PARTITIONED CONSERVATIVE, VARIABLE STEP, SECOND-ORDER METHOD FOR MAGNETO-HYDRODYNAMICS IN ELSÄSSER VARIABLES

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Abstract. Magnetohydrodynamics (MHD) describes the interaction between electrically conducting fluids and electromagnetic fields. We propose and analyze a symplectic, second-order algorithm for the evolutionary MHD system in Elsässer variables. We reduce the computational cost of the iterative non-linear solver, at each time step, by partitioning the coupled system into two subproblems of half size, solved in parallel. We prove that the iterations converge linearly, under a time step restriction similar to the one required in the full space-time error analysis. The variable step algorithm unconditionally conserves the energy, cross-helicity and magnetic helicity, and numerical solutions are second-order accurate in the L^2 and H^1-norms. The time adaptive mechanism, based on a local truncation error criterion, helps the variable step algorithm balance accuracy and time efficiency. Several numerical tests support the theoretical findings and verify the advantage of time adaptivity.

Key words. Magnetohydrodynamics, Elsässer variables, partitioned algorithms, iterative methods, second-order accurate, variable steps, time adaptivity

AMS subject classifications. 76W05, 65M12, 65M60, 35Q30, 35Q61

1. Introduction. The equations of magnetohydrodynamics [1] (MHD) describe the motion of electrically conducting, incompressible flows in the presence of magnetic fields. The understanding of the MHD system is essential to numerous applications in science and engineering, astrophysics [4,16,35], geophysics [14,36], sea water propulsion [31], cooling system design [3,23] and process metallurgy [12]. Given a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ and time interval $[0,T]$, the fluid velocity field u, magnetic field B and the pressure p satisfy

$$\partial_t u + (u\cdot\nabla)u - (B\cdot\nabla)B - \nu\Delta u + \nabla p = f, \qquad \nabla\cdot u = 0, \tag{1.1}$$

$$\partial_t B + (u\cdot\nabla)B - (B\cdot\nabla)u - \nu_m\Delta B = 0, \qquad \nabla\cdot B = 0, \tag{1.2}$$

with Dirichlet boundary conditions: $u|_{\partial\Omega} = 0$, $B|_{\partial\Omega} = B_\circ$. Here ν is the kinematic viscosity, ν_m is the magnetic resistivity, $B_\circ$ is a constant external magnetic field, and f is an external force. The MHD flows entail two distinct physical processes: the motion of fluid - governed by the hydrodynamics equations (1.1), and its interaction with the magnetic fields - governed by the Maxwell equations (1.2). Splitting the magnetic field B into its mean and fluctuation $B = B_\circ + b$, the Elsässer variables [15]

$$z^+ = u + b, \qquad z^- = u - b,$$

merge the physical properties of the Navier-Stokes and Maxwell equations. The Elsässer variables, used in plasma turbulence, differentiate between the MHD waves

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propagating parallel or anti-parallel to the main magnetic field $B_\circ$. The MHD equations (1.1)-(1.2) are equivalently written in terms of the Elsässer variables as

$$\begin{cases} \partial_t z^{\pm} \mp (B_\circ \cdot \nabla) z^{\pm} + (z^{\mp} \cdot \nabla) z^{\pm} - \nu^{+} \Delta z^{\pm} - \nu^{-} \Delta z^{\mp} + \nabla p = f, \\ \nabla \cdot z^{\pm} = 0, \end{cases} \tag{1.3}$$

and $z^{\pm} = 0$ on $\partial\Omega$. Here $\nu^{+} = (\nu + \nu_m)/2$ and $\nu^{-} = (\nu - \nu_m)/2$. The nonlinear interactions between the Alfvénic fluctuations $z^{\pm}$, via the cross-coupling term $(z^{\mp} \cdot \nabla) z^{\pm}$, are basis of the Alfvén effect, a fundamental interaction process [13, 18, 21, 26].

The numerical simulation of the fully-coupled system (1.3) is computationally challenging. The monolithic methods or implicit fully-coupled algorithms [41], which assemble the coupled system at each time step and solve it iteratively, are stable and accurate, but quite demanding in computational speed and storage.

The implicit-explicit (IMEX) partitioned methods [30, 39], at each time step, decouple the computations and treat the subphysics/subdomain variables implicitly in time, while the coupling terms are evaluated explicitly in time (by lagging or extrapolating values), hence solving two subproblems in parallel. The IMEX decoupling highly reduces the computational complexity, but often introduces a Lyapunov-type instability.

Inhere we propose an algorithm which, at each time step, partitions the solve via an iterative process, and prove that the iterations linearly converge to the solution of the fully-coupled implicit monolithic method. The fully-coupled implicit monolithic method is the variable step, one-step, second-order, symplectic, midpoint method. Given $z_n^{\pm}$ at time t_n, the fully-coupled scheme solves for $z^{\pm}$ at time t_{n+1} by

$$\begin{cases} \dfrac{z_{n+1}^{\pm} - z_n^{\pm}}{\tau_n} \mp \left(B_\circ \cdot \nabla\right) z_{n+1/2}^{\pm} + \left(z_{n+1/2}^{\mp} \cdot \nabla\right) z_{n+1/2}^{\pm} \\ \qquad - \nu^{+} \Delta z_{n+1/2}^{\pm} - \nu^{-} \Delta z_{n+1/2}^{\mp} + \nabla p_{n+1/2}^{\pm} = f(t_{n+1/2}), \\ \nabla \cdot z_{n+1/2}^{\pm} = 0, \end{cases} \tag{1.4}$$

where $t_{n+1/2} = (t_{n+1} + t_n)/2$, $\tau_n = t_{n+1} - t_n$ is the local time step, and $z_{n+1/2}^{\pm}$, $p_{n+1/2}^{\pm}$ approximate $z^{\pm}$, p at $t_{n+1/2}$, respectively. The implementation of (1.4) can be simplified by the refactorization on the backward Euler (BE) solver:
Step 1. BE solver for $z_{n+1/2}^{\pm}$ on $[t_n, t_{n+1/2}]$

$$\begin{cases} \dfrac{z_{n+1/2}^{\pm} - z_n^{\pm}}{\tau_n/2} \mp \left(B_\circ \cdot \nabla\right) z_{n+1/2}^{\pm} + \left(z_{n+1/2}^{\mp} \cdot \nabla\right) z_{n+1/2}^{\pm} \\ \qquad - \nu^{+} \Delta z_{n+1/2}^{\pm} - \nu^{-} \Delta z_{n+1/2}^{\mp} + \nabla p_{n+1/2}^{\pm} = f(t_{n+1/2}), \\ \nabla \cdot z_{n+1/2}^{\pm} = 0. \end{cases} \tag{1.5}$$

Step 2. Post-process for $z_{n+1}^{\pm}$: $z_{n+1}^{\pm} = 2 z_{n+1/2}^{\pm} - z_n^{\pm}$.
The post-process is equivalent to the forward Euler (FE) solver on $[t_{n+1/2}, t_{n+1}]$ [8].

We decouple the implicit monolithic midpoint method (1.4) by partitioning the iterations in the refactorized implicit BE solver (1.5) for $z_{n+1/2}^{\pm}$. With an initial guess

$z^{\pm}_{(0)} = \frac{3}{2}z^{\pm}_n - \frac{1}{2}z^{\pm}_{n-1}$, we seek $z^{\pm}_{(k)}$ and $p_{(k)}$ such that

$$\begin{cases} \dfrac{z^{\pm}_{(k)} - z^{\pm}_n}{\tau_n/2} \mp \big(B_\circ \cdot \nabla\big) z^{\pm}_{(k)} + \big(z^{\mp}_{(k-1)} \cdot \nabla\big) z^{\pm}_{(k)} \\ \qquad -\nu^+ \Delta z^{\pm}_{(k)} - \nu^- \Delta z^{\mp}_{(k-1)} + \nabla p^{\pm}_{(k)} = f(t_{n+1/2}), \\ \nabla \cdot z^{\pm}_{(k)} = 0. \end{cases} \tag{1.6}$$

In Theorem 3.1 we prove that the sequence $\{z^{\pm}_{(k)}\}_{k=0}^{\infty}$ in the BE partitioned iteration (1.6) converges linearly to $z^{\pm}_{n+1/2}$, strongly in $H^1(\Omega)$, provided the time step τ_n is small enough. The BE partitioned iteration (1.6) is computationally efficient [38], as it decouples the cross-coupling terms $(z^{\mp} \cdot \nabla) z^{\pm}$. Due to the fact that the symplectic, second-order accurate, one-step implicit midpoint method conserves all quadratic Hamiltonians [2, 5, 8], the algorithm (1.4) conserves

- the model energy: $\mathcal{E}(t) = \dfrac{1}{2}\displaystyle\int_\Omega (|u|^2 + |B|^2) dx$,
- the cross-helicity: $\mathcal{H}_C(t) = \dfrac{1}{2}\displaystyle\int_\Omega u \cdot B dx$,
- the magnetic-helicity: $\mathcal{H}_M(t) = \dfrac{1}{2}\displaystyle\int_\Omega \mathbb{A} \cdot B dx$ for the vector potential $\mathbb{A}$ such that $B = \nabla \times \mathbb{A}$,

and its numerical solutions are second-order accurate in time. The midpoint rule is a special member of the robust Dahlquist-Liniger-Nevanlinna (DLN) family of one-leg second-order variable step methods [11,27–29,33], which are unconditionally stable for any arbitrary sequence of time steps. Also having the smallest error constant in the DLN family makes the midpoint method an ideal choice for time adaptivity, based on various criteria (minimum numerical dissipation, global error or local truncation error (LTE)). Time adaptivity is an effective approach to balances the computational costs and accuracy in the simulation of many stiff ordinary differential equations and fluid models [9, 27, 29, 33, 34, 37]. Here we utilize the explicit AB2-like method (a variant of the two-step Adams-Bashforth method [8]) to estimate the LTE of the algorithm (1.4), with negligible extra computational costs. This adaptive mechanism achieves outstanding performance in numerous highly stiff evolutionary equations [7] and thus deserves a fair trial in the simulations of MHD flows.

The rest of report is organized as follows. In Section 2 we give some necessary notations and preliminary results. In Section 3 we present detailed numerical analysis of the implicit fully-decoupled algorithm (1.4) with partitioned BE iterations (1.6). In subsection 3.1 we prove that the sequence $\{z^{\pm}_{(k)}\}_{k=0}^{\infty}$ in (1.6) converges linearly to $z^{\pm}_{n+1/2}$ in (1.5), strongly in $H^1(\Omega)$. We show in subsection 3.2 that the implicit algorithm (1.4) is unconditionally stable, and (in the ideal case) conserves the quadratic Hamitonians: energy, cross-helicity and magnetic-helicity. The variable step error analysis for the Elsässer variables $z^{\pm}$ is provided in subsection 3.4. Several numerical tests are presented in Section 5 to support main results of the report. Examples of two-dimensional traveling wave problem [40], and Hartmann flows [19] are to verify that the algorithm (1.4) is second-order accurate and has long-term conservations in the model energy, the cross-helicity and the magnetic-helicity. The efficiency of time adaptivity by LTE criterion is confirmed by these two examples with Lindberg time component [32].

2. Preliminaries and Notations. Let $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ be a bounded domain, and denote by $L^p(\Omega)$ the Banach space of Lebesgue measurable function f such that $|f|^p$ is integrable. We denote by $H^\ell(\Omega)$ is the usual Sobolev space $W^{\ell,2}(\Omega)$ with norm $\|\cdot\|_\ell$ and semi-norm $|\cdot|_\ell$. In the case $\ell=0$, $H^0(\Omega)$ reduces to the $L^2(\Omega)$ Hilbert space, with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. The function spaces X for Elsässer variables $z^\pm$ and Q for pressure p are

$$X = \left(H_0^1(\Omega)\right)^d = \Big\{v \in \left(H^1(\Omega)\right)^d : v|_{\partial\Omega} = 0\Big\}, \qquad Q = \Big\{q \in L^2(\Omega) : (q,1) = 0\Big\}.$$

The divergence-free space for X is

$$V = \Big\{v \in X : \left(\nabla \cdot v, q\right) = 0, \quad \forall q \in Q\Big\},$$

and let X' denote the dual space of X with the dual norm

$$\|f\|_{-1} := \sup_{v\in X,\ v\neq 0} \frac{(f,v)}{\|\nabla v\|}, \quad \forall f \in X'. \tag{2.1}$$

For spatial discretization we use the finite element method. Let $\{\mathcal{T}_h\}$ be a family of edge-to-edge triangulation with diameter $h \in (0,1)$. We denote $X^h \subset X$ and $Q^h \subset Q$ as certain finite element spaces for Elsässer variables $z^\pm$ and pressure p respectively. The divergence-free space for X^h is

$$V^h := \Big\{v^h \in X^h : \left(q^h, \nabla \cdot v^h\right) = 0, \ \ \forall q^h \in Q^h\Big\}.$$

