An Elementary Approach to a Model Problem of Lagerstrom

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March 17, 2008

Abstract

The equation studied is $u'' + \frac{n-1}{r}u' + \varepsilon u\ u' = 0$, with boundary conditions $u(1) = 0, u(\infty) = 1$. This model equation has been studied by many authors since it was introduced in the 1950s by P. A. Lagerstrom. We use an elementary approach to show that there is an infinite series solution which is uniformly convergent on $1 \leq r < \infty$. The first few terms are easily derived, from which one quickly deduces the inner and outer asymptotic expansions. We also give a very short and elementary existence and global uniqueness proof which covers all $\varepsilon > 0$ and $n \geq 1$.

1 Introduction

The main problem is to investigate the asymptotics as $\varepsilon \to 0$ of the boundary value problem

$$u'' + \frac{n-1}{r}u' + \varepsilon uu' = 0 \tag{1}$$

with

$$u(1) = 0, u(\infty) = 1. \tag{2}$$

Our interest in this problem, originally due to Lagerstrom in the 1950s [3], was stimulated by two recent papers by Popovic and Szmolyan [4],[5], who adopt a geometric approach to the problem, and there are many papers which use methods of matched asymptotics or multiple scales, with varying degrees of rigor. (The papers [4] and [5] give lists of references.)

Our point is that it is possible to give a completely rigorous and relatively short answer to the problem without making any appeal either to geometric methods or to matched asymptotics. We can express the solution as an infinite series, uniformly convergent for all values of the independent variable, from which one can read off the asymptotics as $\varepsilon \to 0$.

Lagerstrom came up with the problem as a model of viscous flow, so his work centered on $n = 2$ or 3, but the infinite series can be obtained for any real number n. What n controls is the rate of convergence of the series.

We start in section two by showing that the problem does indeed have one and only one solution for any $n \geq 1$ and any $\varepsilon > 0$. This is based on a simple shooting argument plus a comparison principle. The proof is very much shorter than the one for small ε in [4], and gives global uniqueness.

Before proceeding, we make three further remarks. The first is that, for any solution of (1) $-$ (2), $u' > 0$ on $[1, \infty)$. For if at any point $u' = 0$, then also $u'' = 0$ and so the solution is constant and could not satisfy both boundary conditions. Our solutions are therefore monotone, with $u' > 0, 0 < u < 1$.

Secondly, there is an obvious distinction between $n > 2$ and $n \leq 2$. If $n > 2$, then the problem (1) $-$ (2) has the solution

$$u = 1 - \frac{1}{r^{n-2}}, \tag{3}$$

so that the solution with ε small is presumably some sort of perturbation of this. If $n \leq 2$ then there is no such solution. A consequence is that the convergence as $\varepsilon \to 0$ is more subtle when $n \leq 2$ then when $n > 2$. Our analysis will show that there is little prospect of discussing the behavior for small ε if $n < 2$, but fortunately we can handle all $n \geq 2$.

Finally, our methods are not restricted to Lagerstrom's problem. Hinch, who gives a clear discussion of the problem in [2], goes on to introduce a "terrible" problem which he regards as even more complicated than Lagerstrom's. We show in our final section that our methods deal equally well with that, and even with generalizations of it, with little increase in complexity. We note that it is claimed in many cases that the method of matched asymptotics, while perhaps deficient in rigor, is at least efficient in execution, but that claim is dubious for this last example.

2 Existence and uniqueness

Previous existence and uniqueness proofs have been valid only for small ε, and give just local uniqueness. A much stronger result is easily proved.

Theorem 1 *There exists a unique solution to the problem* $(1)-(2)$ *for any* $\varepsilon > 0$ *and* $n \geq 1$.

Proof: We use a shooting method, by considering the initial value problem

$$u'' + \frac{n-1}{r}u' + \varepsilon u\ u' = 0 \tag{4}$$

$$u(1) = 0, u'(1) = c \tag{5}$$

for each $c > 0$. It is easy to see that solutions to this problem are positive and increasing, and exist on $[0, \infty)$.

We write the equation in the form

$$\left(r^{n-1}u'\right)' + \varepsilon u \left(r^{n-1}u'\right) = 0, \tag{6}$$

so that, on integration,

$$u(r) = \int_1^r \frac{c}{s^{n-1}} e^{-\int_1^s \varepsilon u dt} ds. \tag{7}$$

For $n \geq 1$, (7) implies that either $u(2) \geq 1$ or $u(2) \geq \frac{c}{\varepsilon}(1 - e^{-\varepsilon})$. Let $\mu = \min\left\{1, \frac{c}{\varepsilon}(1 - e^{-\varepsilon})\right\}$ Then, for $r > 2$,

$$u(r) < c\int_1^2 \frac{1}{s^{n-1}} ds + c\int_2^r e^{-\varepsilon\mu(s-2)} ds. \tag{8}$$

From this it follows that $u(\infty)$ is finite. Also, from (7),

$$u(r) = c\int_1^\infty \frac{1}{s^{n-1}} e^{-\int_1^s \varepsilon u dt} ds - c\int_r^\infty \frac{1}{s^{n-1}} e^{-\int_1^s \varepsilon u dt} ds,$$

so that $u(r) \to u(\infty)$ exponentially fast.

From (8), we see that $u(\infty) < 1$ if c is sufficiently small. Equation (7) further implies that if $u \leq 1$ on $[1, 2]$, then $u(2) \geq \int_1^2 \frac{c}{s^k} e^{-\varepsilon} ds$, and this gives a contradiction if

$$c > \frac{e^\varepsilon}{\int_1^2 s^{-k} ds}. \tag{9}$$

Hence, in this range of c, $u(\infty) > 1$.

