1. (a) (5 points) Find the unit tangent and unit normal vectors $\mathbf{T}$ and $\mathbf{N}$ to the curve

$$\mathbf{r}(t) = \langle 3\cos t,\, 4t,\, 3\sin t\rangle$$

at the point $P = \left(-\frac{3}{\sqrt{2}},\, 3\pi,\, \frac{3}{\sqrt{2}}\right)$.

(b) (5 points) Find curvature of the curve at the point P.

Solution: (a) $\mathbf{r}'(t) = \langle -3\sin t,\, 4,\, 3\cos t\rangle$, $|\mathbf{r}'(t)| = \sqrt{9+16} = 5$ **(+1 point)**

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle -\frac{3}{5}\sin t,\, \frac{4}{5},\, \frac{3}{5}\cos t\right\rangle$ **(+1 point)**

At the point P: $t = \frac{3\pi}{4}$, $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. **(+1 point)**

Then $\mathbf{T}\left(\frac{3\pi}{4}\right) = \left\langle -\frac{3}{5\sqrt{2}},\, \frac{4}{5},\, -\frac{3}{5\sqrt{2}}\right\rangle = \left\langle -\frac{3\sqrt{2}}{10},\, \frac{4}{5},\, -\frac{3\sqrt{2}}{10}\right\rangle$ **(+1 point)**

$\mathbf{T}'(t) = \left\langle -\frac{3}{5}\cos t,\, 0,\, -\frac{3}{5}\sin t\right\rangle$, $\mathbf{T}'(t) = \frac{3}{5}$.

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos t,\, 0,\, -\sin t\rangle$, $\mathbf{N}\left(\frac{3\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2},\, 0,\, -\frac{\sqrt{2}}{2}\right\rangle = \left\langle \frac{1}{\sqrt{2}},\, 0,\, -\frac{1}{\sqrt{2}}\right\rangle$.
(+2 points)

(b) $\mathbf{r}'\left(\frac{3\pi}{4}\right) = \left\langle -\frac{3}{\sqrt{2}},\, 4,\, -\frac{3}{\sqrt{2}}\right\rangle$,

$\mathbf{r}''(t) = \langle -3\cos t,\, 0,\, -3\sin t\rangle$, $\mathbf{r}''\left(\frac{3\pi}{4}\right) = \left\langle \frac{3}{\sqrt{2}},\, 0,\, -\frac{3}{\sqrt{2}}\right\rangle$,

$$\mathbf{r}'\left(\frac{3\pi}{4}\right) \times \mathbf{r}''\left(\frac{3\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{\sqrt{2}} & 4 & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & 0 & -\frac{3}{\sqrt{2}} \end{vmatrix} = \left\langle -6\sqrt{2},\, -9,\, -6\sqrt{2}\right\rangle, \quad \textbf{(+2 points)}$$

$\left|\mathbf{r}'\left(\frac{3\pi}{4}\right) \times \mathbf{r}''\left(\frac{3\pi}{4}\right)\right| = 15$, $\left|\mathbf{r}'\left(\frac{3\pi}{4}\right)\right|^3 = 125$,

$\kappa = \frac{15}{125} = \frac{3}{25}$. **(+2 points)**

2. (10 points) Use LINEAR approximation to approximate the number $\sqrt{3.04 + e^{-0.08}}$.

Solution: Denote $f(x, y) = \sqrt{x + e^y}$.