We assume that X^h is C^m-space containing polynomials of highest degree r, Q^h is C^m-space containing polynomials of highest degree s, and we have the following approximations for $v \in (H^{r+1})^d \cap X$ and $q \in H^{s+1} \cap Q$

$$\begin{aligned} \inf_{v^h\in X^h} \|v - v^h\|_{\ell_1} &\le C_A^{\ell_1,r} h^{r+1-\ell_1} |v|_{r+1}, \quad 0 \le \ell_1 \le \min\{m+1, r+1\}, \\ \inf_{q^h\in Q^h} \|q - q^h\|_{\ell_2} &\le C_A^{\ell_2,s} h^{s+1-\ell_2} |q|_{s+1}, \quad 0 \le \ell_2 \le \min\{m+1, s+1\}, \end{aligned} \tag{2.2}$$

where the constants $C_A^{\ell_1,r}, C_A^{\ell_2,s} > 0$ are independent of h (see e.g., [6, 10]). We also recall the inverse inequality for X^h is

$$|v^h|_1 \le C_I h^{-1} \|v^h\|, \qquad \forall v^h \in X^h. \tag{2.3}$$

for $C_I > 0$ independent of h. To ensure the robustness of the fully discrete algorithm, we assume that X^h and Q^h satisfy the discrete inf-sup condition (Ladyzhenskaya-Babuška-Brezzi condition)

$$\inf_{q^h\in Q^h} \sup_{v^h\in X^h} \frac{\left(\nabla \cdot v^h, q^h\right)}{\|\nabla v^h\| \|q^h\|} \ge \beta_{\texttt{is}} > 0, \tag{2.4}$$

where $\beta_{\texttt{is}}$ is independent of h. The Taylor-Hood ($\mathbb{P}2$-$\mathbb{P}1$) space and Mini element space are typical examples meeting this criterion. For any pair $(u,p) \in V \times Q$, the Stokes projection $\left(I_{\mathrm{St}} u, I_{\mathrm{St}} p\right) \in V^h \times Q^h$ is defined to be the unique solution to the following Stokes problem

$$\left\{ \begin{array}{c} \left(\nabla u - \nabla I_{\mathrm{St}} u, \nabla v^h\right) = \left(p - I_{\mathrm{St}} p, \nabla \cdot v^h\right) \\ -\left(q^h, \nabla \cdot I_{\mathrm{St}} u\right) = 0 \end{array} \right. , \quad \forall (v^h, q^h) \in X^h \times Q^h. \tag{2.5}$$

We recall that if the pair (X^h,Q^h) satisfies the discrete inf-sup condition in (2.4), the following approximation property of the Stokes projection [20, 25] holds

$$|u-I_{\mathrm{St}}u|_1 \leq 2\Big(1+\frac{1}{\beta_{\mathrm{is}}}\Big)\inf_{v^h\in X^h}|u-v^h|_1+\inf_{q^h\in Q^h}\|p-q^h\|. \tag{2.6}$$

In the error analysis we will make use of the skew-symmetric nonlinear operator

$$\mathcal{N}(u,v,w)=\frac{1}{2}\big(u\cdot\nabla v,w\big)-\frac{1}{2}\big(u\cdot\nabla w,v\big), \qquad \forall u,v,w\in\big(H^1(\Omega)\big)^d, \tag{2.7}$$

which can also be written as

$$\mathcal{N}(u,v,w)=\big(u\cdot\nabla v,w\big)+\frac{1}{2}\big((\nabla\cdot u)v,w\big), \quad \forall u,v,w\in X. \tag{2.8}$$

Finally, we recall the following Sobolev embedding and Ladyzhenskaya inequality [17, p. 52]

$$\|u\|_{L^3(\Omega)} \leq \Big(\frac{1}{d^{1/2}}\Big)^{d/6}\|u\|^{1-d/6}\|\nabla u\|^{d/6}, \tag{2.9}$$

$$\|u\|_{L^4(\Omega)} \leq \Big(\frac{2(d-1)}{d^{3/2}}\Big)^{d/4}\|u\|^{1-d/4}\|\nabla u\|^{d/4}, \qquad \forall u\in\big(H^1(\Omega)\big)^d. \tag{2.10}$$

3. Partitioned, conservative, variable step, second-order method. In this section, first we prove the linear convergence of the iterative solution of the partitioned method (1.6), to the solution at time $t_{n+1/2}$ of the fully coupled midpoint method (1.4). Secondly, we prove the stability of (1.4), and the conservation of three quadratic invariants: the energy in the original variables u and B, the cross-helicity and the magnetic-helicity. Finally, we perform error analysis of the full time-space discretization, using the finite element method.

3.1. Convergence of the subiterates. To simplify the presentation, we introduce the following notations related to the errors at each iteration

$$\begin{aligned} &a_{(k)}=z^+_{n+1/2}-z^+_{(k)}, \qquad b_{(k)}=z^-_{n+1/2}-z^-_{(k)},\\ &\gamma_n=\max\big\{\|\nabla z^+_{n+1/2}\|,\|\nabla z^-_{n+1/2}\|\big\}, \qquad \delta=\Big(\frac{2(d-1)}{d^{3/2}}\Big)^{d/4}. \end{aligned}$$

and a weighted $H^1(\Omega)$ norm of the errors $a_{(k)},b_{(k)}$

$$\begin{aligned} \mathcal{G}_{(k)}=&\frac{4-d}{8}\big(\delta^2\gamma_n\big)^{\frac{4}{4-d}}\Big(\frac{d}{2}\frac{\nu+\nu_m}{\nu\nu_m}\Big)^{\frac{d}{4-d}}\big(\|a_{(k)}\|^2+\|b_{(k)}\|^2\big)\\ &+\frac{\nu^2-\nu\nu_m+\nu_m^2}{4(\nu+\nu_m)}\big(\|\nabla a_{(k)}\|^2+\|\nabla b_{(k)}\|^2\big). \end{aligned}$$

The next result shows the $H^1(\Omega)$ strong convergence of the iterates $z^{\pm}_{(k)}$, under the following time step restriction

$$\tau_n \leq \frac{8}{4-d}\frac{1}{(\delta^2\gamma_n)^{4/(4-d)}}\frac{\nu^2-\nu\nu_m+\nu_m^2}{\nu^2+\nu_m^2}\Big(\frac{2\nu\nu_m}{d(\nu+\nu_m)}\Big)^{d/(4-d)}. \tag{3.1}$$

THEOREM 3.1. *Asume that the time step τ_n satisfies condition (3.1). Then, the (1.6) sequence of iterates $\{z^{\pm}_{(k)}\}_{k\geq 0}$ converges linearly to $z^{\pm}_{n+1/2}$ in the $H^1(\Omega)$*

$$\mathcal{G}_{(k)} \leq \Big(1 - \frac{2\nu\nu_m}{\nu^2+\nu\nu_m+\nu_m^2}\Big)\mathcal{G}_{(k-1)}. \tag{3.2}$$

Proof. First, we subtract (1.6) from (1.5), and test the result with $a_{(k)}$ and $b_{(k)}$, respectively, to obtain

$$\begin{aligned}
&\frac{2}{\tau_n}\big(\|a_{(k)}\|^2+\|b_{(k)}\|^2\big)+\frac{\nu+\nu_m}{2}\big(\|\nabla a_{(k)}\|^2+\|\nabla b_{(k)}\|^2\big)\\
&\quad+\frac{\nu-\nu_m}{2}\Big((\nabla a_{(k)},\nabla b_{(k-1)})+(\nabla a_{(k-1)},\nabla b_{(k)})\Big)\\
&\quad+\int_\Omega b_{(k-1)}\cdot\nabla z^+_{n+1/2}\,a_{(k)}+\int_\Omega a_{(k)}\cdot\nabla z^-_{n+1/2}\,b_{(k)}=0.
\end{aligned}$$

Then, using the polarization identity on the mixed diffusion terms, we have

$$\begin{aligned}
&\frac{\nu-\nu_m}{2}\Big((\nabla a_{(k)},\nabla b_{(k-1)})+(\nabla a_{(k-1)},\nabla b_{(k)})\Big)\\
&=-\frac{\nu+\nu_m}{4}\big(\|\nabla a_{(k)}\|^2+\|\nabla b_{(k)}\|^2\big)-\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(\|\nabla a_{(k-1)}\|^2+\|\nabla b_{(k-1)}\|^2\big)\\
&\quad+\frac{|\nu-\nu_m|}{4}\left\|\sqrt{\frac{\nu+\nu_m}{|\nu-\nu_m|}}\nabla a_{(k)}+\mathrm{sign}(\nu-\nu_m)\sqrt{\frac{|\nu-\nu_m|}{\nu+\nu_m}}\nabla b_{(k-1)}\right\|^2\\
&\quad+\frac{|\nu-\nu_m|}{4}\left\|\sqrt{\frac{\nu+\nu_m}{|\nu-\nu_m|}}\nabla b_{(k)}+\mathrm{sign}(\nu-\nu_m)\sqrt{\frac{|\nu-\nu_m|}{\nu+\nu_m}}\nabla a_{(k-1)}\right\|^2,
\end{aligned}$$

where

$$\mathrm{sign}(x) \coloneqq \begin{cases} -1, & \text{if } x<0,\\ 0, & \text{if } x=0,\\ 1, & \text{if } x>0,\end{cases}$$

and therefore

$$\begin{aligned}
&\frac{2}{\tau_n}\big(\|a_{(k)}\|^2+\|b_{(k)}\|^2\big)+\frac{\nu+\nu_m}{4}\big(\|\nabla a_{(k)}\|^2+\|\nabla b_{(k)}\|^2\big)\\
&\quad+\frac{|\nu-\nu_m|}{4}\left\|\sqrt{\frac{\nu+\nu_m}{|\nu-\nu_m|}}\nabla a_{(k)}+\mathrm{sign}(\nu-\nu_m)\sqrt{\frac{|\nu-\nu_m|}{\nu+\nu_m}}\nabla b_{(k-1)}\right\|^2\\
&\quad+\frac{|\nu-\nu_m|}{4}\left\|\sqrt{\frac{\nu+\nu_m}{|\nu-\nu_m|}}\nabla b_{(k)}+\mathrm{sign}(\nu-\nu_m)\sqrt{\frac{|\nu-\nu_m|}{\nu+\nu_m}}\nabla a_{(k-1)}\right\|^2\\
&\quad+\int_\Omega b_{(k-1)}\cdot\nabla z^+_{n+1/2}\,a_{(k)}+\int_\Omega a_{(k)}\cdot\nabla z^-_{n+1/2}\,b_{(k)}\\
&=\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(\|\nabla a_{(k-1)}\|^2+\|\nabla b_{(k-1)}\|^2\big).
\end{aligned}$$

Now we apply the Ladyzhenskaya inequality (2.10) to the convective terms to obtain

$$\int_\Omega b_{(k-1)}\cdot\nabla z^+_{n+1/2}\,a_{(k)}+\int_\Omega a_{(k-1)}\cdot\nabla z^-_{n+1/2}\,b_{(k)}$$

$$
\begin{aligned}
\geq & -\frac{4-d}{8}\Big(\frac{2(d-1)}{d^{3/2}}\Big)^{\frac{2d}{4-d}}\Big(\frac{d}{2}\Big)^{d/(4-d)}\Big(\frac{\nu+\nu_m}{\nu\nu_m}\Big)^{\frac{d}{4-d}}\gamma_n^{\frac{4}{4-d}}\Big(\|a_{(k)}\|^2+\|b_{(k)}\|^2\Big)\\
& -\frac{4-d}{8}\Big(\frac{2(d-1)}{d^{3/2}}\Big)^{\frac{2d}{4-d}}\Big(\frac{d}{2}\Big)^{d/(4-d)}\Big(\frac{\nu+\nu_m}{\nu\nu_m}\Big)^{\frac{d}{4-d}}\gamma_n^{\frac{4}{4-d}}\Big(\|a_{(k-1)}\|^2+\|b_{(k-1)}\|^2\Big)\\
& -\frac{1}{4}\frac{\nu\nu_m}{\nu+\nu_m}\Big(\|\nabla a_{(k)}\|^2+\|\nabla b_{(k)}\|^2\Big)-\frac{1}{4}\frac{\nu\nu_m}{\nu+\nu_m}\Big(\|\nabla a_{(k-1)}\|^2+\|\nabla b_{(k-1)}\|^2\Big).
\end{aligned}
$$

Finally, using this inequality in the relation above implies

$$
\begin{aligned}
&\Big(\frac{2}{\tau_n}-\frac{4-d}{8}(\delta^2\gamma_n)^{\frac{4}{4-d}}\Big(\frac{d}{2}\frac{\nu+\nu_m}{\nu\nu_m}\Big)^{\frac{d}{4-d}}\Big)(\|a_{(k)}\|^2+\|b_{(k)}\|^2)\\
&\qquad+\frac{\nu^2+\nu\nu_m+\nu_m^2}{4(\nu+\nu_m)}(\|\nabla a_{(k)}\|^2+\|\nabla b_{(k)}\|^2)\\
&\leq \frac{4-d}{8}(\delta^2\gamma_n)^{\frac{4}{4-d}}\Big(\frac{d}{2}\frac{\nu+\nu_m}{\nu\nu_m}\Big)^{\frac{d}{4-d}}(\|a_{(k-1)}\|^2+\|b_{(k-1)}\|^2)\\
&\qquad+\frac{\nu^2-\nu\nu_m+\nu_m^2}{4(\nu+\nu_m)}(\|\nabla a_{(k-1)}\|^2+\|\nabla b_{(k-1)}\|^2).
\end{aligned}
$$

Hence, under the time step restriction (3.1), the above estimate yields the linear convergence relation (3.2). ∎

3.2. Stability and a balance of energy. Now we consider the stability of the algorithm (1.4). We denote the discrete kinetic energy of the system by $\mathcal{E}^N$, the viscous dissipation rate by $\mathcal{D}^N$:

$$
\begin{aligned}
\mathcal{E}^N =&\frac{1}{2}(\|z_N^+\|^2+\|z_N^-\|^2),\\
\mathcal{D}^N =&\min\{\nu,\nu_m\}\sum_{n=1}^{N-1}\tau_n(\|\nabla z_{n+1/2}^+\|^2+\|\nabla z_{n+1/2}^-\|^2)\\
&+|v^-|\sum_{n=1}^{N-1}\tau_n\big\|\nabla z_{n+1/2}^+ +\mathrm{sign}(v^-)\nabla z_{n+1/2}^-\big\|^2,
\end{aligned}
\tag{3.3}
$$