We therefore know that there are c_1 and c_2, with $0 < c_1 < c_2$, such that if $c = c_1$ then $u(\infty) < 1$, while if $c = c_2$, then $u(\infty) > 1$. Further, for any $r > R > 0$,

$$u(r) = u(R) + c \int_R^r \frac{1}{s^{n-1}} e^{-\int_1^s \varepsilon u(t) dt}.$$

If $n > 2$ the second term is bounded by $\frac{c}{(n-2)R^{n-2}}$, while if $1 \leq n \leq 2$, it is bounded by

$$c \int_R^\infty e^{-\varepsilon\mu(s-2)} ds.$$

Hence for any $n \geq 1$ this term tends to zero as $R \to \infty$, uniformly for $c_1 \leq c \leq c_2$. Since $u(R)$ is a continuous function of c, for any R, it follows that $u(\infty)$ also is continuous in c. Hence there is a c with $u(\infty) = 1$, giving a solution to $(1) - (2)$.

For uniqueness we again use the form (6), which we can integrate to obtain

$$r^{n-1} u'(r) = c - \frac{1}{2} \varepsilon r^{n-1} u^2 + \frac{\varepsilon(n-1)}{2} \int_1^r s^{n-2} u^2 ds. \tag{10}$$

From this it follows that if $c_1 > c_2$, then the corresponding solutions satisfy $u_1 > u_2$ on $(0, \infty)$. However this does not quite show the desired uniqueness. Suppose that there are two solutions of $(1) - (2)$, say u_1 and u_2, with $u_1'(1) = c_1 > u_2'(1) = c_2$. Then from (10),

$$r^{n-1}(u_1' - u_2') = (c_1 - c_2) - \frac{1}{2} \varepsilon r^{n-1} \left(u_1^2 - u_2^2\right) + \frac{\varepsilon(n-1)}{2} \int_1^r s^{n-2} \left(u_1^2 - u_2^2\right) ds. \tag{11}$$

We have shown that $u_1' \to 0, u_2' \to 0$, and are assuming that $u_1 \to 1, u_2 \to 1$, all limits being at an exponential rate. Further, $c_1 > c_2$, and $u_1 > u_2 > 0$ on $(1, \infty)$. It follows that as $r \to \infty$, the left side of (11) tends to zero and the right side to a positive limit. This contradiction proves uniqueness.

Remark 1. *The existence theorem in [4] has one added part. It is shown there that as $\varepsilon \to 0$, the solution tends to a so-called "singular" solution obtained by taking a formal limit as $\varepsilon \to 0$. See [4] for details. This limit result follows from our rigorous asymptotic expansions given below.*

Remark 2. *There would seem to be no difficulty in extending the existence proof even to $n < 1$, but the uniqueness proof does use essentially the fact that $n \geq 1$. We return to this point at the end of section 5.*

3 The infinite series (with $n \geq 2$)

Starting again with (1), and $u(1) = 0$, we obtain

$$r^{n-1} u' = B e^{-\varepsilon \int_1^r u(t) dt} \tag{12}$$

for some constant B. Since $u(\infty) = 1$, (12) implies that $u'(r)$ is exponentially small as $r \to \infty$. Hence we can rewrite (12) as

$$r^{n-1} u' = C e^{-\varepsilon r - \varepsilon \int_\infty^r (u-1) dt},$$

so that

$$u - 1 = C \int_\infty^r \frac{1}{t^{n-1}} e^{-\varepsilon t - \varepsilon \int_\infty^t (u-1) ds} dt.$$

Setting $\varepsilon r = \rho$, $\varepsilon t = \tau$, and $\varepsilon s = \sigma$, we obtain

$$u(\rho) - 1 = C \varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} e^{-\int_\infty^\tau (u(\sigma)-1) d\sigma} d\tau, \tag{13}$$

where we use the arguments ρ and σ to indicate that we mean the rescaled version of u. Here C is a constant satisfying

$$-1 = C \varepsilon^{n-2} \int_\infty^\varepsilon \frac{1}{\tau^{n-1}} e^{-\tau} e^{-\int_\infty^\tau (u-1) d\sigma} d\tau. \tag{14}$$

Since for each ε there is a unique solution, this determines a unique C, dependent on ε. Because $u < 1$, we see that the τ-integral term in (14) tends to infinity as $\varepsilon \to 0$, so

$$\lim_{\varepsilon \to 0} C \varepsilon^{n-2} = 0.$$

By differentiating (3) we are led to expect that C is close to $n-2$ for small ε if $n > 2$.

Since both exponents in (13)have negative exponents, we can immediately say that if

$$E_{n-1}(\rho) = \int_\rho^\infty \frac{1}{\tau^{n-1}} e^{-\tau} d\tau,$$

then

$$|u(\rho) - 1| < C \varepsilon^{n-2} E_{n-1}(\rho). \tag{15}$$

For purposes of future estimates, we make the obvious remark that

$$E_{n-1}(\rho) = \begin{cases} O(\rho^{2-n}) \text{ as } \rho \to 0 \text{ if } n > 2 \\ O(\log \rho) \text{ as } \rho \to 0 \text{ if } n = 2 \\ O(\rho^{1-n} e^{-\rho}) \text{ as } \rho \to \infty. \end{cases} \tag{16}$$

Hence if $n > 2$ there is a constant K such that

$$E_{n-1}(\rho) \leq K \min\left(\rho^{2-n}, \rho^{1-n} e^{-\rho}\right). \tag{17}$$

The method now is to work from (3.2). As observed before, since $u'(r)$ is exponentially small as $r \to \infty$, the integral term $\int_\rho^\infty (u-1)\,d\sigma$ converges. Hence, for given $\varepsilon > 0$ and $\rho_0 > 0$, and any $\rho \geq \rho_0$,

$$u(\rho) - 1 = C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left\{ 1 - \int_\infty^\tau (u-1)\,d\sigma + \frac{1}{2}\left(\int_\infty^\tau (u-1)\,d\sigma\right)^2 - \cdots \right\} d\tau, \tag{18}$$

where the series in the integrand converges uniformly for $\rho_0 \leq \tau < \infty$.

