Second-order in time decoupled time stepping methods for heat transfer

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Abstract

Based on a refactorization of the midpoint rule, we present and analyze a conservative, absolutely stable, adaptive, variable time-step, partitioned second-order algorithm for decoupling two-domain parabolic problems. Legacy codes originally using the backward Euler method can be upgraded to this method by inserting a single line of code. We use estimates of local truncation error for time adaptivity.

1 Introduction

This follows up on the papers [4–6] on the coupling between two-domain parabolic problem, motivated by the coupling of atmosphere and the ocean dynamics. We cite here [2, 8, 9] for work on cooling of turbine blades.

To reduce the dynamic core of the coupled atmosphere-ocean problem to its simplest form, which still retains the essential difficulty of the coupling condition, let the domain consist of two subdomains Ω_1 and Ω_2 of $\mathbb{R}^d, d = 2, 3$, with outward unit normal vectors $\hat{n}_1$ and $\hat{n}_2$, respectively, coupled across an interface I. The problem is: *given* $\nu_i > 0, f_i : [0, T] \to H^1(\Omega_i), u_i(0) \in H^1(\Omega_i)$, *and* $\alpha \in \mathbb{R}$, *find (for* $i = 1, 2$*)* $u_i : \overline{\Omega}_i \times [0, T] \to \mathbb{R}^d$ *and* $p_i : \overline{\Omega}_i \times [0, T] \to \mathbb{R}$ *satisfying*

$$u_{i,t} - \nu_i \Delta u_i = f_i \text{ in } \Omega_i, \tag{1.1}$$

$$-\nu_i \nabla u_i \cdot \hat{n}_i = \alpha(u_i - u_j) \text{ on } I \text{ for } i, j = 1, 2 \quad i \neq j, \tag{1.2}$$

$$u_i = 0 \text{ on } \Gamma_i = \partial\Omega_i \backslash I. \tag{1.3}$$

$$u_i(x, 0) = u_i^0(x) \text{ in } \Omega_i, \tag{1.4}$$

The lateral boundary conditions on Ω_i, (1.3), are not essential for our study.

We propose a fully decoupled second-order accurate algorithm, based on the refactorization of the midpoint method, namely a sequential implementation

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consisting in a half-step backward-Euler method and a linear extrapolation (see e.g., [1]).

2 Algorithm

We will restrict ourselves only to the semi-discretization in time. Let us denote the mesh points $\{t_n\}_{n\geq 0}$, the timestep τ_n, such that:

$$t_{n+1} = t_n + \tau_n, \quad t_{n+1/2} = t_n + \tfrac{1}{2}\tau_n,$$

and also denote the semi-discrete in time approximations

$$u_i^n \approx u_i(t_n).$$

For all $n > 1$, we define the starting iterations by extrapolation

$$u_{i,(0)}^{n+\frac{1}{2}} = u_i^{n-\frac{1}{2}} + \frac{\tau_n + \tau_{n-1}}{\tau_{n-1}} \big(u_i^n - u_i^{n-\frac{1}{2}}\big). \tag{2.1}$$

Then, for $\kappa > 1$, we iterate until convergence the partitioned half-step backward-Euler method:

$$\frac{u_{i,(\kappa+1)}^{n+\frac{1}{2}} - u_i^n}{\tau_n/2} - \nu_i \Delta u_{i,(\kappa+1)}^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \tag{2.2}$$

$$- \nu_i \nabla u_{i,(\kappa+1)}^{n+\frac{1}{2}} \cdot \hat{n}_i = \alpha(u_{i,(\kappa+1)}^{n+\frac{1}{2}} - u_{j,(\kappa)}^{n+\frac{1}{2}}) \text{ on } I, \tag{2.3}$$

$$u_{i,(\kappa+1)}^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i, \tag{2.4}$$

to get in the limit

$$\lim_{\kappa\to\infty} u_{i,(\kappa+1)}^{n+\frac{1}{2}} = u_i^{n+\frac{1}{2}} \tag{2.5}$$

the coupled system

$$\begin{cases} \dfrac{u_i^{n+\frac{1}{2}} - u_i^n}{\tau_n/2} - \nu_i \Delta u_i^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \\ -\nu_i \nabla u_i^{n+\frac{1}{2}} \cdot \hat{n}_i = \alpha(u_i^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}) \text{ on } I, \\ u_i^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i. \end{cases} \tag{BE}$$