Theorem 3.2. *The algorithm in (1.4) is unconditionally stable and satisfies*

$$
\begin{aligned}
&\mathcal{E}^N+\frac{1}{2}\mathcal{D}^N+\frac{|v^-|}{2}\sum_{n=1}^{N-1}\tau_n\big\|\nabla z_{n+1/2}^+ +\mathrm{sign}(v^-)\nabla z_{n+1/2}^-\big\|^2\\
&\leq \sum_{n=1}^{N-1}\frac{\tau_n}{\min\{\nu,\nu_m\}}\|f(t_{n+1/2})\|_{-1}^2+\mathcal{E}^0.
\end{aligned}
\tag{3.4}
$$

Moveover, the following energy equality holds in the absence of external force f

$$
\mathcal{E}^N+\mathcal{D}^N=\mathcal{E}^0. \tag{3.5}
$$

Proof. We first take inner product of (1.4) with $2z_{n+1/2}^{\pm}$, and use the skew-symmetry properties (2.7), (2.8) to obtain

$$
\frac{1}{\tau_n}(\|z_{n+1}^{\pm}\|^2-\|z_n^{\pm}\|^2)+2\nu^+\|\nabla z_{n+1/2}^{\pm}\|^2+2\nu^-(z_{n+1/2}^{\mp},z_{n+1/2}^{\pm})
$$

$$= 2(f(t_{n+1/2}), z^{\pm}_{n+1/2}). \tag{3.6}$$

Then we use the following polarization identity

$$\begin{aligned}
&\nu^{-}\big(\nabla z^{\mp}_{n+1/2}, \nabla z^{\pm}_{n+1/2}\big)\\
&= \frac{|\nu^{-}|}{2}\big\|\nabla z^{\mp}_{n+1/2} + \mathrm{sign}(\nu^{-})\nabla z^{\pm}_{n+1/2}\big\|^2 - \frac{|\nu^{-}|}{2}\big(\|\nabla z^{\mp}_{n+1/2}\|^2 + \|\nabla z^{\pm}_{n+1/2}\|^2\big),
\end{aligned}$$

to obtain

$$\begin{aligned}
&\frac{1}{2\tau_n}\big(\|z^{+}_{n+1}\|^2 + \|z^{-}_{n+1}\|^2 - \|z^{+}_{n}\|^2 - \|z^{-}_{n}\|^2\big) \\
&\quad + (\nu^{+} - |\nu^{-}|)\big(\|\nabla z^{+}_{n+1/2}\|^2 + \|\nabla z^{-}_{n+1/2}\|^2\big) + |\nu^{-}|\big\|\nabla z^{+}_{n+1/2} + \mathrm{sign}(\nu^{-})\nabla z^{-}_{n+1/2}\big\|^2 \\
&= (f(t_{n+1/2}), z^{+}_{n+1/2}) + (f(t_{n+1/2}), z^{-}_{n+1/2}).
\end{aligned} \tag{3.7}$$

Using $\nu^{+} - |\nu^{-}| = \min\{\nu, \nu_m\}$, the definition of dual norm (2.1), and Young's inequality we have

$$\begin{aligned}
&\frac{1}{2\tau_n}\big(\|z^{+}_{n+1}\|^2 + \|z^{-}_{n+1}\|^2 - \|z^{+}_{n}\|^2 - \|z^{-}_{n}\|^2\big) \\
&+ \frac{1}{2}\min\{\nu, \nu_m\}\big(\|\nabla z^{+}_{n+1/2}\|^2 + \|\nabla z^{-}_{n+1/2}\|^2\big) + |\nu^{-}|\big\|\nabla z^{+}_{n+1/2} + \mathrm{sign}(\nu^{-})\nabla z^{-}_{n+1/2}\big\|^2 \\
&\leq \frac{1}{\min\{\nu, \nu_m\}}\|f(t_{n+1/2})\|^2_{-1}.
\end{aligned} \tag{3.8}$$

Finally, if we multiply (3.8) by τ_n and sum over n from 1 to $N-1$, we obtain (3.4). In the case when the source function f vanishes, summation of (3.7) over n from 1 to $N-1$ yields (3.5). ◻

3.3. Conservation of quadratic invariants: energy, cross-helicity and magnetic-helicity. We note that the fully coupled system (1.4) can be equivalently written as

$$\begin{cases}
\dfrac{u_{n+1} - u_n}{\tau_n} - \big(B_{n+1/2}\cdot\nabla\big)B_{n+1/2} + \big(u_{n+1/2}\cdot\nabla\big)u_{n+1/2} - \nu\Delta u_{n+1/2} \\
\qquad + \nabla\Big(\dfrac{p^{+}_{n+1/2} + p^{-}_{n+1/2}}{2}\Big) = f(t_{n+1/2}), \\
\dfrac{B_{n+1} - B_n}{\tau_n} + \big(u_{n+1/2}\cdot\nabla\big)B_{n+1/2} - \big(B_{n+1/2}\cdot\nabla\big)u_{n+1/2} - \nu_m\Delta B_{n+1/2} \\
\qquad\qquad\qquad + \nabla\Big(\dfrac{p^{+}_{n+1/2} - p^{-}_{n+1/2}}{2}\Big) = 0, \\
\nabla\cdot u_{n+1/2} = 0, \qquad \nabla\cdot B_{n+1/2} = 0,
\end{cases} \tag{3.9}$$

where by *Step 2* in (1.5) we have $u_{n+1/2} = (u_{n+1} + u_n)/2$ and $B_{n+1/2} = (B_{n+1} + B_n)/2$.

We denote the energy, the cross-helicity and the magnetic-helicity corresponding to the solution at time t_n of algorithm (3.9) by

$$\mathcal{E}_n = \frac{1}{2}\int_\Omega(|u_n|^2 + |B_n|^2)dx, \quad \mathcal{H}_{C_n} = \frac{1}{2}\int_\Omega u_n\cdot B_n dx, \quad \mathcal{H}_{M_n} = \frac{1}{2}\int_\Omega \mathbb{A}_n\cdot B_n dx.$$

Theorem 3.3. *The algorithm in* (1.4) *or* (3.9)*, conserves the energy, cross-helicity and magnetic-helicity*

$$\mathcal{E}_n = \mathcal{E}_0, \qquad \mathcal{H}_{C_n} = \mathcal{H}_{C_0}, \qquad \mathcal{H}_{M_n} = \mathcal{H}_{M_0},$$

in the ideal case, i.e., in the absence of external forcing terms, and for zero kinematic viscosity and magnetic diffusivity.

Proof. We assume that $f = 0$, and $\nu = \nu_m = 0$. First, we take inner product of first equation of (3.9) with $u_{n+1/2}$, the second equation of (3.9) with $B_{n+1/2}$ and use the skew-symmetry properties (2.7), (2.8) to obtain

$$\frac{1}{2}\big(\|u_{n+1}\|^2 + \|B_{n+1}\|^2\big) - \frac{1}{2}\big(\|u_n\|^2 + \|B_n\|^2\big) = 0. \tag{3.10}$$

Summation of (3.10) over n from 1 to $N-1$ and proves the conservation of the model energy, i.e. $\mathcal{E}_N = \mathcal{E}_0$.

Secondly, we take the inner product of the first equation in (3.9) with $B_{n+1/2}$, the second equation of (3.9) with $u_{n+1/2}$

$$\frac{1}{2\tau_n}\big(u_{n+1} - u_n, B_{n+1} + B_n\big) + \big((u_{n+1/2}\cdot\nabla)u_{n+1/2}, B_{n+1/2}\big) = 0,$$
$$\frac{1}{2\tau_n}\big(B_{n+1} - B_n, u_{n+1} + u_n\big) + \big((u_{n+1/2}\cdot\nabla)B_{n+1/2}, u_{n+1/2}\big) = 0.$$

and using (2.7), (2.8) similarly we obtain

$$\frac{1}{\tau_n}\int_\Omega u_{n+1}\cdot B_{n+1}dx - \frac{1}{\tau_n}\int_\Omega u_n\cdot B_n dx = 0, \tag{3.11}$$

which proves the conservation of cross helicity, i.e., $\mathcal{H}_{C_N} = \mathcal{H}_{C_0}$.

For the result on the magnetic helicity, we begin by recalling (see e.g., [20, Theorem 3.2, Chapter I]) that, since $\nabla\cdot B_n = 0$, there exists a function $\mathbb{A}_n \in \big(H^1(\Omega)\big)^d$ such that $B_n = \nabla\times\mathbb{A}_n$ for all n. Moreover, $\nabla\cdot\mathbb{A}_n = 0$ and $\mathbb{A}_n = 0$ on the boundary. Next, we take the inner product of second equation of (3.9) with $\mathbb{A}_{n+1/2} = (\mathbb{A}_{n+1} + \mathbb{A}_n)/2$

$$\begin{aligned}&\frac{1}{\tau_n}\big(B_{n+1} - B_n, \mathbb{A}_{n+1/2}\big)\\ &+ \big((u_{n+1/2}\cdot\nabla)B_{n+1/2}, \mathbb{A}_{n+1/2}\big) - \big((B_{n+1/2}\cdot\nabla)u_{n+1/2}, \mathbb{A}_{n+1/2}\big) = 0.\end{aligned} \tag{3.12}$$

Then, we use the identity

$$\nabla\times(B\times u) = (\nabla\cdot u)B + (u\cdot\nabla)B - (\nabla\cdot B)u - (B\cdot\nabla)u,$$

to write the convective terms as follows

$$\begin{aligned}&\big((u_{n+1/2}\cdot\nabla)B_{n+1/2}, \mathbb{A}_{n+1/2}\big) - \big((B_{n+1/2}\cdot\nabla)u_{n+1/2}, \mathbb{A}_{n+1/2}\big)\\ &= \big(\nabla\times(B_{n+1/2}\times u_{n+1/2}), \mathbb{A}_{n+1/2}\big) - \big((\nabla\cdot u_{n+1/2})B_{n+1/2}, \mathbb{A}_{n+1/2}\big)\\ &\quad + \big((\nabla\cdot B_{n+1/2})u_{n+1/2}, \mathbb{A}_{n+1/2}\big)\\ &= \big(\nabla\times((\nabla\times\mathbb{A}_{n+1/2})\times u_{n+1/2}), \mathbb{A}_{n+1/2}\big)\\ &= \big((\nabla\times\mathbb{A}_{n+1/2})\times u_{n+1/2}, \nabla\times\mathbb{A}_{n+1/2}\big) - \int_{\partial\Omega}\big((\nabla\times\mathbb{A}_{n+1/2})\times u_{n+1/2}\big)\times\mathbb{A}_{n+1/2} = 0,\end{aligned} \tag{3.13}$$

where the last equality comes from the homogeneous Dirichlet boundary condition on the velocity $u|_{\partial\Omega}=0$, and the orthogonality of $\nabla\times\mathbb{A}_{n+1/2}$ and $(\nabla\times\mathbb{A}_{n+1/2})\times u_{n+1/2}$. Using (3.13) in (3.12) gives

$$\int_\Omega \mathbb{A}_{n+1}\cdot B_{n+1}dx+\int_\Omega \mathbb{A}_n\cdot B_{n+1}dx-\int_\Omega \mathbb{A}_{n+1}\cdot B_n dx=\int_\Omega \mathbb{A}_n\cdot B_n dx. \tag{3.14}$$

Finally, by the Gauss' divergence theorem we have

$$\int_\Omega \mathbb{A}_n\cdot B_{n+1}dx-\int_\Omega \mathbb{A}_{n+1}\cdot B_n dx=\int_\Omega \nabla\cdot\big(\mathbb{A}_{n+1}\times\mathbb{A}_n\big)dx=0,$$

and therefore (3.14) implies the conservation of magnetic-helicity $\mathcal{H}_{M_N}=\mathcal{H}_{M_0}$. ☐

3.4. Error Analysis. In this section, $z^{\pm}(t_n)$ represent the true solutions of (1.3) at time t_n. For the fully space-time discretization, $z_n^{\pm,h}\in X^h$ and $p_{n+1/2}^{\pm,h}\in Q^h$ are numerical approximations of $z^{\pm}(t_n)$ and $p(t_{n+1/2})$ respectively.

The variational formulation of the fully-decoupled algorithm in (1.4) is: given $z_n^{\pm,h}\in X^h$, we solve $z_{n+1}^{\pm,h}\in X^h, p_{n+1/2}^{\pm,h}\in Q^h$ such that for all $(v^h,q^h)\in X^h\times Q^h$,

$$\begin{cases}\Big(\dfrac{z_{n+1}^{\pm,h}-z_n^{\pm,h}}{\tau_n},v^h\Big)\mp\mathcal{N}\big(B_\circ,z_{n+1/2}^{\pm,h},v^h\big)+\mathcal{N}\big(z_{n+1/2}^{\mp,h},z_{n+1/2}^{\pm,h},v^h\big)\\ +\nu^+\big(\nabla z_{n+1/2}^{\pm,h},\nabla v^h\big)+\nu^-\big(\nabla z_{n+1/2}^{\mp,h},\nabla v^h\big)-\big(p_{n+1/2}^{\pm,h},\nabla\cdot v^h\big)=\big(f(t_{n+1/2}),v^h\big),\\ \big(\nabla\cdot z_{n+1/2}^{\pm,h},q^{\pm,h}\big)=0.\end{cases} \tag{3.15}$$

The following result recalls the consistency of the midpoint finite difference method.