In fact, we will need to use this series for all $\rho \geq \varepsilon$. Thus we need to check its convergence in this interval. This follows from (15) and (16), which imply that for any $\rho \geq \varepsilon$, if $n \geq 2$, then

$$\left| \int_\rho^\infty (u(s) - 1)\,ds \right| < C\varepsilon^{n-2} \int_\varepsilon^\infty E_{n-1}(s)\,ds \tag{19}$$

and

$$\varepsilon^{n-2} \int_\varepsilon^\infty E_{n-1}(s)\,ds = \begin{cases} o(1) \text{ as } \varepsilon \to 0 \text{ if } n > 2 \\ O(1) \text{ as } \varepsilon \to 0 \text{ if } n = 2 \end{cases}.$$

Hence for $n > 2$ and any C, the series in the integrand of (18) converges uniformly on $[\varepsilon, \infty)$.

Now set

$$\Phi = C\varepsilon^{n-2} \int_\varepsilon^\infty E_{n-1}(s)\,ds.$$

We note that, if $n > 2$, then $\Phi \to 0$ as $\varepsilon \to 0$, while if $n = 2$, then $\Phi \to 0$ as $C \to 0$.

We proceed to solve (18)by iteration. Thus, the first approximation is, from (19),

$$u(\rho) - 1 = C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} d\tau + O\left(\Phi^2\right),$$

and we obtain the second approximation by substituting this back in (18). Repeating this, we reach

$$
\begin{aligned}
u-1 = -C\varepsilon^{n-2}E_{n-1} + \left(C\varepsilon^{n-2}\right)^2 \int_\infty^\rho \tfrac{1}{\tau^{n-1}} e^{-\tau}\left(\int_\infty^\tau E_{n-1}d\sigma\right)d\tau \\
+\tfrac{1}{2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \tfrac{1}{\tau^{n-1}} e^{-\tau}\left(\int_\infty^\tau E_{n-1}d\sigma\right)^2 d\tau \\
-\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \tfrac{1}{\tau^{n-1}} e^{-\tau}\int_\infty^\tau \left\{\int_\infty^\sigma \tfrac{1}{s^{n-1}}e^{-s}\left(\int_\infty^s E_{n-1}dt\right)ds\right\}d\sigma d\tau + O\left(\Phi^4\right),
\end{aligned}
\tag{20}
$$

as $\Phi \to 0$.

To obtain C, we need to be able to evaluate each of these terms for small ρ (in particular, for $\rho = \varepsilon$), and this is a matter of integration by parts. Thus, for non-integral n,

$$
\begin{aligned}
E_{n-1}(\rho) &= \int_\rho^\infty \frac{e^{-\tau}}{\tau^{n-1}}d\tau = -\frac{\rho^{2-n}}{2-n}e^{-\rho} + \frac{1}{2-n}\int_\rho^\infty \frac{e^{-\tau}}{\tau^{n-2}}d\tau \\
&= -\frac{\rho^{2-n}}{2-n}e^{-\rho} + \frac{1}{2-n}E_{n-2},
\end{aligned}
\tag{21}
$$

and this can be repeated to give E_{n-1} as a sum of terms of the form $c_k\rho^k e^{-\rho}$ and E_{n-p}, until $0 < n-p < 1$. Then

$$
\begin{aligned}
E_{n-p} &= \int_0^\infty \frac{e^{-\tau}}{\tau^{n-p}}d\tau - \int_0^\rho \frac{e^{-\tau}}{\tau^{n-p}}d\tau \\
&= \Gamma(p+1-n) - \int_0^\rho \frac{e^{-\tau}}{\tau^{n-p}}d\tau
\end{aligned}
$$

and we can then continue to integrate by parts as far as we like. (If n is an integer, we will reach $\int_\rho^\infty \frac{e^{-\tau}}{\tau}d\tau$, which introduces a logarithm.)

Thus $E_{n-1}(\rho)$ can be expressed as a sum of terms of the form $c_k\rho^k e^{-\rho}$, and so obviously the same is true of E_{n-1}^2, with $e^{-2\rho}$ in place of $e^{-\rho}$. Also,

$$
\begin{aligned}
\int_\infty^\rho E_{n-1}(\tau)\,d\tau &= \int_\infty^\rho \left(\int_\tau^\infty \frac{e^{-\sigma}}{\sigma^{n-1}}d\sigma\right)d\tau \\
&= \left[\tau\left(\int_\tau^\infty \frac{e^{-\sigma}}{\sigma^{n-1}}d\sigma\right)\right]\Big|_\infty^\rho + \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-2}}d\tau \\
&= \rho E_{n-1} - E_{n-2},
\end{aligned}
\tag{22}
$$

so that $\int_\infty^\rho E_{n-1}d\tau$ can be expressed as the same type of sum. Hence the second term in (20) gives a sum of terms of the form $E_k(2\rho)$ and the third and fourth terms a sum involving $E_k(3\rho)$.

We now carry the process through in the most interesting cases, $n = 2, 3$.

