$

where C is independent of ν, K, $\tilde{\nu}$.

Proof. Fix $v^h \in V^h$. Note that $\left(q, \nabla \cdot v^h\right) = 0$ for any $q \in Q^h$. Then,

$$a\left(u^{\delta}, v^h\right) + b\left(p^{\delta}, v^h\right) = l_1\left(v^h\right), \text{ and } a\left(u^{\delta,h}, v^h\right) = l_1\left(v^h\right)$$

Fix $\tilde{u}^h \in V^h_{\phi^h}(g)$. Let $\eta := u^\delta - \tilde{u}^h$, $\phi^h := \tilde{u}^h - u^{\delta,h}$. Subtracting the above equations, we get

$$a\left(\phi^h, v^h\right) = -b\left(p^\delta, v^h\right) - a\left(\eta, v^h\right)$$

Take $v^h = \phi^h$ (indeed, $\phi^h = \tilde{u}^h - u^{\delta,h} \in V^h$). Applying Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}
&\sum_{*=f,p,s} \tilde{\nu}^* \left\|\nabla\phi^h\right\|_*^2 + \frac{\nu}{k^*}\left\|\phi^h\right\|_*^2 = -b\left(p^\delta, \phi^h\right) - a\left(\eta, \phi^h\right) \\
&\le \sum_{*=f,p,s} \sqrt{d}\left\|p^\delta - \tilde{p}^h\right\|_* \left\|\nabla\phi^h\right\|_* + \tilde{\nu}^*\left\|\nabla\eta\right\|_*\left\|\nabla\phi^h\right\|_* + \frac{\nu}{k^*}\left\|\eta\right\|_*\left\|\phi^h\right\|_* \\
&\le \sum_{*=f,p,s} \frac{3d}{2\tilde{\nu}^*}\left\|p^\delta - \tilde{p^\delta}^h\right\|_*^2 + \frac{\tilde{\nu}^*}{6}\left\|\nabla\phi^h\right\|_*^2 + \frac{3\tilde{\nu}^*}{2}\left\|\nabla\eta\right\|_*^2 + \frac{\tilde{\nu}^*}{6}\left\|\nabla\phi^h\right\|_*^2 \\
&\qquad + \frac{3C_P^2\nu}{2(k^f)^2}\left\|\eta\right\|_f^2 + \frac{\nu}{6}\left\|\nabla\phi^h\right\|_f^2 + \sum_{*=p,s}\frac{\nu}{2k^*}\left(\left\|\eta\right\|_*^2 + \left\|\phi^h\right\|_*^2\right)
\end{aligned}$$

Recall $\nu/\tilde{\nu}^s$, $1/k^f$, and $k^s = \delta$.

$$\begin{aligned}
&\nu\left\|\nabla\phi^h\right\|_{fp}^2 + \alpha_0^s\left\|\phi^h\right\|_{1,s}^2 \\
&\le \frac{3d}{\nu}\left\|p^\delta - \tilde{p}^h\right\|_{fp}^2 + 3d\delta\left\|p^\delta - \tilde{p}^h\right\|_s^2 + 3\nu\left(1 + C_p^4\delta + \frac{C_p^2}{k^p}\right)\left\|\nabla\eta\right\|_{fp}^2 + \frac{4\nu}{\delta}\left\|\nabla\eta\right\|_{1,s}^2
\end{aligned}$$

Also, recall that $\nu < \alpha_0^s = \min_{x\in\Omega_s}\{1/\delta, \nu/\delta\}$. Then,

$$\begin{aligned}
\left\|\nabla\phi^h\right\|^2 &\le C\Big[\tfrac{1}{\nu^2}\left\|p^\delta - \tilde{p}^h\right\|_{fp}^2 + \tfrac{\delta}{\nu}\left\|p^\delta - \tilde{p}^h\right\|_s^2 \\
&\qquad + \left(1 + \delta + \tfrac{1}{k^p}\right)\left\|\nabla\eta\right\|_{fp}^2 + \tfrac{1}{\delta}\left\|\nabla\eta\right\|_{1,s}^2\Big] \\
\left\|\phi^h\right\|_{1,s}^2 &\le C\Big[\tfrac{\delta}{\nu^2}\left\|p^\delta - \tilde{p}^h\right\|_{fp}^2 + \tfrac{\delta^2}{\nu}\left\|p^\delta - \tilde{p}^h\right\|_s^2 \\
&\qquad + \left(\delta + \delta^2 + \tfrac{\delta}{k^p}\right)\left\|\nabla\eta\right\|_{fp}^2 + \left\|\nabla\eta\right\|_{1,s}^2\Big]
\end{aligned}$$

Applying the triangle inequality $\left\|u^\delta - u^{\delta,h}\right\|_1 \le \left\|\phi^h\right\|_1 + \|\eta\|_1$ nearly proves the claim. However, since this holds for any $\tilde{p}^h \in Q^h$ but only for $\tilde{u}^h \in V^h_{\phi^h}(g)$, we still must show that this holds for any $\tilde{u}^h \in X^h_{\phi^h}$. This follows from Lemma 3.7. ∎

We notice that the only problematic term remaining is the term

$$\frac{1}{\delta}\left\|u^\delta - v^h\right\|_{1,s}^2$$

We show that this is bounded with respect to $\delta \to 0$.

THEOREM 3.9. *For a suitable approximation $v^h \in X^h_{\phi^h}$ of $u^\delta \in X_\phi$,*

$$\frac{1}{\delta}\left\|u^\delta - v^h\right\|_{1,s}^2 \le C < \infty$$

where, C is a generic constant independent of δ.