Then finally, we use linear extrapolation to obtain the values at $n+1$ in the domain $\Omega_1 \cup I \cup \Omega_2$

$$u_i^{n+1} = 2u_i^{n+\frac{1}{2}} - u_i^n \text{ in } \Omega_i \cup I, \qquad u_i^{n+1} = 0 \text{ on } \Gamma_i. \tag{FE}$$

We note that the extrapolation step is equivalent to the forward Euler method

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\tau_n/2} - \nu_i \Delta u_i^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \\ u_i^{n+1} = 2u_i^{n+\frac{1}{2}} - u_i^n \text{ on } I, \\ u_i^{n+1} = 0 \text{ on } \Gamma_i, \end{cases} \tag{2.6}$$

and we will use this form in Proposition 2.2, to prove stability and the energy equality. Therefore the solution is obtained by solving sequentially the following system:

$$\begin{cases} \dfrac{u_i^{n+\frac{1}{2}} - u_i^n}{\tau_n/2} - \nu_i \Delta u_i^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \\ -\nu_i \nabla u_i^{n+\frac{1}{2}} \cdot \hat{n}_i = \alpha(u_i^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}) \text{ on } I, \\ u_i^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i, \\ u_i^{n+1} = 2u_i^{n+\frac{1}{2}} - u_i^n \text{ in } \Omega_i \cup I, \qquad u_i^{n+1} = 0 \text{ on } \Gamma_i. \end{cases} \tag{midpoint rule}$$

Proposition 2.1. *The iterations of the partitioned problem* (2.2)-(2.4) *converge linearly*

$$u_{i,(\kappa)}^{n+\frac{1}{2}} \xrightarrow{\kappa\to\infty} u_i^{n+\frac{1}{2}} \quad \textit{in} \quad L^2(I)) \cap H^1(\Omega_i)$$

to the solution of the coupled (midpoint rule) *problem.*

Proof. The difference between the solution to the coupled (BE) system and the solution to the partitioned system (2.2)-(2.4) satisfies

$$\begin{aligned} &\frac{u_i^{n+\frac{1}{2}} - u_{i,(\kappa+1)}^{n+\frac{1}{2}}}{\tau_n/2} - \nu_i \Delta\big(u_i^{n+\frac{1}{2}} - u_{i,(\kappa+1)}^{n+\frac{1}{2}}\big) = 0 \text{ in } \Omega_i, \\ &-\nu_i \nabla\big(u_i^{n+\frac{1}{2}} - u_{i,(\kappa+1)}^{n+\frac{1}{2}}\big) \cdot \hat{n}_i = \alpha\Big(\big(u_i^{n+\frac{1}{2}} - u_{i,(\kappa+1)}^{n+\frac{1}{2}}\big) - \big(u_j^{n+\frac{1}{2}} - u_{j,(\kappa)}^{n+\frac{1}{2}}\big)\Big) \text{ on } I, \\ &u_i^{n+\frac{1}{2}} - u_{i,(\kappa+1)}^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i. \end{aligned}$$

Testing with $u_i^{n+\frac{1}{2}} - u_{i,(\kappa+1)}^{n+\frac{1}{2}}$ in $L^2(\Omega_i)$ respectively, and adding up the results from both domains gives