Lemma 3.4. *For any given nodes $\{t_n\}_{n=0}^N$ on the time interval $[0,T]$, if the mapping $u:[0,T]\to H^\ell(\Omega)$ is smooth enough, then*

$$\Big\|\big(u(\cdot,t_{n+1})+u(\cdot,t_n)\big)/2-u(\cdot,t_{n+1/2})\Big\|_\ell\le C\tau_n^3\int_{t_n}^{t_{n+1}}\|u_{tt}(\cdot,t)\|_\ell^2dt, \tag{3.16}$$

$$\Big\|\frac{u(\cdot,t_{n+1})-u(\cdot,t_n)}{\tau_n}-u(\cdot,t_{n+1/2})\Big\|_\ell\le C\tau_n^3\int_{t_n}^{t_{n+1}}\|u_{ttt}(\cdot,t)\|_\ell^2dt. \tag{3.17}$$

Proof. A direct calculation involving the Taylor expansion with integral remainder about $t_{n+1/2}$. ☐

We also need the following Bochner spaces on the time interval $[0,T]$

$$\begin{aligned}L^p\big(0,T;(H^\ell(\Omega))^d\big)=&\Big\{v(\cdot,t)\in(H^\ell(\Omega))^d:\|v\|_{p,\ell}=\Big(\int_0^T\|v(\cdot,t)\|_\ell^p dt\Big)^{1/p}<\infty\Big\},\\ L^\infty\big(0,T;(H^\ell(\Omega))^d\big)=&\Big\{v(\cdot,t)\in(H^\ell(\Omega))^d:\|v\|_{\infty,\ell}=\sup_{0<t<T}\|v(\cdot,t)\|_\ell<\infty\Big\},\\ L^p\big(0,T;X'\big)=&\Big\{v(\cdot,t)\in X':\|v\|_{p,-1}=\Big(\int_0^T\|v(\cdot,t)\|_{-1}^p dt\Big)^{1/p}<\infty\Big\},\end{aligned}$$

and the corresponding discrete Bochner spaces

$$\begin{aligned}&\ell^\infty\big(\{t_n\}_{n=0}^N;(H^\ell(\Omega))^d\big)\\ &:=\Big\{v(\cdot,t_n)\in(H^\ell(\Omega))^d:|||v|||_{\infty,\ell}:=\max_{0\le n\le N}\|v(\cdot,t_n)\|_\ell<\infty\Big\},\end{aligned}$$

$$\ell^{\infty,1/2}\big(\{t_n\}_{n=0}^N;(H^\ell(\Omega))^d\big)$$
$$:=\Big\{v(\cdot,t_{n+1/2})\in(H^\ell(\Omega))^d: |||v|||_{\infty,\ell,1/2}:=\max_{1\le n\le N-1}\|v(\cdot,t_{n+1/2})\|_\ell<\infty\Big\},$$
$$\ell^{p,1/2}\big(\{t_n\}_{n=0}^N;(H^\ell(\Omega))^d\big)$$
$$:=\Big\{v(\cdot,t_{n+1/2})\in(H^\ell(\Omega))^d: |||v|||_{p,\ell,1/2}:=\Big(\sum_{n=1}^{N-1}\tau_n\|v(\cdot,t_{n+1/2})\|_\ell^p\Big)^{1/p}<\infty\Big\},$$
$$\ell^{p,1/2}\big(\{t_n\}_{n=0}^N;X'\big)$$
$$:=\Big\{v(\cdot,t_{n+1/2})\in X': |||v|||_{p,\ell,-1}:=\Big(\sum_{n=1}^{N-1}\tau_n\|v(\cdot,t_{n+1/2})\|_{-1}^p\Big)^{1/p}<\infty\Big\}.$$

We denote the errors in the algorithm (3.15) at time t_n by $e_n^\pm=z_n^{\pm,h}-z^\pm(t_n)$.
This should be $e_n^{\pm,h}$!!!

THEOREM 3.5. *We assume that $(z^\pm,p)$ in (1.3) satisfies the regularity assumptions*

$$\begin{aligned}
&z^\pm\in\ell^\infty\big(\{t_n\}_{n=0}^N;(H^\ell)^d\big)\cap\ell^{\infty,1/2}\big(\{t_n\}_{n=0}^N;(H^\ell)^d\big)\cap\ell^{2,1/2}\big(\{t_n\}_{n=0}^N;(H^{r+1})^d\cap(H^2)^d\big),\\
&z^\pm\in\ell^{4,1/2}\big(\{t_n\}_{n=0}^N;(H^1)^d\big)\cap L^4\big(0,T;(H^1)^d\big)\\
&z_t^\pm\in L^2\big(0,T;(H^{r+1})^d\big),\quad z_{tt}^\pm\in L^2\big(0,T;(H^{r+1})^d\cap(H^1)^d\big),\\
&z_{ttt}^\pm\in L^2\big(0,T;X'\big),\qquad p\in\ell^{2,1/2}\big(\{t_n\}_{n=0}^N;H^{s+1}\big).
\end{aligned}\tag{3.18}$$

and time step size τ_n satisfies

$$\tau_n<\frac{1}{16\big(C_1^*hZ_n^{(1)}+256C_2^*Z_n^{(2)}\big)},\quad\forall n=1,2,\cdots,N-1,\tag{3.19}$$

where

$$\begin{aligned}
&C_1^*=\frac{2(3(d-1))^{2d/3}C_P^{3-d}C_I(C_A^{1,0})^2}{\epsilon\nu^*d^{7d/6}}\big(1+\frac{1}{\beta_{\mathtt{is}}}\big)^2,\quad C_2^*=\frac{(3(d-1))^{4d/3}C_P^{2(3-d)}}{32(\nu^*)^3d^{7d/3}},\\
&Z_n^{(1)}=\tfrac12\max\big\{|z^+(t_{n+1})+z^+(t_n)|_2,|z^-(t_{n+1})+z^-(t_n)|_2\big\},\\
&Z_n^{(2)}=\frac12\max\big\{\|\nabla\big(z^+(t_{n+1})+z^+(t_n)\big)\|^4,\|\nabla\big(z^-(t_{n+1})+z^-(t_n)\big)\|^4\big\},
\end{aligned}$$

$\nu^*=\nu^+-|\nu^-|=\min\{\nu,\nu_m\}$, and C_P represents the constant in the Poincaré inequality. Then the following error estimates hold

$$\begin{aligned}
\|e_N^+\|^2+\|e_N^-\|^2\le{}&Ch^{2r}\big(|||z^+|||_{\infty,r+1}^2+|||z^-|||_{\infty,r+1}^2\big)\\
&+C\exp\Big(C\sum_{n=1}^{N-1}\big(C_1^*hZ_n^{(1)}+C_2^*Z_n^{(2)}\big)\tau_n\Big)F\big(h^{2r},h^{2s+2},\tau_{\max}^4,\|B_\circ\|^2\big),
\end{aligned}\tag{3.20}$$

and

$$\nu^*\sum_{n=0}^{N-1}\tau_n\big(\|\nabla(e_{n+1}^++e_n^+)/2\|^2+\|\nabla(e_{n+1}^-+e_n^-)/2\|^2\big)$$

$$
\le C\nu^* h^{2r}\big(\tau_{\max}^4\|z^\pm\|_{2,r+1}^2+|||z^\pm|||_{2,r+1,1/2}^2\big)
$$
$$
+C\exp\Big(C\sum_{n=1}^{N-1}\big(C_1^* h Z_n^{(1)}+C_2^* Z_n^{(2)}\big)\tau_n\Big)F\big(h^{2r},h^{2s+2},\tau_{\max}^4,\|B_\circ\|^2\big),
$$

where $\tau_{\max}=\max_{0\le n\le N}\{\tau_n\}$ *and*

$$
\begin{aligned}
&F\big(h^{2r},h^{2s+2},\tau_{\max}^4,\|B_\circ\|^2\big)\\
&=\frac{h^{2r+2}}{\nu^*}\big(\|z^+\|_{2,r+1}^2+\|z^-\|_{2,r+1}^2\big)+\frac{\big((\nu^+)^2+(\nu^-)^2\big)h^{2r}\tau_{\max}^4}{\nu^*}\big(\|z_{tt}^+\|_{2,r+1}^2+\|z_{tt}^-\|_{2,r+1}^2\big)\\
&+\frac{\big((\nu^+)^2+(\nu^-)^2\big)h^{2r}}{\nu^*}\big(|||z^+|||_{2,r+1,1/2}^2+|||z^-|||_{2,r+1,1/2}^2\big)\\
&+\frac{(\nu^-)^2\tau_{\max}^4}{\nu^*}\big(\|\nabla z_{tt}^+\|_{2,0}^2+\|\nabla z_{tt}^-\|_{2,0}^2\big)+\frac{h^{2s+2}}{\nu^*}\big(|||p^+|||_{2,s+1,1/2}^2+|||p^-|||_{2,s+1,1/2}^2\big)\\
&+\frac{\tau_{\max}^4}{\nu^*}\big(\|\nabla z_{ttt}^+\|_{2,0}^2+\|\nabla z_{ttt}^-\|_{2,0}^2\big)+\frac{\tau_{\max}^4}{\nu^*}\|B_\circ\|^2\big(\|\nabla z_{tt}^+\|_{2,0}^2+\|\nabla z_{tt}^-\|_{2,0}^2\big)\\
&+\frac{h^{2r}\tau_{\max}^4}{\nu^*}\|B_\circ\|^2\big(\|z_{tt}^\pm\|_{2,r+1}^2+\|z_{tt}^\pm\|_{2,r+1}^2\big)\\
&+\frac{h^{2r}}{\nu^*}\|B_\circ\|^2\big(|||z^+|||_{2,r+1,1/2}^2+|||z^-|||_{2,r+1,1/2}^2\big)\\
&+\frac{\tau_{\max}^4}{\nu^*}\big(|||z^-|||_{\infty,1}^2\|\nabla z_{tt}^+\|_{2,0}^2+|||z^+|||_{\infty,1}^2\|\nabla z_{tt}^-\|_{2,0}^2+|||z^+|||_{\infty,1,1/2}^2\|\nabla z_{tt}^-\|_{2,0}^2\\
&\qquad\qquad+|||z^-|||_{\infty,1,1/2}^2\|\nabla z_{tt}^+\|_{2,0}^2\big)\\
&+\frac{h^{2r}\tau_{\max}^4}{\nu^*}\big(|||\nabla z^-|||_{\infty,0}^2\|z_{tt}^+\|_{2,r+1}^2+|||\nabla z^+|||_{\infty,0}^2\|z_{tt}^-\|_{2,r+1}^2\big)\\
&+\frac{h^{2r}}{\nu^*}\big(|||z^+|||_{2,r+1,1/2}^2+|||z^-|||_{2,r+1,1/2}^2\big)+\big(\|\xi_0^{+,h}\|^2+\|\xi_0^{-,h}\|^2\big).
\end{aligned}
$$

Proof. The true solutions of (1.3) at time $t_{n+1/2}$ satisfies

$$
\begin{aligned}
&\Big(\frac{z^\pm(t_{n+1})-z^\pm(t_n)}{\tau_n},v^h\Big)\mp\mathcal{N}\big(B_\circ,z^\pm(t_{n+1/2}),v^h\big)+\mathcal{N}\big(z^\mp(t_{n+1/2}),z^\pm(t_{n+1/2}),v^h\big)\\
&+\nu^+\big(\nabla z^\pm(t_{n+1/2}),\nabla v^h\big)+\nu^-\big(\nabla z^\mp(t_{n+1/2}),\nabla v^h\big)-\big(p^\pm(t_{n+1/2}),\nabla\cdot v^h\big)\\
&=\big(f(t_{n+1/2}),v^h\big)+\Big(\frac{z^\pm(t_{n+1})-z^\pm(t_n)}{\tau_n}-z_t^\pm(t_{n+1/2}),v^h\Big),\qquad\forall v^h\in V^h.
\end{aligned}\tag{3.21}
$$

We denote by $I_{\rm St}z_n^\pm$ the velocity components of Stokes projection of $(z_n^\pm,0)$ onto $V^h\times Q^h$ and decompose the error $e_n^\pm$ as

$$
e_n^\pm=\xi_n^{\pm,h}+\eta^\pm(t_n),\quad \xi_n^{\pm,h}:=z_n^{\pm,h}-I_{\rm St}z^\pm(t_n),\quad \eta^\pm(t_n):=I_{\rm St}z^\pm(t_n)-z^\pm(t_n).\tag{3.22}
$$

For convenience of the presentation, we denote

$$
\begin{aligned}
&\xi_{n+1/2}^\pm:=(\xi_{n+1}^\pm+\xi_n^\pm)/2,\\
&z_{n+1/2}^\pm:=(z^\pm(t_{n+1})+z^\pm(t_n))/2,\quad \eta_{n+1/2}^\pm:=(\eta^\pm(t_{n+1})+\eta^\pm(t_n))/2.
\end{aligned}
$$