Proof. From approximation theory we have that $\left\|u^\delta - v^h\right\|_{1,s} \le C\left\|u^\delta\right\|_{1,s}$. From Theorem 2.14 we have the stability bound $\left\|u^\delta\right\|_{1,s}^2 \le C\delta$ which proves the claim ∎

We can also conclude the following error estimate for NS-Brinkman, Problem 3.3.

THEOREM 3.10. *(NS-Brinkman error estimate) Suppose that the small data condition Eq. (2.21) is satisfied. Then,*

$$\left\|u^{\delta}-u^{\delta,h}\right\|_{1,s}^{2} \leq C \inf_{q^{h}\in Q^{h}}\left(\frac{\delta}{\nu^{2}}\left\|p^{\delta}-q^{h}\right\|_{fp}^{2}+\frac{\delta^{2}}{\nu}\left\|p^{\delta}-q^{h}\right\|_{s}^{2}\right)$$
$$+\inf_{v^{h}\in X_{\phi^{h}}^{h}}\left[\delta\left(1+\frac{1}{k^{p}}+\frac{C_{\delta,NSE}}{\nu^{4}}\delta\right)\left\|u^{\delta}-v^{h}\right\|_{1,fp}^{2}+\left(1+\frac{C_{\delta,NSE}}{\nu^{4}}\delta^{3}\right)\left\|u^{\delta}-v^{h}\right\|_{1,s}^{2}\right]$$

and

$$\left\|\nabla\left(u^{\delta}-u^{\delta,h}\right)\right\|^{2} \leq C \inf_{q^{h}\in Q^{h}}\left(\frac{1}{\nu^{2}}\left\|p^{\delta}-q^{h}\right\|_{fp}^{2}+\frac{\delta}{\nu}\left\|p^{\delta}-q^{h}\right\|_{s}^{2}\right)$$
$$+\inf_{v^{h}\in X_{\phi^{h}}^{h}}\left[\left(1+\frac{1}{k^{p}}+\frac{C_{\delta,NSE}}{\nu^{4}}\delta\right)\left\|u^{\delta}-v^{h}\right\|_{1,fp}^{2}+\frac{1}{\delta}\left(1+\frac{C_{\delta,NSE}}{\nu^{4}}\delta^{3}\right)\left\|u^{\delta}-v^{h}\right\|_{1,s}^{2}\right]$$

where $C>0$ is independent of ν, $\tilde{\nu}$, and K.

Proof. Fix $v^h \in V^h$. Note that $\left(q, \nabla\cdot v^h\right)=0$ for any $q\in Q^h$. Then,

$$a\left(u^{\delta},v^{h}\right)+b\left(p^{\delta},v^{h}\right)+c^{h}\left(u^{\delta},u^{\delta},v^{h}\right)=l_{1}\left(v^{h}\right)$$
$$a\left(u^{\delta,h},v^{h}\right)+c^{h}\left(u^{\delta,h},u^{\delta,h},v^{h}\right)=l_{1}\left(v^{h}\right)$$

Fix $\tilde{u}^h \in V^h_{\phi^h}(g)$. Define $\eta := u^\delta - \tilde{u}^h$, $\phi^h := \tilde{u}^h - u^{\delta,h}$. Expanding the nonlinear term we obtain:

$$\begin{aligned}&-c^{h}\left(u^{\delta},u^{\delta},v^{h}\right)+c^{h}\left(u^{\delta},u^{\delta},v^{h}\right)\\&=-c^{h}\left(\eta,u^{\delta},v^{h}\right)-c^{h}\left(\phi^{h},u^{\delta},v^{h}\right)-c^{h}\left(u^{\delta,h},\eta,v^{h}\right)-c^{h}\left(u^{\delta,h},\phi^{h},v^{h}\right)\end{aligned}$$

Subtracting the above equations, we get

$$\begin{aligned}a\left(\phi^{h},v^{h}\right)&=-b\left(p^{\delta},v^{h}\right)-a\left(\eta,v^{h}\right)\\&-c^{h}\left(\eta,u^{\delta},v^{h}\right)-c^{h}\left(\phi^{h},u^{\delta},v^{h}\right)-c^{h}\left(u^{\delta,h},\eta,v^{h}\right)-c^{h}\left(u^{\delta,h},\phi^{h},v^{h}\right)\end{aligned}$$

Take $v^h=\phi^h$ (indeed, $\phi^h = \tilde{u}^h - u^{\delta,h}\in V^h$). Then, applying Hölder's and Young's inequalities and the explicit skew-symmetry of $c^h_*(\cdot,\cdot,\cdot)$, we obtain