$$\begin{aligned} 0 = {} & \frac{2}{\tau_n}\|u_1^{n+\frac{1}{2}} - u_{1,(\kappa+1)}^{n+\frac{1}{2}}\|^2 + \frac{2}{\tau_n}\|u_2^{n+\frac{1}{2}} - u_{2,(\kappa+1)}^{n+\frac{1}{2}}\|^2 \\ & + \nu_1\|\nabla\big(u_1^{n+\frac{1}{2}} - u_{1,(\kappa+1)}^{n+\frac{1}{2}}\big)\|^2 + \nu_2\|\nabla\big(u_2^{n+\frac{1}{2}} - u_{2,(\kappa+1)}^{n+\frac{1}{2}}\big)\|^2 \\ & + \frac{\alpha}{2}\int_I \Big(\big(u_1^{n+\frac{1}{2}} - u_{1,(\kappa+1)}^{n+\frac{1}{2}}\big)^2 - \big(u_1^{n+\frac{1}{2}} - u_{1,(\kappa)}^{n+\frac{1}{2}}\big)^2\Big) \\ & + \frac{\alpha}{2}\int_I \Big(\big(u_2^{n+\frac{1}{2}} - u_{2,(\kappa+1)}^{n+\frac{1}{2}}\big)^2 - \big(u_2^{n+\frac{1}{2}} - u_{2,(\kappa)}^{n+\frac{1}{2}}\big)^2\Big) \end{aligned}$$

$$+ \frac{\alpha}{2}\int_I \Big(\big(u_1^{n+\frac12} - u_{1,(\kappa+1)}^{n+\frac12}\big) - \big(u_2^{n+\frac12} - u_{2,(\kappa)}^{n+\frac12}\big)\Big)^2$$
$$+ \frac{\alpha}{2}\int_I \Big(\big(u_2^{n+\frac12} - u_{2,(\kappa+1)}^{n+\frac12}\big) - \big(u_1^{n+\frac12} - u_{1,(\kappa)}^{n+\frac12}\big)\Big)^2.$$

Summation over the iteration indices κ from 0 to $\ell - 1$ yields

$$\frac{\alpha}{2}\int_I \Big(\big(u_1^{n+\frac12} - u_{1,(\ell)}^{n+\frac12}\big)^2 + \big(u_2^{n+\frac12} - u_{2,(\ell)}^{n+\frac12}\big)^2\Big)$$
$$+ \frac{2}{\tau_n}\sum_{\kappa=1}^{\ell}\big(\|u_1^{n+\frac12} - u_{1,(\kappa)}^{n+\frac12}\|^2 + \|u_2^{n+\frac12} - u_{2,(\kappa)}^{n+\frac12}\|^2\big)$$
$$+ \sum_{\kappa=1}^{\ell}\Big(\nu_1\|\nabla\big(u_1^{n+\frac12} - u_{1,(\kappa)}^{n+\frac12}\big)\|^2 + \nu_2\|\nabla\big(u_2^{n+\frac12} - u_{2,(\kappa)}^{n+\frac12}\big)\|^2\Big)$$
$$+ \frac{\alpha}{2}\sum_{\kappa=0}^{\ell}\int_I \Big(\big(u_1^{n+\frac12} - u_{1,(\kappa)}^{n+\frac12}\big) - \big(u_2^{n+\frac12} - u_{2,(\kappa)}^{n+\frac12}\big)\Big)^2$$
$$+ \frac{\alpha}{2}\sum_{\kappa=1}^{\ell}\int_I \Big(\big(u_2^{n+\frac12} - u_{2,(\kappa)}^{n+\frac12}\big) - \big(u_1^{n+\frac12} - u_{1,(\kappa-1)}^{n+\frac12}\big)\Big)^2$$
$$= \frac{\alpha}{2}\int_I \Big(\big(u_1^{n+\frac12} - u_{1,(0)}^{n+\frac12}\big)^2 + \big(u_2^{n+\frac12} - u_{2,(0)}^{n+\frac12}\big)^2\Big).$$

Therefore the sequence of iterates $\{u_{i,(\kappa)}^{n+\frac12}\}_{\kappa\geq 0}$ converges strongly in the $L^2(I)$ and $H_0^1(\Omega_i)$ norms. Moreover, the convergence of iterative process is linear, and the convergence factor is smaller for smaller values of the timestep. □