We subtract (3.21) from the first equation of (3.15) to obtain:

$$
\begin{aligned}
&\Big(\frac{e^{\pm}_{n+1}-e^{\pm}_{n}}{\tau_n}, v^h\Big) + \nu^{+}\big(\nabla(e^{\pm}_{n+1}+e^{\pm}_{n})/2, \nabla v^h\big) + \nu^{-}\big(\nabla(e^{\mp}_{n+1}+e^{\mp}_{n})/2, \nabla v^h\big)\\
&\qquad \mp \mathcal{N}\big(B_\circ, (e^{\mp}_{n+1}+e^{\mp}_{n})/2, v^h\big)\\
&= \nu^{+}\big(\nabla z^{\pm}(t_{n+1/2}) - \nabla((z^{\pm}(t_{n+1})+z^{\pm}(t_n))/2), \nabla v^h\big)\\
&\quad + \nu^{-}\big(\nabla z^{\mp}(t_{n+1/2}) - \nabla((z^{\mp}(t_{n+1})+z^{\mp}(t_n))/2), \nabla v^h\big)\\
&\quad \pm \mathcal{N}\big(B_\circ, (z^{\pm}(t_{n+1})+z^{\pm}(t_n))/2 - z^{\pm}(t_{n+1/2}), v^h\big) - \big(p^{\pm}(t_{n+1/2}) - q^h, \nabla\cdot v^h\big)\\
&\quad + \mathcal{N}\big(z^{\mp}(t_{n+1/2}), z^{\pm}(t_{n+1/2}), v^h\big) - \mathcal{N}\big(z^{\mp,h}_{n+1/2}, z^{\pm,h}_{n+1/2}, v^h\big)\\
&\quad - \Big(\frac{z^{\pm}(t_{n+1})-z^{\pm}(t_n)}{\tau_n} - z^{\pm}_t(t_{n+1/2}), v^h\Big), \quad \forall (v^h, q^h) \in V^h \times Q^h.
\end{aligned}
$$

By the decomposition of the error $e^{\pm}_n$ (3.22), the above can be equivalently written as

$$
\begin{aligned}
&\Big(\frac{\xi^{\pm,h}_{n+1}-\xi^{\pm,h}_{n}}{\tau_n}, v^h\Big) + \nu^{+}\big(\nabla \xi^{\pm,h}_{n+1/2}, \nabla v^h\big) + \nu^{-}\big(\nabla \xi^{\mp,h}_{n+1/2}, \nabla v^h\big) \mp \mathcal{N}\big(B_\circ, \xi^{\pm,h}_{n+1/2}, v^h\big)\\
&= -\Big(\frac{\eta^{\pm}(t_{n+1})-\eta^{\pm}(t_n)}{\tau_n}, v^h\Big) - \nu^{+}\big(\nabla \eta^{\pm}_{n+1/2}, \nabla v^h\big) - \nu^{-}\big(\nabla \eta^{\mp}_{n+1/2}, \nabla v^h\big) \pm \mathcal{N}\big(B_\circ, \eta^{\pm}_{n+1/2}, v^h\big)\\
&\quad + \nu^{+}\big(\nabla z^{\pm}(t_{n+1/2}) - \nabla z^{\pm}_{n+1/2}, \nabla v^h\big) + \nu^{-}\big(\nabla z^{\mp}(t_{n+1/2}) - \nabla z^{\mp}_{n+1/2}, \nabla v^h\big)\\
&\quad - \big(p^{\pm}(t_{n+1/2}) - q^h, \nabla\cdot v^h\big) - \Big(\frac{z^{\pm}(t_{n+1})-z^{\pm}(t_n)}{\tau_n} - z^{\pm}_t(t_{n+1/2}), v^h\Big)\\
&\quad \pm \mathcal{N}\big(B_\circ, z^{\pm}_{n+1/2} - z^{\pm}(t_{n+1/2}), v^h\big) + \mathcal{N}\big(z^{\mp}(t_{n+1/2}), z^{\pm}(t_{n+1/2}), v^h\big)\\
&\quad - \mathcal{N}\big(z^{\mp,h}_{n+1/2}, z^{\pm,h}_{n+1/2}, v^h\big).
\end{aligned}
$$

We set $v^h = \xi^{\pm,h}_{n+1/2}$ and use skew-symmetry of $\mathcal{N}$ to obtain

$$
\begin{aligned}
&\frac{1}{2\tau_n}\big(\|\xi^{\pm,h}_{n+1}\|^2 - \|\xi^{\pm,h}_{n}\|^2\big) + \nu^{+}\|\nabla \xi^{\pm,h}_{n+1/2}\|^2 + \nu^{-}\big(\nabla \xi^{\mp,h}_{n+1/2}, \nabla \xi^{\pm,h}_{n+1/2}\big) \qquad (3.23)\\
&= -\Big(\frac{\eta^{\pm}(t_{n+1})-\eta^{\pm}(t_n)}{\tau_n}, \xi^{\pm,h}_{n+1/2}\Big) - \nu^{+}\big(\nabla \eta^{\pm}_{n+1/2}, \nabla \xi^{\pm,h}_{n+1/2}\big) - \nu^{-}\big(\nabla \eta^{\mp}_{n+1/2}, \nabla \xi^{\pm,h}_{n+1/2}\big)\\
&\quad + \nu^{+}\big(\nabla z^{\pm}(t_{n+1/2}) - \nabla z^{\pm}_{n+1/2}, \nabla \xi^{\pm,h}_{n+1/2}\big) + \nu^{-}\big(\nabla z^{\mp}(t_{n+1/2}) - \nabla z^{\mp}_{n+1/2}, \nabla \xi^{\pm,h}_{n+1/2}\big)\\
&\quad - \big(p^{\pm}(t_{n+1/2}) - q^h, \nabla\cdot \xi^{\pm,h}_{n+1/2}\big) - \Big(\frac{z^{\pm}(t_{n+1})-z^{\pm}(t_n)}{\tau_n} - z^{\pm}_t(t_{n+1/2}), \xi^{\pm,h}_{n+1/2}\Big)\\
&\quad \pm \mathcal{N}\big(B_\circ, \eta^{\pm}_{n+1/2}, \xi^{\pm,h}_{n+1/2}\big) \pm \mathcal{N}\big(B_\circ, z^{\pm}_{n+1/2} - z^{\pm}(t_{n+1/2}), \xi^{\pm,h}_{n+1/2}\big)\\
&\quad + \mathcal{N}\big(z^{\mp}(t_{n+1/2}), z^{\pm}(t_{n+1/2}), \xi^{\pm,h}_{n+1/2}\big) - \mathcal{N}\big(z^{\mp}_{n+1/2}, z^{\pm}_{n+1/2}, \xi^{\pm,h}_{n+1/2}\big)\\
&\quad + \mathcal{N}\big(z^{\mp}_{n+1/2}, z^{\pm}_{n+1/2}, \xi^{\pm,h}_{n+1/2}\big) - \mathcal{N}\big(z^{\mp,h}_{n+1/2}, z^{\pm,h}_{n+1/2}, \xi^{\pm,h}_{n+1/2}\big) = I_1 + \cdots + I_{11}.
\end{aligned}
$$

Now we address each of the terms on the right hand side of (3.23). By the Cauchy-Schwarz, Poincaré and Young's inequalities, the estimate of the Stokes projection (2.6), the approximations in (2.2) and Hölder's inequality we have

$$
I_1 = -\Big(\frac{\eta^{\pm}(t_{n+1})-\eta^{\pm}(t_n)}{\tau_n}, \xi^{\pm,h}_{n+1/2}\Big) \tag{3.24}
$$

$$
\begin{aligned}
&\leq \frac{C(\epsilon)}{\tau_n^2 \nu^*}\|\eta^{\pm}(t_{n+1})-\eta^{\pm}(t_n)\|^2+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2\\
&\leq \frac{C(\epsilon)h^{2r+2}}{\tau_n^2\nu^*}\|z^{\pm}(t_{n+1})-z^{\pm}(t_n)\|_{r+1}^2+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2,\\
&\leq \frac{C(\epsilon)h^{2r+2}}{\tau_n\nu^*}\int_{t_n}^{t_{n+1}}\|z^{\pm}\|_{r+1}^2dt+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2,
\end{aligned}
$$

where $\epsilon>0$ is a constant to be chosen later, and $C(\epsilon)>0$ is a constant depending on ϵ. Using the Cauchy-Schwarz, Young's inequality, the estimate (2.6), the approximations (2.2) and (3.16) we also get

$$
\begin{aligned}
I_2 &= -\nu^+\big(\nabla\eta_{n+1/2}^{\pm},\nabla\xi_{n+1/2}^{\pm,h}\big)\leq C\nu^+h^r\|z_{n+1/2}^{\pm}\|_{r+1}\|\nabla\xi_{n+1/2}^{\pm,h}\| \qquad (3.25)\\
&\leq \frac{C(\epsilon)(\nu^+)^2h^{2r}}{\nu^*}\big(\|z_{n+1/2}^{\pm}-z(t_{n+1/2})\|_{r+1}^2+\|z(t_{n+1/2})\|_{r+1}^2\big)+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2\\
&\leq \frac{C(\epsilon)(\nu^+)^2h^{2r}}{\nu^*}\Big(\tau_{\max}^3\int_{t_n}^{t_{n+1}}\|z_{tt}^{\pm}\|_{r+1}^2dt+\|z^{\pm}(t_{n+1/2})\|_{r+1}^2\Big)+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_3 &= -\nu^-\big(\nabla\eta_{n+1/2}^{\mp},\nabla\xi_{n+1/2}^{\pm,h}\big) \qquad (3.26)\\
&\leq \frac{C(\epsilon)(\nu^-)^2h^{2r}}{\nu^*}\Big(\tau_{\max}^3\int_{t_n}^{t_{n+1}}\|z_{tt}^{\mp}\|_{r+1}^2dt+\|z^{\mp}(t_{n+1/2})\|_{r+1}^2\Big)+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2.
\end{aligned}
$$

By the Cauchy-Schwarz and Young inequalities, using (3.16) we have

$$
\begin{aligned}
I_4 &= \nu^+\big(\nabla z^{\pm}(t_{n+1/2})-\nabla z_{n+1/2}^{\pm},\nabla\xi_{n+1/2}^{\pm,h}\big) \qquad (3.27)\\
&\leq \nu^+\|\nabla z^{\pm}(t_{n+1/2})-\nabla z_{n+1/2}^{\pm}\|\|\nabla\xi_{n+1/2}^{\pm,h}\|\\
&\leq \frac{C(\epsilon)(\nu^+)^2\tau_{\max}^3}{\nu^*}\int_{t_n}^{t_{n+1}}\|\nabla z_{tt}^{\pm}\|^2dt+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_5 &= \nu^-\big(\nabla z^{\mp}(t_{n+1/2})-\nabla z_{n+1/2}^{\mp},\nabla\xi_{n+1/2}^{\pm,h}\big) \qquad (3.28)\\
&\leq \nu^-\|\nabla z^{\mp}(t_{n+1/2})-\nabla z_{n+1/2}^{\mp}\|\|\nabla\xi_{n+1/2}^{\pm,h}\|\\
&\leq \frac{C(\epsilon)(\nu^-)^2\tau_{\max}^3}{\nu^*}\int_{t_n}^{t_{n+1}}\|\nabla z_{tt}^{\mp}\|^2dt+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2.
\end{aligned}
$$

Now we set q^h to be L^2-projection of $p^{\pm}(t_{n+1/2})$, and use the Cauchy-Schwarz inequality, (2.2) and Young's inequality to obtain

$$
\begin{aligned}
I_6 &= -\big(p^{\pm}(t_{n+1/2})-q^h,\nabla\cdot\xi_{n+1/2}^{\pm,h}\big)\leq\sqrt{d}\|p^{\pm}(t_{n+1/2})-q^h\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \qquad (3.29)\\
&\leq \frac{C(\epsilon)h^{2s+2}}{\nu^*}\|p^{\pm}(t_{n+1/2})\|_{s+1}^2+\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2.
\end{aligned}
$$

Using the Cauchy-Schwarz and Poincaré inequalities, (3.17) and Young's inequality we obtain

$$
I_7 = -\Big(\frac{z^{\pm}(t_{n+1})-z^{\pm}(t_n)}{\tau_n}-z_t^{\pm}(t_{n+1/2}),\xi_{n+1/2}^{\pm,h}\Big) \qquad (3.30)
$$

$$\le \frac{C(\epsilon)\tau_{\max}^3}{\nu^*}\int_{t_n}^{t_{n+1}} \|\nabla z_{ttt}^{\mp}\|^2 dt + \epsilon\nu^*\|\nabla \xi_{n+1/2}^{\pm,h}\|^2.$$

By Holder's inequality and the Sobolev embedding theorem, the Poincaré inequality, using $\nabla B_\circ = 0$, the approximations (2.6), (2.2) and (3.16), we have

$$\begin{aligned} I_8 &= \pm\mathcal{N}\big(B_\circ, \eta^{\pm}(t_{n+1/2}), \xi_{n+1/2}^{\pm,h}\big) \\ &\le C\big(\|B_\circ\|\|B_\circ\|_1\big)^{1/2}\|\nabla\eta_{n+1/2}^{\pm}\|\|\xi_{n+1/2}^{\pm,h}\|_1 \\ &\le C\|B_\circ\|\|\nabla\eta_{n+1/2}^{\pm}\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\le Ch^r\|B_\circ\|\big(\|z_{n+1/2}^{\pm} - z^{\pm}(t_{n+1/2})\|_{r+1} + \|z^{\pm}(t_{n+1/2})\|_{r+1}\big)\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\le \frac{C(\epsilon)h^{2r}}{\nu^*}\|B_\circ\|^2\big(\tau_{\max}^3\int_{t_n}^{t_{n+1}}\|z_{tt}^{\pm}\|_{r+1}^2 dt + \|z^{\pm}(t_{n+1/2})\|_{r+1}^2\big) + \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} I_9 &= \pm\mathcal{N}\big(B_\circ, z_{n+1/2}^{\pm} - z^{\pm}(t_{n+1/2}), \xi_{n+1/2}^{\pm,h}\big) \\ &\le C\|B_\circ\|\|\nabla z_{n+1/2}^{\pm} - \nabla z^{\pm}(t_{n+1/2})\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\le \frac{C(\epsilon)\tau_{\max}^3}{\nu^*}\|B_\circ\|^2\int_{t_n}^{t_{n+1}}\|\nabla z_{tt}^{\pm}\|^2 dt + \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2. \end{aligned} \tag{3.32}$$