$$\begin{aligned}&\sum_{*=f,p,s}\tilde{\nu}^{*}\left\|\nabla\phi^{h}\right\|_{*}^{2}+\frac{\nu}{k^{*}}\left\|\phi^{h}\right\|_{*}^{2}\\&=\sum_{*=f,p,s}-b_{*}\left(p^{\delta}-\tilde{p}^{h},\phi^{h}\right)-a_{*}\left(\eta,\phi^{h}\right)\\&\quad+\sum_{*=f,p,s}-c_{*}^{h}\left(\eta,u^{\delta},\phi^{h}\right)-c_{*}^{h}\left(\phi^{h},u^{\delta},\phi^{h}\right)-c_{*}^{h}\left(u^{\delta,h},\eta,\phi^{h}\right)-c_{*}^{h}\left(u^{\delta,h},\phi^{h},\phi^{h}\right)\\&\leq\sum_{*=f,p,s}\sqrt{d}\left\|p^{\delta}-\tilde{p}^{h}\right\|_{*}\left\|\nabla\phi^{h}\right\|_{*}+\tilde{\nu}^{*}\left\|\nabla\eta\right\|_{*}\left\|\nabla\phi^{h}\right\|_{*}+\frac{\nu}{k^{*}}\left\|\eta\right\|_{*}\left\|\phi^{h}\right\|_{*}\\&\quad+\sum_{*=f,p,s}C_{L}\left(\left\|u^{\delta}\right\|_{1,*}+\left\|u^{\delta,h}\right\|_{1,*}\right)\left\|\eta\right\|_{1,*}\left\|\phi^{h}\right\|_{1,*}+C_{L}\left\|u^{\delta}\right\|_{1,*}\left\|\phi^{h}\right\|_{1,*}^{2}\\&\quad+\frac{1}{2}\left\|g\right\|_{p}\left(C_{L}\left\|\nabla\phi^{h}\right\|_{fp}^{2}\right)\end{aligned}$$

We have shown previously that u^δ and $u^{\delta,h}$ are bounded in H^1 in Ω and Ω_s (Theorem 2.21) and denoted

$$\left\|u^\delta\right\|_{1,s} \leq \tfrac{\sqrt{\delta}}{\nu}\sqrt{C_{\delta,NSE}}, \qquad \left\|\nabla u^\delta\right\| \leq \tfrac{1}{\nu}\sqrt{C_{\delta,NSE}}$$

Recall $\nu/\tilde{\nu}^s$, $1/k^f$, and $k^s = \delta$. Then,

$$\begin{aligned}
&\sum_{*=f,p,s} \tilde{\nu}^* \left\|\nabla\phi^h\right\|_*^2 + \frac{\nu}{k^*}\left\|\phi^h\right\|_*^2 \\
&\leq \sum_{*=f,p,s} \frac{4d}{\tilde{\nu}^*}\left\|p^\delta - \tilde{p}^h\right\|_*^2 + \frac{\tilde{\nu}^*}{16}\left\|\nabla\phi^h\right\|_*^2 + 4\tilde{\nu}^*\left\|\nabla\eta\right\|_*^2 + \frac{\tilde{\nu}^*}{16}\left\|\nabla\phi^h\right\|_*^2 \\
&\quad + \sum_{*=f,p,s}\left(\frac{2\nu}{k^*}\left\|\eta\right\|_*^2 + \frac{\nu}{8k^*}\left\|\phi^h\right\|_*^2\right) \\
&\quad + \frac{C_L^2 C_{\delta,NSE}}{\nu^2}\frac{6}{\nu}\left\|\nabla\eta\right\|_{fp}^2 + \frac{\nu}{12}\left(\left\|\nabla\phi^h\right\|_f^2 + \left\|\nabla\phi^h\right\|_p^2\right) \\
&\quad + \frac{C_L^2 C_{\delta,NSE}\delta^2}{\nu^2}\frac{8}{\nu}\left\|\eta\right\|_{1,s}^2 + \frac{\nu}{8\delta}\left\|\phi^h\right\|_{1,s}^2 + \frac{C_L}{2}\left\|g\right\|_p\left\|\phi^h\right\|_{1,p}^2 \\
&\quad + C_L\frac{1}{\nu}\sqrt{C_{\delta,NSE}}\left(\left\|\nabla\phi^h\right\|_f^2 + \left\|\nabla\phi^h\right\|_p^2\right) + C_L\frac{\sqrt{\delta}}{\nu}\sqrt{C_{\delta,NSE}}\left\|\phi^h\right\|_{1,s}^2
\end{aligned}$$

Also, recall that $\nu < \alpha_0^s = \nu/\delta$ and $k^p < k^f = 1/\delta$ such that

$$\alpha_1^{fp} = \max\nolimits_{x\in\Omega_{fp}}\left\{\nu, \nu/k\left(x\right)\right\} \leq \nu/k^p$$

Applying the small data condition (sufficient condition for uniqueness of solutions to NS-Brinkman), Eq. (2.21), absorbing terms, and simplifying we obtain

$$\begin{aligned}
&\alpha_0^{fp}\left\|\phi\right\|_{1,fp} + \alpha_0^s\left\|\phi\right\|_{1,s} \\
&\leq 16d\delta\left\|p^\delta - \tilde{p}^h\right\|_s^2 + \left(16\frac{\delta}{\nu} + 8\frac{\nu}{\delta} + 32C_L^2 C_{\delta,NSE}\frac{\delta^2}{\nu^3}\right)\left\|\eta\right\|_{1,s}^2 \\
&\quad + \frac{16d}{\nu}\left\|p^\delta - \tilde{p}^h\right\|_{fp}^2 + \left(32\nu + 16C_L^2\frac{\nu}{k^p} + 24C_L^2 C_{\delta,NSE}\frac{1}{\nu^3}\right)\left\|\nabla\eta\right\|_{fp}^2
\end{aligned}$$