Proposition 2.2. *The solution to the algorithm* (2.2)-(2.4) *and* (FE) *satisfies the following discrete energy equality*

$$\frac{1}{2}\big(\|u_1^N\|^2 + \|u_2^N\|^2\big) + \sum_{n=1}^{N-1}\tau_n\Big(\nu_1\|\nabla u_1^{n+\frac12}\|^2 + \nu_2\|\nabla u_2^{n+\frac12}\|^2 + \alpha\int_I (u_1^{n+\frac12} - u_2^{n+\frac12})^2\Big)$$
$$= \frac{1}{2}\big(\|u_1^1\|^2 + \|u_2^1\|^2\big) + \sum_{n=1}^{N-1}\tau_n\big(\langle f_1^{n+\frac12}, u_1^{n+\frac12}\rangle + \langle f_2^{n+\frac12}, u_2^{n+\frac12}\rangle\big).$$

Proof. We begin by testing both the momentum equation in (BE), and the momentum equation in (2.6) by $u_i^{n+\frac12}$, add, use the polarized identity, integrate by parts, use the boundary conditions, divide by 2, and use again the momentum equations to obtain in each domain Ω_i

$$\langle f_i^{n+\frac12}, u_i^{n+\frac12}\rangle = \frac{1}{2\tau_n}\big(\|u_i^{n+1}\|^2 - \|u_i^n\|^2\big) + \alpha\int_I (u_i^{n+\frac12} - u_j^{n+\frac12})u_i^{n+\frac12} + \nu_i\|\nabla u_i^{n+\frac12}\|^2.$$

Adding the relations from both domains, multiplying by τ_n, and summing over the time intervals from $n = 1$ to $N - 1$ yields the conclusion. □

We recall that Proposition 2.1 establishes the linear convergence of the iterations which partition the calculations (2.2)-(2.4) in each domain, which yield the coupled system (BE). Moreover, the details in the proof show that the number of iterations can be reduced provided the time-step τ_n is small, or if τ_n chosen in a time-adaptive framework, which also ensures a good initial guess (2.1). There are several ways of implementing time-adaptivity (see e.g. [1, 11, 13] and the references therein), among which we present here one based on the estimate of the temporal local truncation error $\widehat{T}_{n+1}$, using the difference between the (midpoint rule) solution and v_i^{n+1}, a variable-step Adams-Bashforth solution:

$$v_i^{n+1} = u_i^n \frac{(\tau_n + \tau_{n-1})(\tau_n + \tau_{n-1} + \tau_{n-2})}{\tau_{n-1}(\tau_{n-1} + \tau_{n-2})} - u_i^{n-1} \frac{\tau_n(\tau_n + \tau_{n-1} + \tau_{n-2})}{\tau_{n-1}\tau_{n-2}} \qquad \text{(AB2-like)}$$
$$+ u_i^{n-2} \frac{\tau_n(\tau_n + \tau_{n-1})}{\tau_{n-2}(\tau_{n-1} + \tau_{n-2})}.$$

We denote

$$\widehat{T}_{n+1} = \frac{1}{1 - 1/(24\mathcal{R}_n)} \sum_{i=1}^{2} \|u_i^{n+1} - v_i^{n+1}\|_{H^1(\Omega_i)}, \qquad \text{(LTE-AB2)}$$

where

$$\mathcal{R}_n = \frac{1}{24} + \frac{1}{8}\Big(1 + \frac{\tau_{n-1}}{\tau_n}\Big)\Big(1 + 2\frac{\tau_{n-1}}{\tau_n} + \frac{\tau_{n-2}}{\tau_n}\Big).$$

If $\widehat{T}_{n+1} > \texttt{tol}$, then τ_n is decreased, and the algorithm repeats the midpoint rule step with a reduced time-step. Respectively, if $\widehat{T}_{n+1} \le \texttt{tol}$, then τ_n is increased, and the computation moves to the next time interval, with the increased time step. The safety factor $\texttt{S} < 1$ reduces the probability of the new time-steps being rejected in the [**if** $\|\widehat{T}_{n+1}\| \le \texttt{tol}$] test in the Algorithm 1