Similarly to (3.32) we obtain

$$\begin{aligned} I_{10} &= \mathcal{N}\big(z^{\mp}(t_{n+1/2}), z^{\pm}(t_{n+1/2}), \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(z_{n+1/2}^{\mp}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \\ &\le \mathcal{N}\big(z^{\mp}(t_{n+1/2}) - z_{n+1/2}^{\mp}, z^{\pm}(t_{n+1/2}), \xi_{n+1/2}^{\pm,h}\big) + \mathcal{N}\big(z_{n+1/2}^{\mp}, z^{\pm}(t_{n+1/2}) - z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \\ &\le C\|\nabla(z^{\mp}(t_{n+1/2}) - z_{n+1/2}^{\mp})\|\|\nabla z^{\pm}(t_{n+1/2})\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\quad + C\|\nabla z_{n+1/2}^{\mp}\|\|\nabla(z^{\pm}(t_{n+1/2}) - z_{n+1/2}^{\pm})\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\le \frac{C(\epsilon)\tau_{\max}^3}{\nu^*}\Big(|||z^{\mp}|||_{\infty,1}^2\int_{t_n}^{t_{n+1}}\|\nabla z_{tt}^{\pm}\|^2 dt + |||z^{\pm}|||_{\infty,1,1/2}^2\int_{t_n}^{t_{n+1}}\|\nabla z_{tt}^{\mp}\|^2 dt\Big) \\ &\quad + \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2. \end{aligned} \tag{3.33}$$

By the skew-symmetriy property (2.7) we can write I_{11} as

$$\begin{aligned} I_{11} &= \mathcal{N}\big(z_{n+1/2}^{\mp}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(z_{n+1/2}^{\mp,h}, z_{n+1/2}^{\pm,h}, \xi_{n+1/2}^{\pm,h}\big) \\ &= -\mathcal{N}\big(\xi_{n+1/2}^{\mp,h}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(\eta_{n+1/2}^{\mp}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(\xi_{n+1/2}^{\mp,h}, \eta_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \\ &\quad - \mathcal{N}\big(\eta_{n+1/2}^{\mp}, \eta_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(z_{n+1/2}^{\mp}, \eta_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) = I_{11}^{(1)} + \cdots + I_{11}^{(5)}. \end{aligned}$$

Now we evaluate each term in the right hand side above as follows. By Hölder's inequality, (2.9), Poincaré and Young's inequality we have

$$\begin{aligned} I_{11}^{(1)} &= -\mathcal{N}\big(\xi_{n+1/2}^{\mp,h}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \\ &\le \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2 + \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\mp,h}\|^2 + \frac{(3(d-1))^{4d/3}C_P^{2(3-d)}}{64(\epsilon\nu^*)^3 d^{7d/3}}\|\nabla z_{n+1/2}^{\pm}\|^4\|\xi_{n+1/2}^{\mp,h}\|^2. \end{aligned} \tag{3.34}$$

Similarly,

$$\begin{aligned} I_{11}^{(2)} &= -\mathcal{N}\big(\eta_{n+1/2}^{\mp}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \\ &\leq \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2 + \frac{C(\epsilon)}{\nu^*}\Big(\|\nabla z_{n+1}^{\pm}\|^2 + \|\nabla z_n^{\pm}\|^2\Big)\|\nabla\eta_{n+1/2}^{\mp}\|^2. \end{aligned} \tag{3.35}$$

Using Hölder's inequality, (2.9), Poincaré and Young's inequality, (2.6), (2.2) and the inverse inequality (2.3) we get

$$\begin{aligned} I_{11}^{(3)} &= -\mathcal{N}\big(\xi_{n+1/2}^{\mp,h}, \eta_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \\ &\leq \frac{(3(d-1))^{d/3}}{d^{7d/12}} C_P^{(3-d)/2}\big(\|\xi_{n+1/2}^{\mp,h}\|\|\nabla\xi_{n+1/2}^{\mp,h}\|\big)^{1/2}\|\nabla\eta_{n+1/2}^{\pm}\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\leq \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2 + \frac{(3(d-1))^{2d/3}C_P^{3-d}C_I(C_A^{1,0})^2}{\epsilon\nu^* d^{7d/6}}\big(1+\frac{1}{\beta_{\mathtt{is}}}\big)^2 h|z_{n+1/2}^{\pm}|_2\|\xi_{n+1/2}^{\mp,h}\|^2, \end{aligned} \tag{3.36}$$

also

$$\begin{aligned} I_{11}^{(4)} &= -\mathcal{N}\big(\eta_{n+1/2}^{\mp}, \eta_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \leq C\big(\|\nabla\eta_{n+1}^{\mp}\| + \|\nabla\eta_n^{\mp}\|\big)\|\nabla\eta_{n+1/2}^{\pm}\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\leq \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2 + \frac{C(\epsilon)}{\nu^*}\big(\|\nabla z_{n+1}^{\mp}\|^2 + \|\nabla z_n^{\mp}\|^2\big)\|\nabla\eta_{n+1/2}^{\pm}\|^2, \end{aligned} \tag{3.37}$$

and finally

$$\begin{aligned} I_{11}^{(5)} &= -\mathcal{N}\big(z_{n+1/2}^{\mp}, \eta_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) \leq C\big(\|\nabla z_{n+1}^{\mp}\| + \|\nabla z_n^{\mp}\|\big)\|\nabla\eta_{n+1/2}^{\pm}\|\|\nabla\xi_{n+1/2}^{\pm,h}\| \\ &\leq \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2 + \frac{C(\epsilon)}{\nu^*}\big(\|\nabla z_{n+1}^{\mp}\|^2 + \|\nabla z_n^{\mp}\|^2\big)\|\nabla\eta_{n+1/2}^{\pm}\|^2. \end{aligned} \tag{3.38}$$

Therefore, we combine (3.34)-(3.38) to obtain

$$\begin{aligned} I_{11} &= \mathcal{N}\big(z_{n+1/2}^{\mp}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(z_{n+1/2}^{\mp,h}, z_{n+1/2}^{\pm,h}, \xi_{n+1/2}^{\pm,h}\big) \\ &\leq \frac{(3(d-1))^{2d/3}C_P^{3-d}C_I(C_A^{1,0})^2}{\epsilon\nu^* d^{7d/6}}\big(1+\frac{1}{\beta_{\mathtt{is}}}\big)^2 h|z_{n+1/2}^{\pm}|_2\|\xi_{n+1/2}^{\mp,h}\|^2 \\ &\quad + \frac{(3(d-1))^{4d/3}C_P^{2(3-d)}}{64(\epsilon\nu^*)^3 d^{7d/3}}\|\nabla z_{n+1/2}^{\pm}\|^4\|\xi_{n+1/2}^{\mp,h}\|^2 \\ &\quad + \frac{C(\epsilon)}{\nu^*}\Big(\|\nabla z_{n+1}^{\pm}\|^2 + \|\nabla z_n^{\pm}\|^2\Big)\|\nabla\eta_{n+1/2}^{\mp}\|^2 \\ &\quad + \frac{C(\epsilon)}{\nu^*}\big(\|\nabla z_{n+1}^{\mp}\|^2 + \|\nabla z_n^{\mp}\|^2\big)\|\nabla\eta_{n+1/2}^{\pm}\|^2 + 5\epsilon\nu^*\|\nabla\xi_{n+1/2}^{\pm,h}\|^2 + \epsilon\nu^*\|\nabla\xi_{n+1/2}^{\mp,h}\|^2. \end{aligned} \tag{3.39}$$

Now we use (2.6), (2.2), the triangle inequality and (3.16) to evaluate

$$\|\nabla\eta_{n+1/2}^{\pm}\|^2 \leq Ch^{2r}\|z_{n+1/2}^{\pm}\|_{r+1}^2 \leq Ch^{2r}\tau_{\max}^3\int_{t_n}^{t_{n+1}}\|z_{tt}^{\pm}\|_{r+1}^2 dt + Ch^{2r}\|z^{\pm}(t_{n+1/2})\|_{r+1}^2,$$

to get

$$I_{11} = \mathcal{N}\big(z_{n+1/2}^{\mp}, z_{n+1/2}^{\pm}, \xi_{n+1/2}^{\pm,h}\big) - \mathcal{N}\big(z_{n+1/2}^{\mp,h}, z_{n+1/2}^{\pm,h}, \xi_{n+1/2}^{\pm,h}\big) \tag{3.40}$$

$$
\begin{aligned}
&\leq \frac{(3(d-1))^{2d/3}C_P^{3-d}C_I(C_A^{1,0})^2}{\epsilon\nu^* d^{7d/6}}\big(1+\frac{1}{\beta_{\tt is}}\big)^2 h|z^{\pm}_{n+1/2}|_2\|\xi^{\mp,h}_{n+1/2}\|^2\\
&\quad+\frac{(3(d-1))^{4d/3}C_P^{2(3-d)}}{64(\epsilon\nu^*)^3 d^{7d/3}}\|\nabla z^{\pm}_{n+1/2}\|^4\|\xi^{\mp,h}_{n+1/2}\|^2+5\epsilon\nu^*\|\nabla\xi^{\pm,h}_{n+1/2}\|^2+\epsilon\nu^*\|\nabla\xi^{\mp,h}_{n+1/2}\|^2\\
&\quad+\frac{C(\epsilon)h^{2r}}{\nu^*}|||\nabla z^{\mp}|||^2_{\infty,0}\Big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\pm}_{tt}\|^2_{r+1}dt+\|z^{\pm}(t_{n+1/2})\|^2_{r+1}\Big)\\
&\quad+\frac{C(\epsilon)h^{2r}}{\nu^*}|||\nabla z^{\pm}|||^2_{\infty,0}\Big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\mp}_{tt}\|^2_{r+1}dt+\|z^{\mp}(t_{n+1/2})\|^2_{r+1}\Big).
\end{aligned}
$$

We combine (3.23)-(3.33) and (3.40) to obtain the following bound for (3.23)

$$
\begin{aligned}
&\frac{1}{2\tau_n}\big(\|\xi^{\pm,h}_{n+1}\|^2-\|\xi^{\pm,h}_{n}\|^2\big)+\nu^+\|\nabla\xi^{\pm,h}_{n+1/2}\|^2-\frac{|\nu^-|}{2}\big(\|\nabla\xi^{\mp,h}_{n+1/2}\|^2+\|\nabla\xi^{\pm,h}_{n+1/2}\|^2\big)\\
&\quad+\frac{|\nu^-|}{2}\big\|\nabla\xi^{\mp,h}_{n+1/2}+\mathrm{sign}(\nu^-)\nabla\xi^{\pm,h}_{n+1/2}\big\|^2-15\epsilon\nu^*\|\nabla\xi^{\pm,h}_{n+1/2}\|^2-\epsilon\nu^*\|\nabla\xi^{\mp,h}_{n+1/2}\|^2\\
&\leq \frac{(3(d-1))^{2d/3}C_P^{3-d}C_I(C_A^{1,0})^2}{\epsilon\nu^* d^{7d/6}}\big(1+\frac{1}{\beta_{\tt is}}\big)^2 h|z^{\pm}_{n+1/2}|_2\|\xi^{\mp,h}_{n+1/2}\|^2\\
&\quad+\frac{(3(d-1))^{4d/3}C_P^{2(3-d)}}{64(\epsilon\nu^*)^3 d^{7d/3}}\|\nabla z^{\pm}_{n+1/2}\|^4\|\xi^{\mp,h}_{n+1/2}\|^2+\frac{C(\epsilon)h^{2r+2}}{\tau_n\nu^*}\int_{t_n}^{t_{n+1}}\|z^{\pm}\|^2_{r+1}dt\\
&\quad+\frac{C(\epsilon)(\nu^+)^2h^{2r}}{\nu^*}\Big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\pm}_{tt}\|^2_{r+1}dt+\|z^{\pm}(t_{n+1/2})\|^2_{r+1}\Big)\\
&\quad+\frac{C(\epsilon)(\nu^-)^2h^{2r}}{\nu^*}\Big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\mp}_{tt}\|^2_{r+1}dt+\|z^{\mp}(t_{n+1/2})\|^2_{r+1}\Big)\\
&\quad+\frac{C(\epsilon)(\nu^-)^2\tau^3_{\max}}{\nu^*}\Big(\int_{t_n}^{t_{n+1}}\|\nabla z^{\mp}_{tt}\|^2dt+\int_{t_n}^{t_{n+1}}\|\nabla z^{\mp}_{tt}\|^2dt\Big)+\frac{C(\epsilon)h^{2s+2}}{\nu^*}\|p^{\pm}(t_{n+1/2})\|^2_{s+1}\\
&\quad+\frac{C(\epsilon)\tau^3_{\max}}{\nu^*}\int_{t_n}^{t_{n+1}}\|\nabla z^{\mp}_{ttt}\|^2dt+\frac{C(\epsilon)\tau^3_{\max}}{\nu^*}\|B_\circ\|^2\int_{t_n}^{t_{n+1}}\|\nabla z^{\pm}_{tt}\|^2dt\\
&\quad+\frac{C(\epsilon)h^{2r}}{\nu^*}\|B_\circ\|^2\big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\pm}_{tt}\|^2_{r+1}dt+\|z^{\pm}(t_{n+1/2})\|^2_{r+1}\big)\\
&\quad+\frac{C(\epsilon)\tau^3_{\max}}{\nu^*}\Big(|||z^{\mp}|||^2_{\infty,1}\int_{t_n}^{t_{n+1}}\|\nabla z^{\pm}_{tt}\|^2dt+|||z^{\pm}|||^2_{\infty,1,1/2}\int_{t_n}^{t_{n+1}}\|\nabla z^{\mp}_{tt}\|^2dt\Big)\\
&\quad+\frac{C(\epsilon)h^{2r}}{\nu^*}|||\nabla z^{\mp}|||^2_{\infty,0}\Big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\pm}_{tt}\|^2_{r+1}dt+\|z^{\pm}(t_{n+1/2})\|^2_{r+1}\Big)\\
&\quad+\frac{C(\epsilon)h^{2r}}{\nu^*}|||\nabla z^{\pm}|||^2_{\infty,0}\Big(\tau^3_{\max}\int_{t_n}^{t_{n+1}}\|z^{\mp}_{tt}\|^2_{r+1}dt+\|z^{\mp}(t_{n+1/2})\|^2_{r+1}\Big).
\end{aligned}
\tag{3.41}
$$