Apply the triangle inequality to obtain estimate for $u^\delta - u^{\delta,h}$ (e.g. $\left\|u^\delta - u^{\delta,h}\right\| \leq \left\|\phi^h\right\| + \left\|\eta\right\|$). This nearly proves the claim. However, since this holds for any $\tilde{p}^h \in Q^h$ but only for $\tilde{u}^h \in V_{\phi^h}^h\left(g\right)$, we still must show that this holds for any $\tilde{u}^h \in X_{\phi^h}^h$. This follows easily from Lemma 3.7. ☐

3.2. Approximating slow, viscous flow around solid obstacles. We assume that the Stokes equation for fluid flow in Ω_f be the true flow velocity:

Problem 3.11. *(Stokes) Find* $(u,p) \in X_{f,\phi} \times Q$ *where*

$$X_{f,\phi} := \left\{v \in H^1\left(\Omega_f\right) : v|_{\partial\Omega} = \phi \text{ and } v|_{\partial\Omega_s} = 0\right\}$$

with boundary data $u|_{\partial\Omega} = \phi \in H^{1/2}\left(\partial\Omega\right)$ *and* $u|_{\partial\Omega_s} = 0$ *such that* $\tau\left(u,p\right) \in L^2\left(\partial\Omega_s\right)$ *satisfying*

$$\begin{aligned}
&\forall v \in H^1\left(\Omega\right), \qquad \int_{\Omega_f}\nu\nabla u : \nabla v - \int_{\Omega_f} p\nabla\cdot v + \int_{\partial\Omega_s}\left(\tau\left(u,p\right)\cdot\hat{n}\right)\cdot v = \left\langle f, v\right\rangle \\
&\forall q \in L_0^2\left(\Omega\right), \qquad \int_{\Omega_f} q\nabla\cdot u = 0
\end{aligned}$$

where

$$\tau(u,p)\cdot\hat{n} := \nu\nabla u\cdot\hat{n} - p\hat{n}$$

Note that $g \equiv 0$ here. Also, we only require velocity test functions to vanish on $\partial\Omega$ and *not* on $\partial\Omega_s$. Hence, we require the inclusion of the boundary integral to properly model Stokes flow around solid obstacles.

In order to establish well-posedness of solutions to this variational Stokes problem, we insist that solutions satisfy $(u,p) \in H^2(\Omega_f) \times H^1(\Omega_f)$ and with extension $u \equiv 0$, $p \equiv 0$ on Ω_s, that $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)$. This will be guaranteed for polygonal boundaries $\partial\Omega$, $\partial\Omega_s$ and $f \in L^2(\Omega)$.

Consider approximations to this flow given by the discrete Stokes-Brinkman equation, $\left(u^{\delta,h}, p^{\delta,h}\right) \in H_0^1(\Omega) \times L_0^2(\Omega)$ solving Problem 3.5. First, we construct a priori estimates for u in Ω_f and $u^{\delta,h}$ in Ω_s.

LEMMA 3.12. *For $f \in L^2(\Omega)$, any solution u of the variational Stokes problem such that $\sigma(u,p) \in L^2(\partial\Omega_s)$ satisfies*

$$\|\nabla u\|_{\Omega_f} \leq C\frac{1}{\nu}\|f\|_f$$

Here, $C > 0$ is a constant depending solely on the domain geometry.

Proof. Taking $v = u$ in the variational Stokes equation, the result follows easily. ∎

PROPOSITION 3.13. *Fix $0 < \delta << 1$. Assume k^s_{min}, $k^s_{max} = \delta$ and $\tilde{\nu}^s = \nu/\delta$. Any solution $u^{\delta,h}$ of the discrete Stokes-Brinkman problem satisfies*

$$\frac{1}{\delta}\left\|\nabla u^{\delta,h}\right\|_s^2 + \frac{\nu}{\delta}\left\|u^{\delta,h}\right\|_s^2 \leq C\left(\frac{1}{\nu^2}\|f\|_f^2 + \frac{\delta}{\nu^2}\|f\|_s^2 + \frac{1}{\nu}\gamma\|\phi\|_{H^{1/2}(\partial\Omega)}\right)$$

Proof. Follows from work in previous section. ∎

We want to exploit the relationship between approximations $u^{\delta,h}$ to Stokes-Brinkman in Ω and solutions to the Stokes equations u in Ω_f with extension $u|_{\Omega_s} \equiv 0$. To this end, we reference Angot's fundamental paper [2] in which he provides a clever proof to establish a sharp estimate for $\left\|u^\delta - u\right\|_1$ where u^δ is the solution to continuous Stokes-Brinkman in Ω. We state the theorem here in a slightly modified form to extract exact dependency of estimate on problem data.

THEOREM 3.14. *(Angot) Fix $0 < \delta << 1$. Assume k^s_{min}, $k^s_{max} = \delta$ and $\tilde{\nu}^s = \nu/\delta$. Let (u,p) be a solution of the Stokes problem.*

$$\left\|u^\delta - u\right\|_1 \leq C\frac{\delta}{\nu}\left(\|f\| + \|\tau\|_{H^{1/2}(\partial\Omega_s)}\right)$$

where we write $\tau := \tau(u,p)$ and $C > 0$ is a constant independent of problem data ν, $\tilde{\nu}$, K, f, ϕ.