Then $\dfrac{\partial f}{\partial x}(x, y) = \dfrac{1}{2\sqrt{x + e^y}}$, $\dfrac{\partial f}{\partial y}(x, y) = \dfrac{e^y}{2\sqrt{x + e^y}}$. **(+2 points)**

We are looking for a linear approximation near the point $(x_0, y_0) = (3, 0)$. **(+1 point)**

$f(3, 0) = 2$, $\dfrac{\partial f}{\partial x}(3, 0) = \dfrac{1}{4}$, $\dfrac{\partial f}{\partial y}(3, 0) = \dfrac{1}{4}$. **(+1 point)**

Let $L(x, y)$ be the linearization of $f(x, y)$ near $(3, 0)$. Then

$$f(x, y) \approx L(x, y) = 2 + \frac{1}{4}(x - 3) + \frac{1}{4}y. \quad \textbf{(+3 points)}$$

Therefore,

$$\begin{aligned}\sqrt{3.04 + e^{-0.08}} &= f(3.04, -0.08) \approx L(3.04, -0.08) \\ &= 2 + \frac{1}{4}(3.04 - 3) + \frac{1}{4}(-0.08) = 2 + 0.01 - 0.02 \\ &= 1.99\end{aligned}$$

or

$$\sqrt{3.04 + e^{-0.08}} \approx 1.99 \quad \textbf{(+3 points)}$$

3. (10 points) Find all critical points of the function $f(x, y) = 4x - 3x^3 - 2xy^2$. For each critical point determine if it is a local maximum, local minimum or a saddle point.

Solution: $f_x(x, y) = 4 - 9x^2 - 2y^2$, $f_y(x, y) = -4xy$. **(+1 point)**
The equation $f_y(x, y) = 0$ gives two solutions $x = 0$ or $y = 0$. **(+1 point)**

Case $x = 0$: $f_x(0, y) = 4 - 2y^2 = 0$ gives solutions $y = -\sqrt{2}$ or $y = \sqrt{2}$ and points $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. **(+1 point)**

Case $y = 0$: $f_x(x, 0) = 4 - 9x^2 = 0$ gives solutions $x = -\frac{2}{3}$ or $x = \frac{2}{3}$ and points $\left(-\frac{2}{3}, 0\right)$ and $\left(\frac{2}{3}, 0\right)$. **(+1 point)**

Critical points are $(0, -\sqrt{2})$, $(0, \sqrt{2})$, $\left(-\frac{2}{3}, 0\right)$, and $\left(\frac{2}{3}, 0\right)$. **(+1 point)**

$f_{xx}(x, y) = -18x$, $f_{xy}(x, y) = -4y$, $f_{yy}(x, y) = -4x$. **(+1 point)**
$D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - f_{xy}^2(x, y) = 72x^2 - 16y^2 = 8(9x^2 - 2y^2)$. **(+1 point)**

$D(0, -\sqrt{2}) = D(0, \sqrt{2}) = -32 < 0$.
Therefore, $(0, -\sqrt{2})$ and $(0, \sqrt{2})$ are saddle points. **(+1 point)**

$D\left(-\frac{2}{3}, 0\right) = 32 > 0$, $f_{xx}\left(-\frac{2}{3}, 0\right) = 12 > 0$.

Therefore, $\left(-\frac{2}{3}, 0\right)$ is a point of a local minimum. **(+1 point)**

$D\left(\frac{2}{3}, 0\right) = 32 > 0$, $f_{xx}\left(\frac{2}{3}, 0\right) = -12 < 0$.

Therefore, $\left(-\frac{2}{3}, 0\right)$ is a point of a local maximum. **(+1 point)**

4. (10 points) Find the volume of the solid E bounded by $y = x^2$, $x = y^2$, $z = x + y + 5$, and $z = 0$.

Solution: $E = \{(x, y, z) | \ 0 \le x \le 1, \ x^2 \le y \le \sqrt{x}, \ 0 \le z \le x + y + 5\}$. **(+2 points)**

The volume V of the solid E is

$$V = \iiint_E dV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y+5} dz\,dy\,dx \quad \textbf{(+2 points)}$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x + y + 5)\,dy\,dx$$

$$= \int_0^1 \left[(x+5)y + \frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{3/2} + 5x^{1/2} + \frac{1}{2}x - x^3 - 5x^2 - \frac{1}{2}x^4\right) dx \quad \textbf{(+3 points)}$$

$$= \left[\frac{2}{5}x^{5/2} + \frac{10}{3}x^{3/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{5}{3}x^3 - \frac{1}{10}x^5\right]_0^1$$

$$= \frac{2}{5} + \frac{10}{3} + \frac{1}{4} - \frac{1}{4} - \frac{5}{3} - \frac{1}{10}$$

$$= \frac{59}{30}. \quad \textbf{(+3 points)}$$

5. (10 points) Find the y coordinate of the center of mass of a lamina that occupies the region bounded by $y^2 = x + 4$, $x = 0$, and $y \geq 0$ and has density $\rho(x, y) = y$. Simplify your answer as much as possible.