Now adding the two equations above, and summing over n from 0 to $N-1$ yields

$$
\begin{aligned}
&\Big[1-\Big(\frac{C_1^*hZ^1_{N-1}}{\epsilon}+\frac{C_2^*Z^2_{N-1}}{\epsilon^3}\Big)\tau_{N-1}\Big]\big(\|\xi^{+,h}_N\|^2+\|\xi^{-,h}_N\|^2\big)\\
&\quad+2\nu^*\big(1-16\epsilon\big)\sum_{n=1}^{N-1}\tau_n\big(\|\nabla\xi^{+,h}_{n+1/2}\|^2+\|\nabla\xi^{-,h}_{n+1/2}\|^2\big)
\end{aligned}
\tag{3.42}
$$

$$\leq \sum_{n=1}^{N-1}\Big(\frac{C_1^* h Z_n^1}{\epsilon}+\frac{C_2^* Z_n^2}{\epsilon^3}\Big)\tau_n\big(\|\xi_n^{+,h}\|^2+\|\xi_n^{-,h}\|^2\big)+C(\epsilon)F\big(h^{2r},h^{2s+2},\tau_{\max}^4,\|B_\circ\|^2\big).$$

Here we impose the positivity conditions

$$\begin{aligned}
&1-\Big(\frac{C_1^* h Z_{N-1}^1}{\epsilon}+\frac{C_2^* Z_{N-1}^2}{\epsilon^3}\Big)\tau_{N-1}>0,\\
&1-16\epsilon>0,
\end{aligned}$$

which imply the time step restrictions (3.19). The discrete Grönwall inequality [24, p. 369] applied to (3.42) gives

$$\begin{aligned}
&\big(\|\xi_N^{+,h}\|^2+\|\xi_N^{-,h}\|^2\big)+\nu^*\sum_{n=1}^{N-1}\tau_n\big(\|\nabla\xi_{n+1/2}^{+,h}\|^2+\|\nabla\xi_{n+1/2}^{-,h}\|^2\big) \qquad (3.43)\\
&\leq C\exp\Big(C\sum_{n=1}^{N-1}\big(C_1^* h Z_n^1+C_2^* Z_n^2\big)\tau_n\Big)F\big(h^{2r},h^{2s+2},\tau_{\max}^4,\|B_\circ\|^2\big).
\end{aligned}$$

We use the triangle inequality, (2.6), (2.2), (3.16) and (3.43) to obtain

$$\|e_N^{\pm}\|^2\leq 2\|\xi_N^{\pm,h}\|^2+2\|\eta_N^{\pm}\|^2\leq 2\|\xi_N^{\pm,h}\|^2+Ch^{2r}|||z^{\pm}|||_{\infty,r+1}^2, \qquad (3.44)$$

and

$$\begin{aligned}
&\nu^*\sum_{n=1}^{N-1}\tau_n\|\nabla(e_{n+1}^{\pm}+e_n^{\pm})/2\|^2\leq 2\nu^*\sum_{n=1}^{N-1}\tau_n\big(\|\nabla\xi_{n+1/2}^{\pm,h}\|^2+\|\nabla\eta_{n+1/2}^{\pm}\|^2\big)\\
&\leq 2\nu^*\sum_{n=1}^{N-1}\tau_n\|\nabla\xi_{n+1/2}^{\pm,h}\|^2+C\nu^* h^{2r}\tau_n\sum_{n=1}^{N-1}\big(\|z_{n+1/2}^{\pm}-z^{\pm}(t_{n+1/2})\|_{r+1}^2+\|z^{\pm}(t_{n+1/2})\|_{r+1}^2\big)\\
&\leq 2\nu^*\sum_{n=1}^{N-1}\tau_n\|\nabla\xi_{n+1/2}^{\pm,h}\|^2+C\nu^* h^{2r}\big(\tau_{\max}^4\|z^{\pm}\|_{2,r+1}^2+|||z^{\pm}|||_{2,r+1,1/2}^2\big).
\end{aligned}$$

Finally, we apply (2.6), (2.2) in (3.44) and use (3.42) to derive (3.20). ∎

4. Time Adaptivity. To construct the time adaptive mechanism for the fully-coupled implicit monolithic method (1.4) with the BE partitioned iteration (1.6) (PIM algorithm), we estimate its local truncation error (LTE) by the fully-explicit two-step AB2-like scheme[1] and adopt the improved time step controller proposed in [22] to adjust time steps. he AB2-like scheme for the ordinary differential equation $y'=f(t,y)$ is

$$\begin{aligned}
y_{n+1}^{\texttt{AB2-like}}=y_n+\frac{t_{n+1}-t_n}{2(t_{n-1/2}-t_{n-3/2})}\Big[&(t_{n+1}+t_n-2t_{n-3/2})g^{\texttt{Mid}}(t_{n-1/2},y_{n-1/2}) \qquad (4.1)\\
&-(t_{n+1}+t_n-2t_{n-1/2})g^{\texttt{Mid}}(t_{n-3/2},y_{n-3/2})\Big],
\end{aligned}$$

where $g^{\texttt{Mid}}(t_{n-1/2},y_{n-1/2})$ and $g^{\texttt{Mid}}(t_{n-3/2},y_{n-3/2})$ are calculated by the midpoint rule at time step t_n, t_{n-1} respectively, i.e.

$$g^{\texttt{Mid}}(t_{n-1/2},y_{n-1/2})=\frac{y_n-y_{n-1}}{k_{n-1}},\quad g^{\texttt{Mid}}(t_{n-3/2},y_{n-3/2})=\frac{y_{n-1}-y_{n-2}}{k_{n-2}}.$$

[1] The derivation of scheme is similar to that of two-step Adam-Bashforth (AB2) scheme, hence we call it AB2-like scheme.

From (4.1), we see that $y_{n+1}^{\texttt{AB2-like}}$ is certain interpolant of the previous three midpoint rule solutions $\{y_n, y_{n-1}, y_{n-2}\}$. Hence, the AB2-like scheme (4.1) is fully explicit. We refer to [8] for the derivation of the explicit AB2-like scheme (4.1) and the following estimator of LTE $\widehat{T}_{n+1}$ of the midpoint rule by AB2-like scheme

$$\begin{aligned}
\widehat{T}_{n+1} =& \frac{1/24}{\mathcal{R}^{(n)} - 1/24}\big(y_{n+1}^{\texttt{Mid}} - y_{n+1}^{\texttt{AB2-like}}\big),\\
\mathcal{R}^{(n)} =& \frac{1}{12}\Big[2+\frac{3}{\tau_n}\Big(1+\frac{1}{2\tau_{n-1}}\Big)\Big(1+\frac{1}{2\tau_n}\Big)+\frac{3}{2\tau_n}\Big(1+\frac{1}{\tau_n}+\frac{1}{2}\frac{1}{\tau_{n-1}}\frac{1}{\tau_n}\Big)\Big],
\end{aligned}$$

and $\tau_n = k_n/k_{n-1}$ is ratio of time step. The AB2-like solutions for the MHD system in Elsässer variables (1.3) at time t_{n+1} is

$$\begin{aligned}
z_{n+1}^{\pm,\texttt{AB2-like}} = z_n^{\pm} + \frac{t_{n+1}-t_n}{2(t_{n-1/2}-t_{n-3/2})}\Big[&(t_{n+1}+t_n-2t_{n-3/2})\frac{z_n^{\pm}-z_{n-1}^{\pm}}{k_{n-1}}\\
&-(t_{n+1}+t_n-2t_{n-1/2})\frac{z_{n-1}^{\pm}-z_{n-2}^{\pm}}{k_{n-2}}\Big],
\end{aligned} \tag{4.2}$$

where $\{z_n^{\pm}, z_{n-1}^{\pm}, z_{n-2}^{\pm}\}$ are three previous solutions by the PIM algorithm in (1.4). The corresponding estimator of LTE for the PIM algorithm (1.4) is

$$\widehat{T}_{n+1} = \max\left\{\frac{|1/24|\big\|z_{n+1}^{+} - z_{n+1}^{+,\texttt{AB2-like}}\big\|}{|\mathcal{R}^{(n)}-1/24|\big\|z_{n+1}^{+}\big\|}, \frac{|1/24|\big\|z_{n+1}^{-} - z_{n+1}^{-,\texttt{AB2-like}}\big\|}{|\mathcal{R}^{(n)}-1/24|\big\|z_{n+1}^{-}\big\|}\right\}. \tag{4.3}$$

The improved time step controller is

$$k_{n+1} = k_n \cdot \min\Big\{1.5, \max\Big\{0.2, \kappa\big(\texttt{Tol}/\|\widehat{T}_{n+1}\|\big)^{\frac{1}{3}}\Big\}\Big\}, \tag{4.4}$$

where $\kappa \in (0,1]$ is the safety factor and $\texttt{Tol}$ is the required tolerance for the LTE. The step controller in (4.4) increases the step size if the estimator for LTE $\widehat{T}_{n+1}$ is small with respect to $\texttt{Tol}$ for efficiency. Meanwhile the controller bounds k_{n+1} between $0.2k_n$ and $1.5k_n$ for the robustness of computing. We summarize the time adaptivity mechanism in the following algorithm.

Algorithm 1: Adaptivity of the PIM algorithm (1.4)

Input: tolerance `Tol`, previous solutions $z_n^{\pm}, z_{n-1}^{\pm}, z_{n-2}^{\pm}$ by the PIM algorithm in (1.4), current time step k_n, two previous time step k_{n-1}, k_{n-2}, safety factor κ, ;

compute the current solution $z_{n+1}^{\pm,\texttt{PIM}}$ and $p_{n+1}^{\pm}$ by the refactorization process ;

compute the AB2-like solution $z_{n+1}^{\pm,\texttt{AB2-like}}$ by (4.2) ;

use k_n, k_{n-1}, k_{n-2} to update τ_n, τ_{n-1} ;

compute $\widehat{T}_{n+1}$ by (4.3) ; `// relative estimator`

if $\widehat{T}_{n+1} < \mathrm{Tol}$ **then**

$z_{n+1}^{\pm} \Leftarrow z_{n+1}^{\pm,\texttt{PIM}}$; `// accept the result`

$k_{n+1} \Leftarrow k_n \cdot \min\left\{1.5, \max\left\{0.2, \kappa\left(\frac{\texttt{Tol}}{\widehat{T}_{n+1}}\right)^{1/3}\right\}\right\}$; `// adjust step by` (4.4)

else

`// adjust current step to recompute`

$k_n \Leftarrow k_n \cdot \min\left\{1.5, \max\left\{0.2, \kappa\left(\frac{\texttt{Tol}}{\widehat{T}_{n+1}}\right)^{1/3}\right\}\right\}$;

5. Numerical Tests. For all numerical tests, we use Taylor-Hood($\mathbb{P}2-\mathbb{P}1$) finite element space for spacial discretization and software Freefem++ for programming. We apply the PIM algorithm (1.4) to all numerical tests in this section and compare it with other three commonly-used algorithms:

The first-order,backward-Euler/forward-Euler (BEFE) method

$$\begin{cases} \dfrac{z_{n+1}^{\pm}-z_n^{\pm}}{\tau_n} \mp (B_\circ\cdot\nabla) z_{n+1}^{\pm} + (z_n^{\mp}\cdot\nabla) z_{n+1}^{\pm} - \nu^+ \Delta z_{n+1}^{\pm} - \nu^- \Delta z_n^{\mp} + \nabla p_{n+1}^{\pm} = 0, \\ \qquad \nabla\cdot z_{n+1}^{\pm} = 0. \end{cases} \tag{5.1}$$

The fully-implicit BE method with BEFE iteration (BEFE-Ite)

$$\begin{cases} \dfrac{z_{n+1}^{\pm}-z_n^{\pm}}{\tau_n} \mp (B_\circ\cdot\nabla) z_{n+1}^{\pm} + (z_{n+1}^{\mp}\cdot\nabla) z_{n+1}^{\pm} - \nu^+ \Delta z_{n+1}^{\pm} - \nu^- \Delta z_{n+1}^{\mp} + \nabla p_{n+1}^{\pm} = 0, \\ \qquad \nabla\cdot z_{n+1}^{\pm} = 0, \end{cases} \tag{5.2}$$

where BEFE iteration is

$$\begin{cases} \dfrac{z_{(k)}^{\pm}-z_n^{\pm}}{\tau_n} \mp (B_\circ\cdot\nabla) z_{(k)}^{\pm} + (z_{(k-1)}^{\mp}\cdot\nabla) z_{(k)}^{\pm} - \nu^+ \Delta z_{(k)}^{\pm} - \nu^- \Delta z_{(k-1)}^{\mp} + \nabla p_{(k)}^{\pm} = 0, \\ \qquad \nabla\cdot z_{(k)}^{\pm} = 0. \end{cases}$$

The constant time-stepping second order BDF2-AB2 method [30]

$$\begin{cases} \dfrac{3z_{n+1}^{\pm}-4z_n^{\pm}+z_{n-1}^{\pm}}{\tau} \mp (B_\circ\cdot\nabla) z_{n+1}^{\pm} + \left((2z_n^{\mp}-z_{n-1}^{\mp})\cdot\nabla\right) z_{n+1}^{\pm} \\ \qquad -\nu^+ \Delta z_{n+1}^{\pm} - \nu^- \Delta\left(2z_n^{\mp}-z_{n-1}^{\mp}\right) + \nabla p_{n+1}^{\pm} = 0, \\ \nabla\cdot z_{n+1}^{\pm} = 0, \end{cases} \tag{5.3}$$

where $\tau > 0$ is the constant time step size. The stopping criterion for the k-th iteration in (1.6) and (5.2)

$$\|z^{\pm}_{(k)} - z^{\pm}_{(k-1)}\| / \|z^{\pm}_{(k)}\| \leq \texttt{Tol},$$

where `Tol` is the required tolerance.