Proof. Fix $0 < \delta < 1$. Let $\tilde{\nu}^f = \nu$, $\tilde{\nu}^s = \nu/\delta$, $k^f = 1/\delta$, and $k^s = \delta$. Start with the variational Brinkman problem: find $u^\delta \in V$ satisfying $a\left(u^\delta, v\right) = \langle f, v\rangle$ for all $v \in V$. Subtracting the variational Stokes problem from this, writing $w = u^\delta - u$ we have

$$
\begin{aligned}
\int_{\Omega_f} |\nabla w|^2 + \frac{1}{\delta}\int_{\Omega_s} |\nabla w|^2 + \delta \int_{\Omega_f} |w|^2 + \frac{1}{\delta}\int_{\Omega_s} |w|^2 \\
&= \frac{1}{\nu}\int_{\Omega_s} f\cdot w - \frac{1}{\nu}\int_{\partial\Omega_s} \left(\tau\left(u,p\right)\cdot\hat{n}\right)\cdot w - \delta\int_{\Omega_f} u\cdot w \\
&\leq \frac{1}{\nu^2}\frac{\delta}{2}\left\|f\right\|_s^2 + \frac{1}{2\delta}\left\|w\right\|^2 + \frac{C}{\nu^2}\frac{\delta}{2}\left\|\tau\left(u,p\right)\right\|^2_{H^{1/2}(\partial\Omega_s)} \\
&\quad + \frac{1}{2\delta}\left\|\nabla w\right\|_s^2 + \frac{\delta}{2}\left\|u\right\|_f^2 + \frac{\delta}{2}\left\|w\right\|_f^2
\end{aligned}
$$

Here, we bounded the right-hand side by successive applications of the Cauchy-Schwarz and Young inequalities. Absorbing terms right to left sides we obtain

$$
\begin{aligned}
\left\|\nabla\left(u^\delta - u\right)\right\|_1^2 &\leq C\delta\left(\tfrac{1}{\nu^2}\left\|f\right\|^2 + \tfrac{1}{\nu^2}\left\|\tau\right\|^2_{H^{1/2}(\partial\Omega_s)}\right) \\
\left\|u^\delta - u\right\|_{1,s} &\leq C\delta^2\left(\tfrac{1}{\nu^2}\left\|f\right\|^2 + \tfrac{1}{\nu^2}\left\|\tau\right\|^2_{H^{1/2}(\partial\Omega_s)}\right)
\end{aligned}
$$

To recover optimal convergence in $H^1\left(\Omega\right)$ (i.e. to match the $\mathcal{O}\left(\delta\right)$ convergence rate obtained in $H^1\left(\Omega_s\right)$), the idea is to avoid Young's inequality and hence bound the left-hand side by a factor of $\left\|w\right\|_1$. To this end, Angot considers the following auxiliary problem: Find (ω,θ), where we write $\omega_s := \omega|_{\Omega_s}$, $\omega_f := \omega|_{\Omega_f}$, $\theta_s := p^*|_{\Omega_s}$, and $\theta_f := p^*|_{\Omega_f}$, satisfying

$$
\begin{aligned}
&-\nu\Delta\omega_s + \nabla\theta_s + \nu\omega_s = f_s, \qquad \nabla\cdot\omega_s = 0, \qquad \text{in } \Omega_s \\
&\tau\left(\omega_s,\theta_s\right)\cdot\hat{n}|_{\partial\Omega_s} = \tau\left(u,p\right)\cdot\hat{n}|_{\partial\Omega_s}
\end{aligned}
$$

and

$$
\begin{aligned}
&-\nu\Delta\omega_f + \nabla\theta_f = 0, \qquad \nabla\cdot\omega_f = 0, \qquad \text{in } \Omega_f \\
&\omega_f|_{\partial\Omega} = 0, \qquad \omega_f|_{\partial\Omega_s} = \omega_s|_{\partial\Omega_s}
\end{aligned}
$$

The problem in Ω_f is obviously well-posed from classical Stokes theory. The well-posedness of problem in Ω_s is more subtle due to the boundary conditions on $\partial\Omega_s$. Considering the variational problem for ω_s and ω_f and the decomposition $u^\delta = u + \delta\omega - z$, a weak formulation for z can be established and through usual techniques, we can recover an energy equation for z

$$
\begin{aligned}
\left\|\nabla z\right\|_f^2 + \frac{1}{\delta}\left\|\nabla z\right\|_s^2 + \delta\left\|z\right\|_f^2 + \frac{1}{\delta}\left\|z\right\|_s^2 \\
&= \delta\int_{\Omega_f}\nabla\omega:\nabla z + \delta\int_{\Omega_f} u\cdot z + \delta^2\int_{\Omega_f}\omega\cdot z \\
&\leq \left(\delta\left\|\nabla\omega\right\|_f + \delta\left\|u\right\|_f + \delta^2\left\|\omega\right\|_f\right)\left\|\nabla z\right\|_{1,f}
\end{aligned}
$$

We note in addition to Angot's work, that

$$
\left\|\nabla\omega\right\|_f \leq C\left\|\omega_s\right\|_1 \leq \frac{1}{\nu}\left\|f\right\|_s + \frac{1}{\nu}\left\|\tau\left(u,p\right)\right\|_{H^{1/2}(\partial\Omega_s)}
$$

So, it follows that, with an application of Poincare-Friedrich's inequality,

$$
\left\|u^\delta - u\right\|_1 \leq \delta\left\|\omega\right\|_1 + \left\|z\right\|_1 \leq C\frac{\delta}{\nu}\left(\left\|f\right\| + \left\|\tau\right\|_{H^{1/2}(\partial\Omega_s)}\right)
$$

h	$\left\|u^{\delta,h} - u^{\delta}\right\|$	$\left\|u^{\delta,h} - u^{\delta}\right\|_1$	Rate (H^1)
0.09428	3.2033e-6	4.2548e-3	—
0.04714	4.0143e-7	1.0663e-4	5.3
0.02357	5.0215e-8	2.6674e-5	2.0
0.01179	6.2781e-9	6.6696e-6	2.0