4 The case $n = 2$

When $n = 2$ we are interested in

$$\begin{aligned}
E_1(\rho) &= \int_\rho^\infty \frac{1}{\tau} e^{-\tau} d\tau \\
&= -e^{-\rho}\log\rho + \int_\rho^\infty e^{-\tau}\log\tau d\tau \\
&= -e^{-\rho}\log\rho + \int_0^\infty e^{-\tau}\log\tau d\tau - \int_0^\rho e^{-\tau}\log\tau d\tau \\
&= -e^{-\rho}\log\rho - \gamma - \rho(\log\rho - 1)e^{-\rho} + O\left(\rho^2\log\rho\right), \text{ for small } \rho, \\
&= -\log\rho - \gamma + \rho + O\left(\rho^2\log\rho\right). \qquad (23)
\end{aligned}$$

(See, for example, [1], Chapter 1.) Also, for future purposes, using (21) we obtain

$$\begin{aligned}
E_2(\rho) &= \frac{e^{-\rho}}{\rho} - E_1(\rho) \qquad (24) \\
&= \frac{1}{\rho} + \log\rho + (\gamma - 1) - \frac{1}{2}\rho + O\left(\rho^2\log\rho\right) \text{ as } \rho \to 0. \qquad (25)
\end{aligned}$$

Looking now at (20), with $\rho = \varepsilon$, we see that as $\varepsilon \to 0$,

$$C\log\varepsilon \to -1$$

and

$$C = \frac{1}{\log\frac{1}{\varepsilon}} + O\left(\frac{1}{\left(\log\frac{1}{\varepsilon}\right)^2}\right).$$

Hence the series in (20) is in powers of $\frac{1}{\log\frac{1}{\varepsilon}}$.

Also, we will work our approximations (in order to compare the results with those of Hinch) to order $\frac{1}{\log^2\left(\frac{1}{\varepsilon}\right)}$, so that (for example)

$$u = \frac{a(r)}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{b(r)}{\log^2\left(\frac{1}{\varepsilon}\right)} + O\left(\log^{-3}\frac{1}{\varepsilon}\right)$$

for any fixed value of r (ρ of order ε). This, as we shall see, necessitates finding

$$C=\frac{1}{\log\left(\frac{1}{\varepsilon}\right)}\left\{1+\frac{A}{\log\left(\frac{1}{\varepsilon}\right)}+\frac{B}{\log^{2}\left(\frac{1}{\varepsilon}\right)}+O\left(\log^{-3}\left(\frac{1}{\varepsilon}\right)\right)\right\},$$

and requires use of all the terms in (20).

With this in mind, we look at the second term of (20). Thus from (22),

$$\int_{\rho}^{\infty}E_1 d\tau=-\rho E_1+e^{-\rho}, \tag{26}$$

so that the second term is

$$\begin{aligned}C^2\int_{\infty}^{\rho}\frac{1}{\tau}e^{-\tau}\left(\tau E_1-e^{-\tau}\right)d\tau &= C^2\left\{\int_{\infty}^{\rho}e^{-\tau}E_1 d\tau-\int_{\infty}^{\rho}\frac{e^{-2\tau}}{\tau}d\tau\right\}\\ &= C^2\left\{\left[-e^{-\tau}E_1\right]|_{\infty}^{\rho}-2\int_{\infty}^{\rho}\frac{e^{-2\tau}}{\tau}d\tau\right\}\\ &= C^2\left(-e^{-\rho}E_1(\rho)+2E_1(2\rho)\right).\end{aligned} \tag{27}$$

From (23), the second term is therefore

$$\begin{aligned}&C^2\left(\log\rho+\gamma-2\log 2\rho-2\gamma+O(\rho)\right)\\ &=C^2\left(-\log\rho-\gamma-2\log 2+O(\rho)\right)\end{aligned} \tag{28}$$

as $\rho\to 0$.

In the third and fourth terms of (20) we need only the leading terms, i.e. we can ignore the equivalent of $\gamma+2\log 2$ in (28). Using (26) the third term becomes

$$\frac{1}{2}C^3\int_{\infty}^{\rho}\frac{e^{-\tau}}{\tau}\left(e^{-\tau}-\tau E_1\right)^2 d\tau=\frac{1}{2}C^3\left(\log\rho+O(1)\right)\ \text{ as } \rho\to 0. \tag{29}$$

Finally, in the fourth term, the integrand in the τ-integral is just the second term, (as a function of σ), so that from (26), the fourth term is

$$M=-C^3\int_{\infty}^{\rho}\frac{e^{-\tau}}{\tau}\left[\int_{\infty}^{\tau}\left\{-e^{-\sigma}E_1(\sigma)+2E_1(2\sigma)\right\}d\sigma\right]d\tau. \tag{30}$$

It is seen from (26) that for any $\tau \leq \infty$, $\int_0^\tau E_1(\sigma)\, d\sigma$ converges. Hence we can write the inner integral above in the form $\int_\infty^0 - \int_0^\tau$, and it follows that

$$M = -C^3 \int_\infty^\rho \frac{e^{-\tau}}{\tau} \{K + r(\tau)\}\, d\tau$$

where K is a constant, r is bounded and $r(\tau) = O(\tau \log \tau)$ as $\tau \to 0$. It further follows that

$$M = C^3 (K E_1(\rho) + O(1)) \text{ as } \rho \to 0.$$

We can evaluate K using (26) and (23):

$$\int_0^\infty E_1(2\sigma)\, d\sigma = \frac{1}{2} \int_0^\infty E_1(u)\, dv = \frac{1}{2},$$

$$\begin{aligned}\int_0^\infty e^{-\sigma} E_1(\sigma)\, d\sigma &= \left[-\left(e^{-\sigma} - 1\right) E_1\right]_0^\infty - \int_0^\infty \left(e^{-\sigma} - 1\right) \frac{e^{-\sigma}}{\sigma} d\sigma \\ &= \lim_{\sigma \to 0} \{-E_1(2\sigma) + E_1(\sigma)\} = \lim_{\sigma \to 0} (\log 2\sigma - \log \sigma) = \log 2. \qquad (31)\end{aligned}$$

Hence, from (30), the fourth term of (20) is

$$C^3 \{E_1(\rho)(\log 2 - 1) + O(1)\} = -C^3 \{(\log 2 - 1) \log \rho + O(1)\} \quad \text{as } \rho \to 0. \qquad (32)$$