<u>Solution:</u> The region is $R = \{(x, y)|\ 0 \leq y \leq 2,\ y^2 - 4 \leq x \leq 0\}$. **(+2 points)**

The mass of the lamina is

$$m = \iint\limits_R \rho(x, y)\, dA \quad \textbf{(+1 point)}$$

$$= \int_0^2 \int_{y^2-4}^{0} y\, dx\, dy = \int_0^2 (-y^3 + 4y)\, dy$$

$$= \left[-\frac{1}{4}y^4 + 2y^2\right]_0^2$$

$$= -4 + 8 = 4. \quad \textbf{(+2 points)}$$

The y coordinate of the center of mass of the lamina is

$$\bar{y} = \frac{1}{m} \iint\limits_R y\, \rho(x, y)\, dA \quad \textbf{(+1 point)}$$

$$= \frac{1}{4} \int_0^2 \int_{y^2-4}^{0} y^2\, dx\, dy = \frac{1}{4} \int_0^2 (-y^4 + 4y^2)\, dy \quad \textbf{(+2 points)}$$

$$= \frac{1}{4} \left[-\frac{1}{5}y^5 + \frac{4}{3}y^3\right]_0^2$$

$$= \frac{1}{4} \left[-\frac{32}{5} + \frac{32}{3}\right]$$

$$= \frac{16}{15}. \quad \textbf{(+2 points)}$$

6. (10 points) Evaluate the integral

$$\iint_R e^{x-2y}\,dA$$

where R is the parallelogram $ABCD$ with vertices $A=(0,0)$, $B=(4,1)$, $C=(7,4)$, and $D=(3,3)$ using the transformation $x=4u+3v$ and $y=u+3v$. Simplify your answer as much as possible.

Solution: $T:\ \ x=4u+3v, \quad y=u+3v.$

The inverse transformation is $T^{-1}:\ \ u=\frac{1}{3}(x-y), \quad v=\frac{1}{9}(-x+4y).$ **(+2 points)**

In the uv-plane the region that corresponds to the parallelogram $ABCD$ can be found if we apply T^{-1} to its vertices:

$A_1=T^{-1}(A)=(0,0)$, $B_1=T^{-1}(B)=(1,0)$, $C_1=T^{-1}(C)=(1,1)$, $D_1=T^{-1}(D)=(0,1)$, which is the square $R_1=A_1B_1C_1D_1=\{(u,v)|\,0\le u\le 1,\ 0\le v\le 1\}$. **(+2 pts)**

The Jacobian of the transformation is

$$J=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}=\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}=12-3=9. \quad \textbf{(+2 points)}$$

$$\iint_R e^{x-2y}\,dA=\iint_{R_1} e^{2u-3v}\cdot 9\cdot dA \quad \textbf{(+1 point)}$$

$$=9\int_0^1\int_0^1 e^{2u-3v}\,du\,dv=\frac{9}{2}\int_0^1 \left[e^{2u-3v}\right]_0^1\,dv$$

$$=\frac{9}{2}\int_0^1 \left(e^2-1\right)\,e^{-3v}\,dv=-\frac{3}{2}\left[\left(e^2-1\right)e^{-3v}\right]_0^1$$

$$=-\frac{3}{2}\left(\left(e^2-1\right)\left(e^{-3}-1\right)\right.$$

$$=\frac{3}{2}\left(e^2-1-e^{-1}+e^{-3}\right). \quad \textbf{(+3 points)}$$

7. (10 points) Evaluate the line integral

$$\oint_C e^{2x+y}\,dx + e^{-y}\,dy$$

along the **negatively** oriented closed curve C, where C is the boundary of the triangle with the vertices $(0,0)$, $(0,1)$, and $(1,0)$.