TABLE 4.1
Convergence rate data for the first experiment

$\square$

Now we can conclude the following

THEOREM 3.15. *Let the finite element discretization conform to obstacle boundaries such that for any $E_h \in \mathcal{T}_h$ either $E_h \in \Omega_f$ or $E_h \in \Omega_s$. Let u be a solution of the Stokes problem in Ω_f extend to 0 in Ω_s, u^δ a solution of the Stokes-Brinkman problem in Ω, and $u^{\delta,h}$ a solution of the discrete Stokes-Brinkman problem in Ω. Then,*

$$\left\|u^{\delta,h} - u\right\|_1 \leq C\frac{\delta}{\nu}\left(\|f\| + \|\tau\|_{H^{1/2}(\partial\Omega_s)}\right) + \left\|u^{\delta} - u^{\delta,h}\right\|_1$$

where bound for $\left\|u^{\delta} - u^{\delta,h}\right\|_1$ is given in Theorem 3.8.

Proof. Apply the triangle inequality and estimate from Theorem 3.14. $\square$

4. Numerical results. We consider three distinct numerical experiments in this section. First, we confirm the convergence rate ($h \to 0$) for NS-Brinkman suggested in Section 3.1. Next, we demonstrate the robust capability of our proposed FE-discretization of NS-Brinkman to handle a source and non-homogeneous boundary conditions ($\nabla \cdot u \neq 0$ and $\int_{\partial\Omega} u \cdot \hat{n} \neq 0$). Lastly, we consider flow past a non-uniform array of solid obstacles to test the rate of convergence ($h, \delta \to 0$) for Stokes-Brinkman to Stokes with no-slip velocity condition imposed at each obstacle interface.

We utilize Taylor-Hood mixed finite elements (piecewise quadratics for velocity and piecewise linear pressure) for the discretization. Note that the optimal convergence rate for steady Navier-Stokes and Stokes velocity approximations is of order $\mathcal{O}(h^2)$ in $H^1(\Omega)$ and $\mathcal{O}(h^3)$ in $L^2(\Omega)$. We use a Picard iteration to solve the nonlinear NS-Brinkman equation: i.e. set $u^{(0)} = 0$, solve for $u = u^{(n+1)}$ lagging the convective term by $u^{(n)} \cdot \nabla u^{(n+1)}$. We use the FreeFem++ software for each of our simulations.

Experiment 1: For the first experiment, we consider $\Omega = [0,1]^2$ with $\Omega_p = ([0,0.5] \times [0,0.5]) \cup ([0.5,1] \times [0.5,1])$, $\nu = 10^{-2}$, $\delta = 10^{-2}$, $\tilde{\nu}^p = \nu/\delta$, $\tilde{\nu}^f = \nu$, $K_f = 1/\delta$, $K_p = \delta$ and true velocity and pressure given by

$$u = \begin{bmatrix} 0.01\pi sin(\pi x)cos(\pi y) \\ -0.01\pi cos(\pi x)sin(\pi y) \end{bmatrix}, \qquad p = 0.25\,(x-0.5)\,(y-0.5)$$

Note that since the velocity is smooth and K, $\tilde{\nu}$ are discontinuous, it follows that f must be discontinuous.

A uniform triangular mesh is used. The results for this experiment are compiled in Table 4.1. Notice that the H^1-convergence rate is optimal $\mathcal{O}(h^2)$ supporting the basic effectiveness of the proposed FE-discretization of the NS-Brinkman equation and confirming the predictions of the convergence analysis.

Experiment 2: Now we consider $\Omega = [0,2] \times [0,1]$, $\nu = 10^{-2}$, $\tilde{\nu}^s = 1/\delta$, $K_f = 1/\delta$, $K_s = \delta$. Here, we consider a 0.2×0.2 source $g = 1$ centered in the domain Ω and the resulting flow around two square obstacles as shown in Figure 4.1 with imposed Dirichlet boundary conditions