Now setting $\rho = \varepsilon$ and using (28), (29), and (32), we obtain that

$$\begin{aligned}-1 = &-C(-\log \varepsilon - \gamma + O(\varepsilon)) + C^2(-\log \varepsilon - \gamma - 2\log 2 + O(\varepsilon)) \\ &+ \frac{1}{2} C^3 (\log \varepsilon + O(1)) - C^3 \{(\log 2 - 1) \log \varepsilon + O(1)\}\end{aligned}$$

as $\varepsilon \to 0$. Hence,

$$\frac{1}{\log\left(\frac{1}{\varepsilon}\right)} = C\left(1 - \frac{\gamma}{\log\left(\frac{1}{\varepsilon}\right)}\right) - C^2\left(1 - \frac{\gamma + 2\log 2}{\log\left(\frac{1}{\varepsilon}\right)}\right) + C^3\left(\frac{3}{2} - \log 2\right) + O\left(\log^{-4}\left(\frac{1}{\varepsilon}\right)\right), \qquad (33)$$

and

$$C = \frac{1}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{A}{\log^2\left(\frac{1}{\varepsilon}\right)} + \frac{B}{\log^3\left(\frac{1}{\varepsilon}\right)} + O\left(\frac{1}{\log^4\left(\frac{1}{\varepsilon}\right)}\right),$$

where

$$\begin{aligned}-\gamma + A - 1 &= 0, \\ B - \gamma A - 2A + (\gamma + 2\log 2) + \frac{3}{2} - \log 2 &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}A &= \gamma + 1 \\ B &= \gamma^2 + 2\gamma + \frac{1}{2} - \log 2.\end{aligned}$$

Thus, for fixed r, ρ of order ε, we have, with $\lambda = \log\left(\frac{1}{\varepsilon}\right)$,

$$\begin{aligned}u - 1 = &\left(\frac{1}{\lambda} + \frac{\gamma+1}{\lambda^2} + \frac{(\gamma+1)^2 - \frac{1}{2} - \log 2}{\lambda^3}\right)(\log r + \log\varepsilon + \gamma) \\ &+ \frac{1}{\lambda^2}\left(1 + \frac{2(\gamma+1)}{\lambda}\right)(-\log r - \log\varepsilon - \gamma - 2\log 2) \\ &+ \frac{1}{\lambda^3}\left(\frac{3}{2} - \log 2\right)(\log r + \log\varepsilon) + O\left(\lambda^{-4}\right),\end{aligned}$$

so that, after cancellation,

$$u = \frac{\log r}{\lambda} + \frac{\gamma \log r}{\lambda^2} + O\left(\lambda^{-3}\right).$$

This is the "inner expansion". For the "outer expansion", i.e. fixed ρ, r of order $\frac{1}{\varepsilon}$, we have

$$u - 1 = -E_1(\rho)\left(\frac{1}{\lambda} + \frac{\gamma+1}{\lambda^2}\right) + \frac{1}{\lambda^2}\left(2E_1(2\rho) - e^{-\rho}E_1(\rho)\right) + O\left(\lambda^{-3}\right).$$

These results are in accordance with those of Hinch and of others on this problem.

5 The case $n = 3$

Here we are interested in (from (23) and (24))

$$E_2(\rho) = \frac{e^{-\rho}}{\rho} - E_1(\rho) = \frac{1}{\rho} + \log\rho + (\gamma - 1) - \frac{1}{2}\rho + O\left(\rho^2\log\rho\right) \text{ as } \rho \to 0.$$

Thus, the first term on the right of (20) evaluated at $\rho = \varepsilon$ is

$$-C\left(1 + \varepsilon\log\varepsilon + (\gamma - 1) + O\left(\varepsilon^2\right)\right) \text{ as } \varepsilon \to 0.$$

The second term is

$$
\begin{aligned}
&(C\varepsilon)^2 \int_\infty^\rho \frac{1}{\tau^2} e^{-\tau} \left(\int_\infty^\tau E_2 d\sigma \right) d\tau \\
&= (C\varepsilon)^2 \left\{ \left[-E_2(\tau) \int_\infty^\tau E_2(\sigma)\, d\sigma \right]_\infty^\rho + \int_\infty^\rho E_2^2 d\tau \right\} \\
&= (C\varepsilon)^2 \left\{ -E_2(\rho) \int_\infty^\rho E_2(\tau)\, d\tau + \int_\infty^\rho E_2^2 d\tau \right\}.
\end{aligned}
$$

From (24) we see that

$$
\int_\infty^\rho E_2^2 d\tau = -\frac{1}{\rho} + \log^2 \rho + O(\log \rho) \text{ as } \rho \to 0,
$$

while from (22),

$$
\begin{aligned}
\int_\infty^\rho E_2 d\tau &= \rho E_2 - E_1 = 1 + \log \rho + \gamma + O(\rho \log \rho), \\
E_2 \int_\infty^\rho E_2 d\tau &= \frac{1}{\rho} \log \rho + \frac{\gamma + 1}{\rho} + O\left(\log^2 \rho\right).
\end{aligned}
$$

In all, the second term is

$$
(C\varepsilon)^2 \left\{ -\frac{1}{\rho} \log \rho - \frac{\gamma + 2}{\rho} + O\left(\log^2 \rho\right) \right\}.
$$

It is readily verified that the third and fourth terms in (20)give $O\left\{C^3\varepsilon^3\left(\frac{1}{\rho}\log^2\rho\right)\right\}$, which is negligible. Thus, evaluating (20) at $\rho = \varepsilon$, we have

$$
-1 = -C\varepsilon \left(\frac{1}{\varepsilon} + \log \varepsilon + \gamma - 1 \right) + (C\varepsilon)^2 \left(-\frac{1}{\varepsilon} \log \varepsilon - \frac{\gamma + 2}{\varepsilon} \right) + O\left(C^3 \varepsilon^2 \log^2 \varepsilon\right),
$$

so that

$$
C = 1 - 2\varepsilon \log \varepsilon - \varepsilon(2\gamma + 1) + O\left(\varepsilon^2 \log^2 \varepsilon\right).
$$