<u>Solution:</u> $P(x,y) = e^{2x+y}$, $Q(x,y) = e^{-y}$. Hence $\dfrac{\partial P}{\partial y}(x,y) = e^{2x+y}$, $\dfrac{\partial Q}{\partial x}(x,y) = 0$. **(+2 points)**

The triangle D bounded by C is $D = \{(x,y)|\, 0 \le x \le 1,\ 0 \le y \le -x+1\}$. **(+2 points)**

Using Green's Theorem **(+1 point)** and negative orientation of C we get

$$\oint_C e^{2x+y}\,dx + e^{-y}\,dy = -\iint_D \left(0 - e^{2x+y}\right)\,dA \quad \textbf{(+2 points)}$$

$$= \int_0^1 \int_0^{-x+1} e^{2x+y}\,dy\,dx = \int_0^1 \left[e^{2x+y}\right]_0^{-x+1}\,dx$$

$$= \int_0^1 \left(e^{x+1} - e^{2x}\right)\,dx = \left[e^{x+1} - \frac{1}{2}e^{2x}\right]_0^1$$

$$= e^2 - \frac{1}{2}e^2 - e + \frac{1}{2}$$

$$= \frac{1}{2}e^2 - e + \frac{1}{2}. \quad \textbf{(+3 points)}$$

8. (10 points) Evaluate the integral

$$\iint_S (10 - 2z)\, dS,$$

where S is the part of the surface $z = 5 - \frac{x^2}{2} - \frac{y^2}{2}$ inside the cylinder $x^2 + y^2 = 1$.

<u>Solution:</u> $z = g(x,y) = 5 - \frac{x^2}{2} - \frac{y^2}{2}$, $\frac{\partial g}{\partial x}(x,y) = -x$, $\frac{\partial g}{\partial y}(x,y) = -y$, **(+1 point)**

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA = \sqrt{1 + x^2 + y^2}\, dA.$$ **(+2 points)**

The domain in the xy-plane is the disk $D = \{(x,y) |\, x^2 + y^2 \leq 1\}$. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and $D = \{(r,\theta) |\, 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$. **(+1 point)** Hence

$$\begin{aligned}\iint_S (10 - 2z)\, dS &= \iint_D \left(10 - (10 - x^2 - y^2)\right) \sqrt{1 + x^2 + y^2}\, dA \\ &= \iint_D \left(x^2 + y^2\right) \sqrt{1 + x^2 + y^2}\, dA \\ &= \int_0^{2\pi} \int_0^1 r^2 \sqrt{1 + r^2} \cdot r \cdot dr\, d\theta = 2\pi \int_0^1 r^2 \sqrt{1 + r^2} \cdot r \cdot dr\end{aligned}$$ **(+3 points)**

Substitution: $u = 1 + r^2$ gives $r^2 = u - 1$, $du = 2r\, dr$, $r\, dr = \frac{1}{2}\, du$. Then

$$2\pi \int_0^1 r^2 \sqrt{1 + r^2} \cdot r \cdot dr = 2\pi \int_1^2 (u - 1) u^{1/2}\, \frac{1}{2}\, du$$ **(+1 point)**

$$= \pi \int_1^2 (u^{3/2} - u^{1/2})\, du = \pi \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2}\right]_1^2$$

$$= \pi \left(\frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3}\right) = \frac{4}{15}\left(\sqrt{2} + 1\right)\pi.$$

Hence

$$\iint_S (10 - 2z)\, dS = \frac{4}{15}\left(\sqrt{2} + 1\right)\pi.$$ **(+2 points)**

9. (10 points) Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

for the vector field $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} - z\,\mathbf{k}$, where the closed curve C is the boundary of the triangle with vertices $(0, 0, 5)$, $(2, 0, 1)$, and $(0, 3, 2)$ traced in this order.