5.1. Electrically Conducted 2D Traveling Wave. In this section we verify the rate of convergence of constant time-stepping PIM algorithm on an electrically conducted two-dimensional traveling wave problem [40] on the domain $\Omega = [0.5, 1.5]^2$. The exact solutions (in Elsässer variables) are

$$\begin{aligned}
z^+ &= \begin{bmatrix} \frac{3}{4} + \frac{1}{4}\cos(2\pi(x-t))\sin(2\pi(x-t))e^{-8\pi^2\nu t} + \frac{1}{10}(y+1)^2 e^{\nu_m t} \\ -\frac{1}{4}\sin(2\pi(x-t))\cos(2\pi(x-t))e^{-8\pi^2\nu t} + \frac{1}{10}(x+1)^2 e^{\nu_m t} \end{bmatrix}, \\
z^- &= \begin{bmatrix} \frac{3}{4} + \frac{1}{4}\cos(2\pi(x-t))\sin(2\pi(x-t))e^{-8\pi^2\nu t} - \frac{1}{10}(y+1)^2 e^{\nu_m t} \\ -\frac{1}{4}\sin(2\pi(x-t))\cos(2\pi(x-t))e^{-8\pi^2\nu t} - \frac{1}{10}(x+1)^2 e^{\nu_m t} \end{bmatrix}, \\
p &= -\frac{1}{64}\big(\cos(4\pi(x-t)) + \cos(4\pi(y-t))\big)e^{-16\pi^2\nu t}.
\end{aligned} \tag{5.4}$$

We set $\nu = \nu_m = 2.5 \times 10^{-4}$, the time step $\Delta t = h$, and the required tolerance $\texttt{Tol} = \texttt{1.e}-\texttt{6}$ for iterations at each time step. We test over three values of $B_\circ$: $(0,0)$, $(1,1)$, $(10,10)$, and simulate the problem over time interval $[0,1]$. The initial conditions and boundary conditions are given by exact solutions. The results are presented in Tables 5.1 to 5.3.

PIM algorithm					
$\Delta t = h$	$\|z^+ - z^{+,h}\|_{\infty,0}$	Rate	$\|z^- - z^{-,h}\|_{\infty,0}$	Rate	Average iteration
1/16	1.6803e − 2		2.5740e − 2		9.31
1/32	2.2706e − 3	2.8876	5.9990e − 3	2.1012	6.19
1/64	5.3284e − 4	2.0913	1.4732e − 3	2.0258	4.12
1/128	1.3197e − 4	2.0135	3.6695e − 4	2.0053	3.02
BDF2-AB2 algorithm					
$\Delta t = h$	$\|z^+ - z^{+,h}\|_{\infty,0}$	Rate	$\|z^- - z^{-,h}\|_{\infty,0}$	Rate	
1/16	4.2726e − 2		6.5633e − 2		
1/32	1.0623e − 2	2.0079	2.2256e − 2	1.5602	
1/64	9.7834e − 3	0.1188	7.9360e − 3	1.4877	
1/128	5.0295e − 3	0.9600	3.2317e − 3	1.2961	

Table 5.1: Error and rate of convergence with $B_\circ = (0,0)$

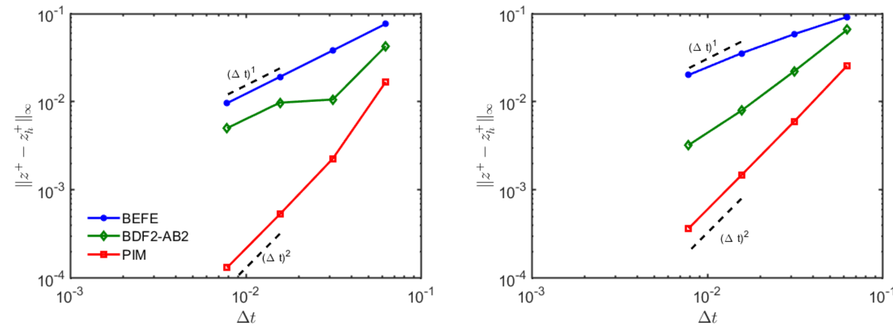

Fig. 5.1: Log-log plot of the errors as a function of time step Δt.

PIM algorithm					
$\Delta t = h$	$\|z^+ - z^{+,h}\|_{\infty,0}$	Rate	$\|z^- - z^{-,h}\|_{\infty,0}$	Rate	Average iteration
1/16	1.0656e − 2		1.5860e − 2		9.44
1/32	2.0089e − 3	2.4072	3.3290e − 3	2.2523	6.09
1/64	5.1407e − 4	1.9663	7.9070e − 4	2.0739	4.05
1/128	1.2962e − 4	1.9877	1.9466e − 4	2.0739	3.01
BDF2-AB2 algorithm					
$\Delta t = h$	$\|z^+ - z^{+,h}\|_{\infty,0}$	Rate	$\|z^- - z^{-,h}\|_{\infty,0}$	Rate	
1/16	2.5927e − 2		4.0365e − 2		
1/32	5.1292e − 3	2.3376	1.0111e − 2	1.9972	
1/64	1.2879e − 3	1.9937	2.4751e − 3	2.0304	
1/128	3.2148e − 4	2.0023	6.0824e − 4	2.0247	

Table 5.2: Error and rate of convergence with $B_\circ = (1,1)$

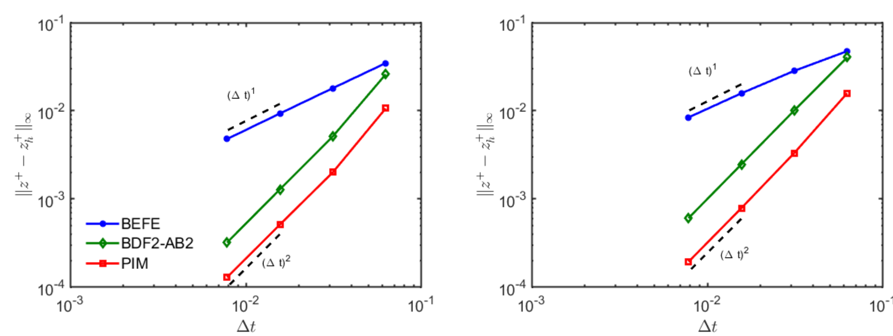

Fig. 5.2: Log-log plot of the errors as a function of time step Δt.

$\Delta t = h$	$\|z^+ - z^{+,h}\|_{\infty,0}$	Rate	$\|z^- - z^{-,h}\|_{\infty,0}$	Rate	Average iteration
PIM algorithm					
1/16	4.0128e−3		3.7020e−3		7.00
1/32	1.0391e−3	2.0952	1.0789e−3	2.1519	5.91
1/64	3.6672e−4	2.2101	3.5414e−4	2.1153	3.75
1/128	1.1268e−4	1.9066	1.1073e−4	1.8706	3.02
BDF2-AB2 algorithm					
$\Delta t = h$	$\|z^+ - z^{+,h}\|_{\infty,0}$	Rate	$\|z^- - z^{-,h}\|_{\infty,0}$	Rate	
1/16	7.2273e−3		5.6705e−3		
1/32	1.0877e−3	2.7322	1.0801e−3	2.3923	
1/64	1.9431e−4	2.4849	2.2144e−4	2.2862	
1/128	4.7211e−5	2.0412	5.5349e−5	2.0003	

Table 5.3: Error and rate of convergence with $B_\circ = (10, 10)$

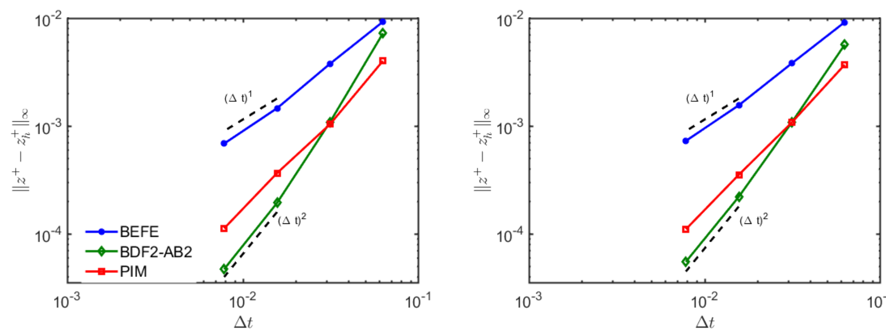

Fig. 5.3: Log-log plot of the errors as a function of time step Δt.

5.2. Hartmann Flows Problem. The 2D Hartmann Flows problem on the domain $\Omega = [0, L] \times [-1, 1]$ has the following exact solutions

$$u = (u_1(y), 0), \qquad B = (b_1(y), M), \qquad B_\circ = (0, M)$$
$$u_1(y) = \frac{G}{\nu \cdot Ha \cdot \tanh(Ha)}\Big(1 - \frac{\cosh(y \cdot Ha)}{\cosh(Ha)}\Big),$$
$$b_1(y) = \frac{G}{S}\Big(\frac{\sinh(y \cdot Ha)}{\sinh(Ha)} - y\Big), \qquad p(x, y) = -Gx - \frac{1}{2} S b_1^2(y),$$

where $M > 0$ is the parameter of external magnetic field, $Ha > 0$ is the Hartmann number and $L, G, S > 0$ are other parameters of the problems. We first verify the convergence rate of PIM algorithm by setting $L = 6$, $G = S = 1$, $Ha = 5$ and required tolerance $\texttt{Tol} = 1.\mathrm{e}-6$. We simulate the problem by the constant time-stepping PIM algorithm with four values of $B_\circ$: $(0,1)$, $(0,10)$, $(0,100)$, $(0,1000)$ over time interval $[0, 1]$.

$B_\circ = (0,1)$					
$\Delta t = h$	$\Vert z^+ - z^{+,h}\Vert_{\infty,0}$	Rate	$\Vert z^- - z^{-,h}\Vert_{\infty,0}$	Rate	Average iteration
1/16	3.5146e − 3		3.5166e − 3		2.87
1/32	4.2923e − 4	3.0336	4.2927e − 4	3.0343	2
1/64	5.1377e − 5	3.0625	5.1377e − 5	3.0627	1.37
1/128	5.7723e − 5	3.2866	6.0922e − 5	3.3215	1
$B_\circ = (0,10)$					
$\Delta t = h$	$\Vert z^+ - z^{+,h}\Vert_{\infty,0}$	Rate	$\Vert z^- - z^{-,h}\Vert_{\infty,0}$	Rate	Average iteration
1/16	8.8998e − 3		8.9317e − 3		3
1/32	7.8380e − 4	3.5052	7.8431e − 4	3.5094	2
1/64	6.7809e − 5	3.5309	6.7806e − 5	3.5319	1.89
1/128	6.8948e − 6	3.2979	6.8948e − 6	3.2978	1
$B_\circ = (0,100)$					
$\Delta t = h$	$\Vert z^+ - z^{+,h}\Vert_{\infty,0}$	Rate	$\Vert z^- - z^{-,h}\Vert_{\infty,0}$	Rate	Average iteration
1/16	3.5855e − 2		3.5794e − 2		3.07
1/32	4.9008e − 3	2.8711	4.9041e − 3	2.8676	2.19
1/64	4.4118e − 4	3.4736	4.4140e − 4	3.4738	2
1/128	3.1643e − 5	3.8014	3.1652e − 5	3.8017	2
$B_\circ = (0,1000)$					
$\Delta t = h$	$\Vert z^+ - z^{+,h}\Vert_{\infty,0}$	Rate	$\Vert z^- - z^{-,h}\Vert_{\infty,0}$	Rate	Average iteration
1/16	6.5272e − 2		6.5278e − 2		3
1/32	1.2988e − 2	2.3293	1.2982e − 2	2.3301	3
1/64	2.0417e − 3	2.6693	2.0417e − 3	2.6686	2
1/128	2.5951e − 4	2.9759	2.5959e − 4	2.9755	2

Table 5.4: Error and rate of convergence for PIM

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