Then, for fixed r, ρ of order ε, we have

$$
\begin{aligned}
u - 1 = &-\varepsilon\left(1 - 2\varepsilon \log \varepsilon - \varepsilon(2\gamma + 1)\right) \left(\frac{1}{\varepsilon r} + \log \varepsilon + \log r + \gamma - 1 \right) \\
&+ \varepsilon^2 \left(-\frac{1}{\varepsilon r} (\log \varepsilon + \log r) - \frac{\gamma + 2}{\varepsilon r} \right) + O\left(\varepsilon^2 \log^2 \varepsilon\right),
\end{aligned}
$$

$$u = 1 - \frac{1}{r} - \varepsilon \log \varepsilon \left(1 - \frac{1}{r}\right) - \varepsilon \left(\log r + \frac{\log r}{r}\right) + \varepsilon (1-\gamma) \left(1 - \frac{1}{r}\right) + O\left(\varepsilon^2 \log^2 \varepsilon\right).$$

For fixed ρ, r of order ε^{-1}, we have

$$u - 1 = -\varepsilon \left(1 - 2\varepsilon \log \varepsilon - \varepsilon (2\gamma + 1)\right) E_2(\rho) + \varepsilon^2 \left\{ E_1(\rho) E_2(\rho) - \rho E_2^2(\rho) - \int_\rho^\infty E_2^2 d\tau \right\} + O\left(\varepsilon^3\right). \tag{34}$$

Again, these results are in agreement with those of Hinch, and others, although (34) gives one term further.

Remark 3. *It is of interest to consider what happens when $n < 2$, since, at least for $n \geq 1$, there still exists a unique solution. The equation* (20) *is still valid at $\rho = \varepsilon$, but since $E_{n-1}(\rho)$ is no longer singular at $\rho = 0$ for $n < 2$,* (20) *with $\rho = \varepsilon$ becomes merely an implicit equation for $C\varepsilon^{n-2}$. This tells us that $C \to 0$, since $\varepsilon^{n-2} \to \infty$, but we no longer get an asymptotic expansion. In particular, it is no longer obvious that C is unique. Of course, we know this from Theorem 1 if $n \geq 1$. For $n < 1$, this uniqueness may fail.*

6 Hinch's terrible problem

In [2], Hinch introduces a further extension of Lagerstrom's problem. This is

$$u'' + \frac{n-1}{r} u' + u'^2 + \varepsilon u \, u' = 0, \tag{35}$$

with

$$u(1) = 0, u(\infty) = 1.$$

Existence for any $n \geq 1$ and any $\varepsilon > 0$ can be proved as before. We focus here on the asymptotics. In Hinch's work it is seen that the method of matched asymptotic expansions is more complicated in this case than in the standard Lagerstrom model.

We can in fact treat a generalization which causes no further difficulties,

$$u'' + \frac{n-1}{r} u' + f(u) u'^2 + \varepsilon u \, u' = 0, \tag{36}$$

with the same boundary conditions.

As remarked in the introduction to the paper, the solution will necessarily have $u' > 0$ so that conditions on $f(u)$ are necessary only for $0 \leq u \leq 1$. We require only that f be continuous and positive in this interval.

Then (36) can be written as

$$\frac{\left(r^{n-1}u'\right)'}{r^{n-1}u'} + f(u)\,u' + \varepsilon u = 0,$$

so that

$$\log\left(r^{n-1}u'\right) = -F(u) - \varepsilon\int_1^r u\,dt + A$$

for some constant A, where

$$F(u) = \int_0^u f(s)\,ds.$$

This becomes

$$e^{F(u)}u' = \frac{C}{r^{n-1}}e^{-\varepsilon r - \varepsilon\int_\infty^r (u-1)ds},$$

or, on integration,

$$G(u) - G(1) = C\int_\infty^r \frac{1}{t^{n-1}}e^{-\varepsilon t - \varepsilon\int_\infty^t (u-1)ds}dt,$$

where

$$G(u) = \int_0^u e^{F(u)}dv.$$

In order to keep the manipulations simple and effect comparisons, we will consider the case considered by Hinch, where $f(u) = 1$, $F(u) = u$, $G(u) = e^u - 1$. Then, with $\varepsilon r = \rho$, $\varepsilon t = \tau$, we have

$$e^u - e = C\varepsilon^{n-2}\int_\infty^\rho \frac{1}{\tau^{n-1}}e^{-\tau}e^{-\int_\infty^\tau (u-1)d\sigma}d\tau, \tag{37}$$

and writing

$$u - 1 = \frac{u-1}{e^u - e}\left(e^u - e\right),$$

we get

$$e^u - e = C\varepsilon^{n-2}\int_\infty^\rho \frac{1}{\tau^{n-1}}e^{-\tau}e^{-\int_\infty^\tau \frac{u-1}{e^u-e}(e^u-e)d\sigma}d\tau. \tag{38}$$

As in section 3, we can integrate by parts, and since $0 \le \frac{u-1}{e^u-e} \le 1$ in $0 \le u < 1$, we will develop a convergent series as before. To get the first three terms (necessary to give Hinch's accuracy when $n = 2$), we have from (38) that

$$e^u - e$$
$$= C\varepsilon^{n-2} \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \left\{ 1 - \int_\infty^\tau \frac{u-1}{e^u-e} \left(e^u - e\right) d\sigma + \frac{1}{2} \left(\int_\infty^\tau \frac{u-1}{e^u-e} \left(e^u - e\right) d\sigma \right)^2 + \cdots \right\} d\tau. \tag{39}$$

As before, since $e^u - e \to 0$ exponentially fast as $\rho \to \infty$, the series in the integrand converges uniformly for large τ, so that (39) is valid for large ρ. But again we need to extend it down to $\rho = \varepsilon$. From (37)we have

$$e^u - e \le C\varepsilon^{n-2} E_{n-1}(\rho)$$

and so the convergence proof is the same as that preceding (20) .