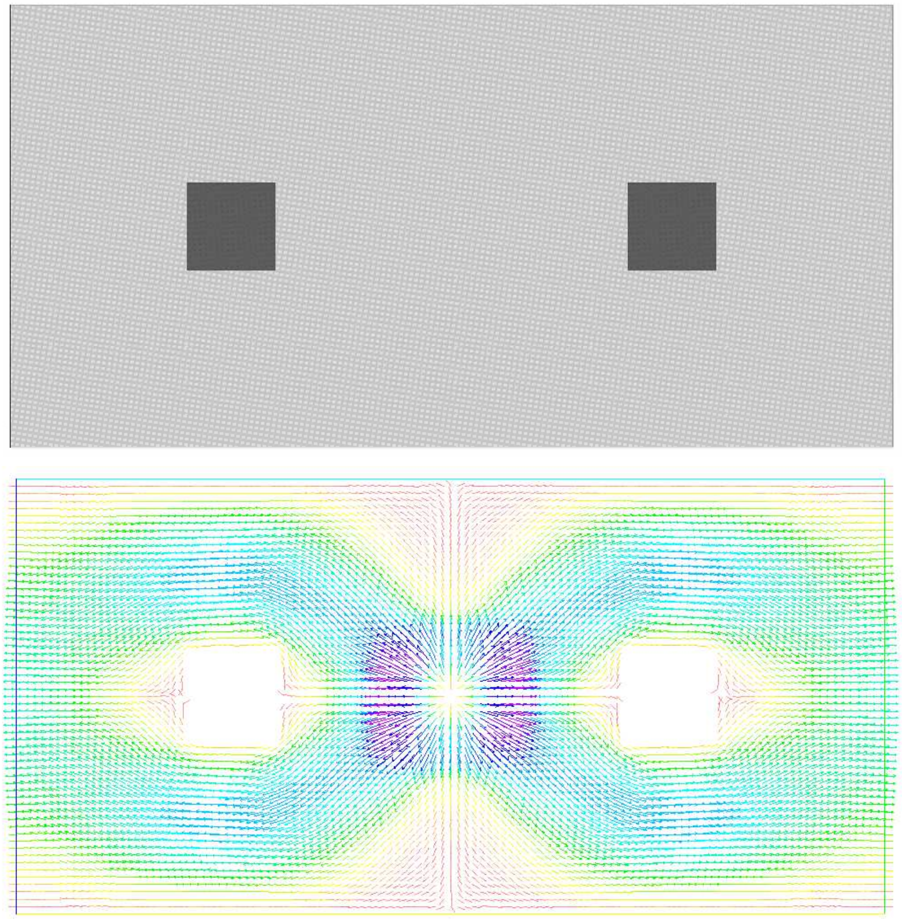

Fig. 4.1. *Experiment 2: (top) problem domain, dark squares represent solid obstacles, (bottom) NS-Brinkman velocity approximation*

$$u|_{x=0} = -0.12y(1-y), \qquad u|_{x=2} = 0.12y(1-y), \qquad u_{y=0,1} = 0$$

A uniform triangular mesh is used. The velocity plot in Figure 4.1 shows the NS-Brinkman approximation to the proposed flow for Experiment 2 corresponding with our intuition. To quantify the accuracy of the approximation, we list the L^2 norm of $u^{\delta,h}$ in Ω_s and H^1 semi-norm in Ω_s and Ω for several combinations of h and δ-values in Table 4.2. Notice that $\left\|u^{\delta,h}\right\|$ and $\left|u^{\delta,h}\right|_{1,s}$ converge at a rate $\mathcal{O}(\delta)$ for each indicated h. This is better than the $\mathcal{O}(\sqrt{\delta})$ suggested by our theory. Also note that $\left|u^{\delta,h}\right|_1$ remains bounded (relatively constant in fact) with h and δ.

	h	$\lVert u^{\delta,h} \rVert_s$	$\lvert u^{\delta,h} \rvert_{1,s}$	$\lvert u^{\delta,h} \rvert_1$
$\delta = 10^{-5}$	0.1414	4.6299e-5	2.1236e-6	0.2973
	0.07071	4.7124e-5	2.3095e-6	0.2992
$\delta = 10^{-10}$	0.1414	4.6543e-10	2.1334e-11	0.2980
	0.07071	4.4738e-10	2.3206e-11	0.2999
$\delta = 10^{-15}$	0.1414	4.6543e-15	2.1335e-16	0.2980
	0.07071	4.7377e-15	2.3296e-16	0.2999

TABLE 4.2

Convergence rate data for the second experiment; $\delta = 10^{-15}$, $h = 0.07071$

	h	$\lVert u^{\delta,h} - u \rVert$	Rate L^2	$\lvert u^{\delta,h} - u \rvert_1$	Rate H^1
$\delta = 10^{-5}$	0.09428	2.6968e-2	—	1.4324e-0	—
	0.04714	1.1757e-2	1.2	8.6499e-1	0.73
	0.02357	7.0509e-3	0.74	4.9880e-1	0.79
$\delta = 10^{-10}$	0.09428	2.4180e-2	—	1.4413e-0	—
	0.04714	7.9136e-3	1.6	8.8644e-1	0.70
	0.02357	2.0529e-3	1.9	4.8989e-1	0.86
$\delta = 10^{-15}$	0.09428	2.4180e-2	—	1.4413e-0	—
	0.04714	7.9154e-3	1.6	8.6437e-1	0.74
	0.02357	2.0529e-3	1.9	4.8989e-1	0.82

TABLE 4.3

Convergence rate data for the third experiment

Experiment 3: Lastly, we consider the relation between the velocity field predicted by Stokes-Brinkman and that predicted by Stokes with no-slip boundary condition imposed at each solid interface. We consider $\Omega = [0,2] \times [0,1]$, $\nu = 10^2$, $f = 0$, $g = 0$, $\tilde{\nu}^s = 1/\delta$, $K_f = 1/\delta$, $K_s = \delta$. Here, we consider the non-uniform array of square obstacles as shown in Figure 4.2 with imposed Dirichlet boundary conditions

$$u|_{x=0} = y(1-y), \qquad u|_{x=2} = y(1-y), \qquad u_{y=0,1} = 0$$

The Stokes velocity used for comparison is obtained by approximating the Stokes equation with the Taylor-Hood mixed finite elements for pressure and velocity with a fine mesh, $h_{max} = 0.018760$. The mesh is constructed by FreeFem++ based on the Delaunay triangulation. We solve Stokes-Brinkman on a coarser, uniform triangular mesh. As illustrated in Table 4.3, there appears to be a degradation in the convergence rate of $u^{\delta,h} \to u$ in L^2 as $h \to 0$ for larger $\delta = 10^{-5}$. For $\delta = 10^{-10}$ and 10^{-15}, the Stokes-Brinkman velocity appears to converge to the Stokes velocity with $h \to 0$ twice as fast in the L^2 norm than H^1 semi-norm, as one would expect. Our results compiled in Table 4.4 also indicates that $u^{\delta,h} \to 0$ in Ω_s as $\delta \to 0$ at a rate $\mathcal{O}(\delta)$. This suggests, once again, a rate of convergence with respect to δ greater than predicted by our theory.