Algorithm 1: Factorization of the midpoint rule for heat-transfer

Input : Ω_i, ν_i, κ
Output: u_i

initialization: set `tol` and τ_0, such that τ_0 is in the convergence range (see e.g. [7, p. 367]), compute u_i^1 with a one step second-order accurate method, set the safety coefficient `S`=0.8;
compute u_i^2 and τ_1 with a second-order accurate method, $t_2 = t_1 + \tau_1$;
$t^{\text{new}} = t_2, \tau^{\text{new}} = \tau_1$;
for $n \geq 2$ (i.e., $t^{\text{new}}, \tau^{\text{new}}, u_i^n, u_i^{n-1}, u_i^{n-2}$ are given);
while $t^{\text{new}} \leq T$ **do**
  $\tau_n \leftarrow \tau^{\text{new}}$;
  evaluate u_i^{n+1} with the midpoint rule (midpoint rule);
  evaluate $\widehat{T}_{n+1}$ with (LTE-AB2) ;
  $\tau^{\text{new}} \leftarrow \tau_n \cdot \min\{1.5, \max\{0.2, \kappa \big| \texttt{tol}/\|\widehat{T}_{n+1}\| \big|^{\frac{1}{3}}\}\}$;
  if $^*\|\widehat{T}_{n+1}\| \leq$ `tol` **then**
    $t_{n+1} \leftarrow t_n + \tau^{\text{new}}$, $t^{\text{new}} \leftarrow t_{n+1}$, $n \leftarrow n+1$
  end
end

3 Nonlinear Interface Condition

We consider the two-domain parabolic problem described above changing the interface condition to a nonlinear one. For ease of notation, let $\Psi = \alpha|u_i - u_j|^s$, where $s \in$ ~~$\{0,1,3\}$~~$[0,1]$. The problem becomes: *given* $\nu_i > 0, f_i : [0,T] \to H^1(\Omega_i)$*, and* $u_i(0) \in H^1(\Omega_i)$*, find (for* $i = 1,2$*)* $u_i : \overline{\Omega}_i \times [0,T] \to \mathbb{R}^d$ *and* $p_i : \overline{\Omega}_i \times [0,T] \to \mathbb{R}$ *satisfying*

$$u_{i,t} - \nu_i \Delta u_i = f_i \text{ in } \Omega_i, \tag{3.1}$$

$$-\nu_i \nabla u_i \cdot \hat{n}_i = \Psi(u_i - u_j) \text{ on } I \text{ for } i,j = 1,2 \quad i \neq j, \tag{3.2}$$

$$u_i = 0 \text{ on } \Gamma_i = \partial\Omega_i \backslash I. \tag{3.3}$$

$$u_i(x,0) = u_i^0(x) \text{ in } \Omega_i. \tag{3.4}$$

Again, we consider only the semi-discretization in time. Let us denote the mesh points $\{t_n\}_{n\geq 0}$, the timestep τ_n, such that:

$$t_{n+1} = t_n + \tau_n, \quad t_{n+1/2} = t_n + \tfrac{1}{2}\tau_n,$$

and also denote the semi-discrete in time approximations

$$u_i^n \approx u_i(t_n).$$

For all $n > 1$, we define the starting iterations by extrapolation

$$u_{i,(0)}^{n+\frac{1}{2}} = u_i^n, \qquad u_{i,(1)}^{n+\frac{1}{2}} = u_i^{n-\frac{1}{2}} + \frac{\tau_n + \tau_{n-1}}{\tau_{n-1}}\big(u_i^n - u_i^{n-\frac{1}{2}}\big). \tag{3.5}$$

Then, for $\kappa \geq 1$, we iterate until convergence the partitioned half-step backward-Euler method:

$$\frac{u_{i,(\kappa+1)}^{n+\frac{1}{2}} - u_i^n}{\tau_n/2} - \nu_i \Delta u_{i,(\kappa+1)}^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \tag{3.6}$$

$$-\nu_i \nabla u_{i,(\kappa+1)}^{n+\frac{1}{2}} \cdot \hat{n}_i = \Psi_{(\kappa)}^{n+\frac{1}{2}} u_{i,(\kappa+1)}^{n+\frac{1}{2}} - \left|\Psi_{(\kappa)}^{n+\frac{1}{2}}\right|^{1/2} \left|\Psi_{(\kappa-1)}^{n+\frac{1}{2}}\right|^{1/2} u_{j,(\kappa)}^{n+\frac{1}{2}} \text{ on } I, \tag{3.7}$$

$$u_{i,(\kappa+1)}^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i, \tag{3.8}$$