Before proceeding further with $n = 2$, we make a couple of remarks about the simpler case $n > 2$. Then, as we saw in subsection 5, only two terms are necessary to give the required accuracy, and then (39) gives

$$e^u - u = C\varepsilon^{n-2} \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-1}} \left\{ 1 - \int_\infty^\tau \frac{u-1}{e^u-e} \left(e^u - e\right) d\sigma + \cdots \right\}$$

and since $\frac{u-1}{e^u-e}$ appears in what is already the highest order term, we can replace it by its limit as $u \to 1$, i.e. $\frac{1}{e}$. Thus we get, to the required order,

$$e^u - e = -C\varepsilon^{n-2} E_{n-1} - \left(\frac{C\varepsilon^{n-2}}{e} \right) \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-1}} \int_\infty^\tau \left(e^u - e\right) d\sigma d\tau.$$

This, apart from the factor $\frac{1}{e}$, is the same equation as we dealt with in section 5 (with $e^u - e$ in place of $u - 1$), and the solution can be written down from there. (If we had a general function f in place of 1, we would get

$$e^{F(u)} - e^{F(1)} = -C\varepsilon^{n-2} E_{n-1} - \frac{C\varepsilon^{n-2}}{e^{F(1)} f(1)} \int_\infty^\rho \frac{e^{-\tau}}{\tau^{n-1}} \left(\int_\infty^\tau \left(e^{F(u)} - e^{F(1)}\right) d\sigma \right) d\tau.)$$

Turning now to the case $n = 2$, and $F(u) = u$, we need three terms on the right of (39) . Thus,

$$\frac{u-1}{e^u-e} = \frac{1}{e} - \frac{1}{2e^2} \left(e^u - e\right) + O\left(e^u - e\right)^2 \text{ as } u \to 1. \tag{40}$$

We follow the method used just before (20) and obtain from (39) that

$$
\begin{aligned}
e^u - e &= -C\varepsilon^{n-2}E_{n-1} + \frac{1}{e}\left(C\varepsilon^{n-2}\right)^2 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau}\left(\int_\infty^\tau E_{n-1} d\sigma\right) d\tau \\
&+ \frac{1}{2e^2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau}\left(\int_\infty^\tau E_{n-1}^2 d\sigma\right) d\tau + \frac{1}{2e^2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau}\left(\int_\infty^\tau E_{n-1} d\sigma\right)^2 d\tau \\
&- \frac{1}{e^2}\left(C\varepsilon^{n-2}\right)^3 \int_\infty^\rho \frac{1}{\tau^{n-1}} e^{-\tau} \int_\infty^\tau \left\{\int_\infty^\sigma \frac{1}{s^{n-1}} e^{-s}\left(\int_\infty^s E_{n-1} dt\right) ds\right\} d\sigma d\tau + O\left(\Phi^4\right) \\
&= -C\varepsilon^{n-2}E_{n-1} + F_1 + F_2 + F_3 + F_4 + O\left(\Phi^4\right), \quad \text{say.}
\end{aligned}
$$

As before, if $n > 2$ then this is valid for any C as $\varepsilon \to 0$, uniformly in $\rho \geq \varepsilon$, while if $n = 2$, it is valid as $C \to 0$.

For $n = 2$ we can continue to follow the argument in section 4. Thus, as $\rho \to 0$,

$$
\begin{aligned}
F_1 &= \frac{1}{e}C^2\left(-\log\rho - \gamma - 2\log 2 + O(\rho)\right) \\
F_3 &= \frac{1}{2e^2}C^3\left(\log\rho + O(1)\right) \\
F_4 &= -\frac{1}{e^2}C^3\left[(\log 2 - 1)\log\rho + O(1)\right].
\end{aligned}
$$

The term F_2 did not appear before. Only the highest order term is needed for our expansion and this is

$$
-\frac{1}{2e^2}C^3\left(\int_\infty^\rho \frac{1}{\tau} e^{-\tau} d\tau\right)\int_0^\infty E_1^2 d\sigma.
$$

Now

$$
\begin{aligned}
\int_0^\infty E_1^2 d\sigma &= \left[\tau E_1^2\right]_0^\infty + 2\int_0^\infty \tau \frac{e^{-\tau}}{\tau} E_1 d\tau \\
&= 2\int_0^\infty e^{-\tau} E_1 d\tau = 2\log 2, \text{ from } (31).
\end{aligned}
$$

Thus,

$$
F_2 = \frac{1}{e^2}C^3 E_1\left(\log 2 + O\left(\rho\log^2\rho\right)\right) = -\frac{1}{e^2}C^3\left((\log 2)\log\rho + O(1)\right),
$$

and, evaluating at $\rho = \varepsilon$, we have

$$
\begin{aligned}
1 - e &= C\left(\log\varepsilon + \gamma + O\left(\varepsilon\right)\right) \\
&\quad - \frac{1}{e}C^2\left(\log\varepsilon + \gamma + 2\log 2 + O\left(\varepsilon\right)\right) + \frac{1}{2e^2}C^3\log\varepsilon\left(-2\log 2 + 1 - 2\log 2 + 2\right) + O\left(C^3\right), \\
\frac{e-1}{\log\frac{1}{\varepsilon}} &= C\left(1 - \frac{\gamma}{\log\left(\frac{1}{\varepsilon}\right)}\right) - \frac{1}{e}C^2\left(1 - \frac{\gamma + 2\log 2}{\log\left(\frac{1}{\varepsilon}\right)}\right) \\
&\quad + \frac{1}{2e^2}C^3\left(3 - 4\log 2 + O\left(\frac{C\varepsilon}{\log\varepsilon}\right) + O\left(\frac{C^2\varepsilon}{\log\varepsilon}\right) + O\left(\frac{C^3}{\log\varepsilon}\right)\right).
\end{aligned}
$$