5. Conclusion. The Brinkman model for fluid flow is simple to implement and integrate into existing computing platforms. The uniform stability of Brinkman velocities as $\delta \to 0$ suggests that the finite element Brinkman approximations are dependably accurate representations of Stokes and Navier-Stokes flows, but avoids the cumbersome and often times infeasible task of enforcing no-slip boundary conditions

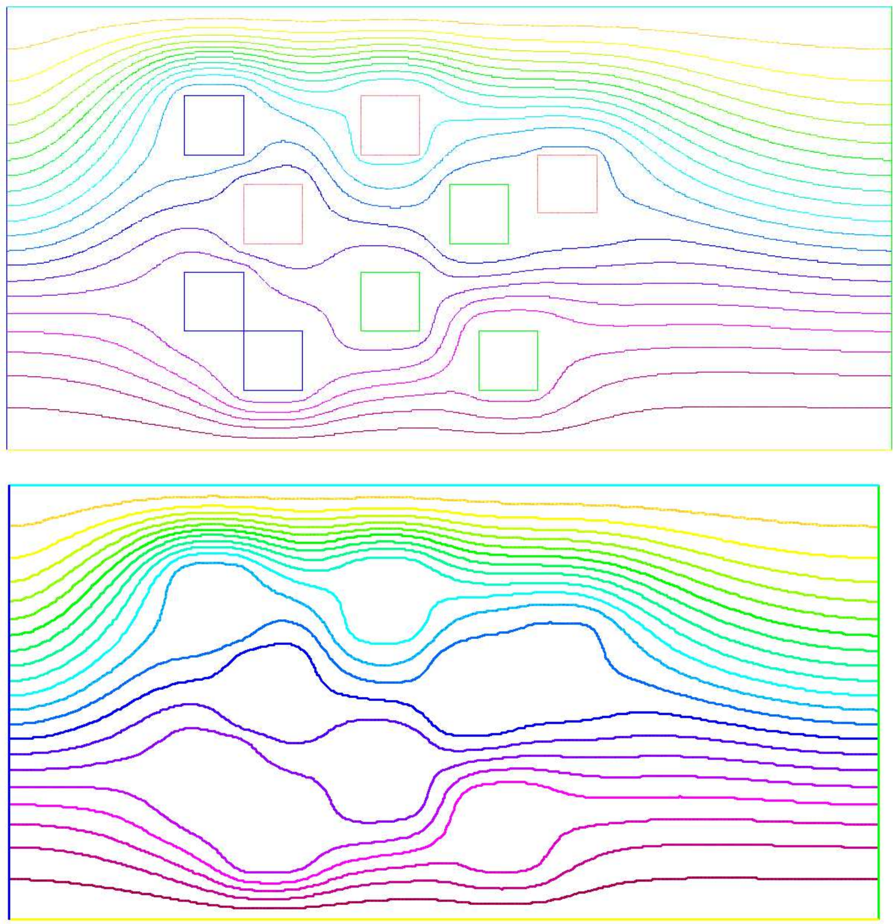

FIG. 4.2. *Experiment 3: Squares in top plot represent outlines of solid obstacles, (top) Stokes velocity approximation, streamlines , (bottom) Stokes-Brinkman velocity approximation, streamlines*

	$\delta = 10^{-5}$	$\delta = 10^{-10}$	$\delta = 10^{-15}$
$\left\|u^{\delta,h}\right\|_s =$	2.0353e-3	2.0983e-08	2.0983e-13
$\left\|u^{\delta,h}\right\|_{1,s} =$	7.8208e-5	8.0846e-10	8.0846e-15

TABLE 4.4
Convergence rate data for the third experiment; $h = 0.02357$

on all interior solid obstacle boundaries.

We have shown sufficient (non-trivial) conditions require for well-posedness of non-solenoidal flows ($\nabla \cdot u^\delta \neq 0$). Moreover, we have shown that the finite element velocity $u^{\delta,h}$ of the NS-Brinkman model converges in $H^1(\Omega_s)$ at a rate $\mathcal{O}(\sqrt{\delta})$ as $\delta \to 0$ and has optimal convergence behavior (approximation theory) in $H^1(\Omega)$ as $h \to 0$ and uniform as $\delta \to 0$. Our numerical experiments confirm, for Taylor-Hood elements, velocity rates of convergence $\mathcal{O}(\delta h^2)$ in $H^1(\Omega_s)$ and $\mathcal{O}(h^2)$ in $H^1(\Omega)$, both uniform as $\delta \to 0$.

Motivated by the ambitious task of accurately modeling the flow of fluids in gascooled, pebble-bed nuclear reactors, we are interested in extending the Brinkman model to the case of compressible fluids and coupling Brinkman flow with the equations of convective and radiative heat transfer. Our preliminary finite element analysis for the steady NS-Brinkman provides encouragement for these advances.

Acknowledgment Thank you, Prof. William Layton, for providing endless ideas, discussions, and encouragement.

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