<u>Solution:</u> Use Stokes' Theorem $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$. **(+1 point)**

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -z \end{vmatrix} = 2\mathbf{k} = \langle\, 0,\ 0,\ 2\rangle. \quad \textbf{(+2 points)}$$

S is the triangle with vertices $P(0, 0, 5)$, $Q(2, 0, 1)$, $R(0, 3, 2)$. To find the equation of S we consider vectors $\overrightarrow{PQ} = \langle 2,\ 0,\ -4\rangle$ and $\overrightarrow{PR} = \langle 0,\ 3,\ -3\rangle$. A normal vector $\mathbf{n}$ to the surface S is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -4 \\ 0 & 3 & -3 \end{vmatrix} = \langle\, 12,\ 6,\ 6\rangle. \quad \textbf{(+1 point)}$$

S lies in the plane with the equation $12x + 6y + 6(z - 5) = 0$ or $z = 5 - 2x - y$. **(+1 pt)** Using x and y as parameters we define S by $\mathbf{r}(x, y) = \langle\, x,\ y,\ 5 - 2x - y\rangle$, **(+1 pt)** where $(x, y) \in D$ and D is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$ in the xy-plane. $\mathbf{r}_x(x, y) = \langle\, 1,\ 0,\ -2\rangle$, $\mathbf{r}_y(x, y) = \langle\, 0,\ 1,\ -1\rangle$.

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \langle\, 2,\ 1,\ 1\rangle. \quad \textbf{(+2 points)}$$

Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \langle\, 0,\ 0,\ 2\rangle \cdot \langle\, 2,\ 1,\ 1\rangle\, dA = 2 \iint_D dA = 2\,A(D) = 2 \cdot 3 = 6,$$

where $A(D)$ is the area of the triangle D.

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 6. \quad \textbf{(+2 points)}$$

10. (10 points) Evaluate the flux of $\mathbf{F}(x, y, z) = z^2 y\,\mathbf{i} + x^2 y\,\mathbf{j} + (x+y)\,\mathbf{k}$ over S, where S is the closed surface consisting of the coordinate planes and the part of the sphere $x^2+y^2+z^2 = 4$ in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$, with the normal pointing outward.

Solution: By the Divergence Theorem the flux is

$$\iint_S \mathbf{F}\, d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV, \quad \textbf{(+1 point)}$$

where $\operatorname{div} \mathbf{F} = x^2$ **(+1 point)** and E is the region bounded by S. In the spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$ the region is

$E = \{(\rho, \theta, \phi) |\, 0 \leq \rho \leq 2,\ 0 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi/2\}$. **(+2 points)** Then

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \rho^2 \sin^2\phi \cos^2\theta \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^3\phi\, d\phi \int_0^{2} \rho^4\, d\rho. \quad \textbf{(+1 point)}$$

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{\cos 2\theta + 1}{2}\, d\theta = \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi/2} = \frac{\pi}{4}, \quad \textbf{(+1 point)}$$

$$\int_0^{\pi/2} \sin^3\phi\, d\phi = \int_0^{\pi/2} \left(1 - \cos^2\phi\right) \sin\phi\, d\phi = [u = \cos\phi] = \int_0^{1} \left(1 - u^2\right)\, du = \frac{2}{3}, \quad \textbf{(+2 points)}$$

$$\int_0^{2} \rho^4\, d\rho = \frac{32}{5}. \quad \textbf{(+1 point)}$$

Hence

$$\int_0^{\pi/2} \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^4\phi\, d\phi \int_0^{2} \rho^3\, d\rho = \frac{\pi}{4} \cdot \frac{2}{3} \cdot \frac{32}{5} = \frac{16}{15}\,\pi. \quad \textbf{(+1 point)}$$

Therefore, the flux is $\dfrac{16}{15}\,\pi$.