Hence if

$$
C = \frac{e-1}{\log\left(\frac{1}{\varepsilon}\right)} + \frac{A}{\log^2\left(\frac{1}{\varepsilon}\right)} + \frac{B}{\log^3\left(\frac{1}{\varepsilon}\right)} + O\left(\log^{-4}\left(\frac{1}{\varepsilon}\right)\right),
$$

then

$$
-\gamma\left(e-1\right) + A - \frac{\left(e-1\right)^2}{e} = 0,
$$

$$
A = \frac{e-1}{e}\left(\gamma e + e - 1\right),
$$

$$
B - A\gamma + \frac{\left(e-1\right)^2}{e}\left(\gamma + 2\log 2\right) - \frac{2A\left(e-1\right)}{e} + \frac{1}{2e^2}\left(e-1\right)^3\left(3 - 4\log 2\right) = 0.
$$

We can of course calculate B, but in fact its value will be be irrelevant to the level of approximation that we take.

Then, for fixed r (ρ of order ε), we have, with $l = \log\left(\frac{1}{\varepsilon}\right)$,

$$
\begin{aligned}
e^u - e &= \left(e-1\right)\left\{\frac{1}{l} + \frac{\gamma + 1 - \frac{1}{e}}{l^2} + \frac{B/\left(e-1\right)}{l^3}\right\}\left(\log\varepsilon + \log r + \gamma\right) \\
&\quad + \frac{1}{e}\left(e-1\right)^2\left\{\frac{1}{l^2} + \frac{2\left(\gamma + 1 - \frac{1}{e}\right)}{l^3}\right\}\left(-\log\varepsilon - \log r - \gamma - 2\log 2\right) \\
&\quad + \frac{1}{2e^2}\frac{\left(e-1\right)^3}{l^3}\left(3 - 4\log 2\right)\left(\log\varepsilon + \log r\right) + O\left(l^{-3}\right) \\
&= 1 - e + \frac{\left(e-1\right)\log r}{l} + \frac{\gamma\left(e-1\right)\log r}{l^2} + O\left(l^{-3}\right). \qquad (41)
\end{aligned}
$$

(Note that the definitions of A and B were such that $u = 0$ at $r = 1$ up to and including order l^{-2}, so that to that order there can be only terms in $\log r$, not constant terms. We do not need the explicit value of B.) To obtain u, we have to invert, so that

$$u = \log\left\{1 + \frac{e-1}{l}\log r + \frac{\gamma\left(e-1\right)}{l^2}\log r + O\left(l^{-3}\right)\right\}.$$

For fixed ρ, r of order ε^{-1}, we have

$$e^u - e = -\frac{e-1}{l}\left(1 + \frac{\gamma + 1 - \frac{1}{e}}{l}\right)E_1\left(\rho\right) + \frac{\left(e-1\right)^2}{el^2}\left(2E_1\left(2\rho\right) - e^{-\rho}E_1\left(\rho\right)\right) + O\left(l^{-3}\right).$$

Thus

$$\begin{aligned} u - 1 &= \frac{1}{e}\left(e^u - e\right) - \frac{1}{2e^2}\left(e^u - e\right)^2 + \cdots \\ &= -\frac{e-1}{e}\left(1 + \frac{\gamma + 1 - \frac{1}{e}}{l}\right)\frac{E_1\left(\rho\right)}{l} + \frac{\left(e-1\right)^2}{e^2}\frac{\left(2E_1\left(2\rho\right) - e^{-\rho}E_1\left(\rho\right)\right)}{l^2} \\ &\quad - \frac{\left(e-1\right)^2}{2e^2}\frac{E_1^2\left(\rho\right)}{l^2} + O\left(l^{-3}\right). \end{aligned} \tag{42}$$

Again, these results are consistent with those of Hinch, except that Hinch has an algebraic mistake which in (42) replaces $\gamma + 1 - \frac{1}{e}$by $\gamma - 1 + \frac{1}{e}$.

7 Remark on "switchback"

Starting with Lagerstrom, the terms involving $\log\varepsilon$ in the inner expansions have been considered strange, and difficult to explain. They are often called "switchback" terms, because, starting with an expansion in powers of ε, one finds inconsistent results which are only resolved by adding terms of lower order, that is, powers of $\varepsilon\log\varepsilon$. The recent approach to the problem by geometric perturbation theory explains this by reference to a "resonance phenomenon", which is too complicated for us to describe here [4],[5].

In our work, the necessity for such terms is seen already from the equation (13) and the resulting expansion (18) :

$$u\left(\rho\right) - 1 = C\varepsilon^{n-2}\int_\infty^\rho \frac{1}{\tau^{n-1}}e^{-\tau}\left\{1 - \int_\infty^\tau\left(u-1\right)d\sigma + \frac{1}{2}\left(\int_\infty^\tau\left(u-1\right)d\sigma\right)^2 - \cdots\right\}d\tau.$$

In the existence proof it was seen in (9) that $C = O(1)$ as $\varepsilon \to 0$. On the right of (18) the first term is simply $C\varepsilon^{n-2}E_{n-1}(\rho)$, and the simple expansions given for E_1 and E_2 show immediately the need for the logarithmic terms. There is no "switchback", because the procedure does not start with any assumption about the nature of the expansion.

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