where $\Psi_{(\kappa)}^{n+\frac{1}{2}} = \alpha \left|u_{i,(k)}^{n+\frac{1}{2}} - u_{j,(k)}^{n+\frac{1}{2}}\right|^s$ to get in the limit

$$\lim_{\kappa \to \infty} u_{i,(\kappa+1)}^{n+\frac{1}{2}} = u_i^{n+\frac{1}{2}} \tag{3.9}$$

the coupled system

$$\begin{cases} \dfrac{u_i^{n+\frac{1}{2}} - u_i^n}{\tau_n/2} - \nu_i \Delta u_i^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \\ -\nu_i \nabla u_i^{n+\frac{1}{2}} \cdot \hat{n}_i = \Psi^{n+\frac{1}{2}} (u_i^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}) \text{ on } I, \\ u_i^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i. \end{cases} \tag{BE}$$

Then finally, we use linear extrapolation to obtain the values at $n+1$ in the domain $\Omega_1 \cup I \cup \Omega_2$

$$u_i^{n+1} = 2u_i^{n+\frac{1}{2}} - u_i^n \text{ in } \Omega_i \cup I, \qquad u_i^{n+1} = 0 \text{ on } \Gamma_i. \tag{FE}$$

Therefore the solution is obtained by solving sequentially the following system:

$$\begin{cases} \dfrac{u_i^{n+\frac{1}{2}} - u_i^n}{\tau_n/2} - \nu_i \Delta u_i^{n+\frac{1}{2}} = f_i^{n+\frac{1}{2}} \text{ in } \Omega_i, \\ -\nu_i \nabla u_i^{n+\frac{1}{2}} \cdot \hat{n}_i = \Psi^{n+\frac{1}{2}} (u_i^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}) \text{ on } I, \\ u_i^{n+\frac{1}{2}} = 0 \text{ on } \Gamma_i, \\ u_i^{n+1} = 2u_i^{n+\frac{1}{2}} - u_i^n \text{ in } \Omega_i \cup I, \qquad u_i^{n+1} = 0 \text{ on } \Gamma_i. \end{cases} \tag{midpoint rule}$$

Proposition 3.1. *The iterations of the partitioned problem* (3.6)-(3.8) *converge linearly*

$$u_{i,(\kappa)}^{n+\frac{1}{2}} \xrightarrow{\kappa \to \infty} u_i^{n+\frac{1}{2}} \quad in \quad L^2(I)) \cap H^1(\Omega_i)$$

to the solution of the coupled (midpoint rule) *problem.*

Proof. Denote $\langle\ ,\ \rangle$ as the integral at the interface, and

$$
\begin{aligned}
E_{i,(k+1)}^{n+\frac{1}{2}} &:= u_i^{n+\frac{1}{2}} - u_{i,(k+1)}^{n+\frac{1}{2}}, \qquad i = 1,2 \\
\Psi_{(k)}^{n+\frac{1}{2}} &= \alpha \big| u_{1,(k)}^{n+\frac{1}{2}} - u_{2,(k)}^{n+\frac{1}{2}} \big|^s.
\end{aligned}
$$

Testing the coupled system (BE) and the solution to the partitioned system (3.6)-(3.8) with $v_i \in L^2(\Omega_i)$ and subtracting we get

$$
\begin{aligned}
&\frac{1}{\tau_n/2}\big(u_1^{n+\frac{1}{2}} - u_{1,(k+1)}^{n+\frac{1}{2}}, v_1\big)_{\Omega_1} + \nu_1\big(\nabla(u_1^{n+\frac{1}{2}} - u_{1,(k+1)}^{n+\frac{1}{2}}), \nabla v_1\big)_{\Omega_1} \\
&\quad + \Big\langle \big(u_1^{n+\frac{1}{2}} - u_{1,(k+1)}^{n+\frac{1}{2}}\big)\Psi^{n+\frac{1}{2}} + u_{1,(k+1)}^{n+\frac{1}{2}}\big(\Psi^{n+\frac{1}{2}} - \Psi_{(k)}^{n+\frac{1}{2}}\big) \\
&\qquad - \big(u_2^{n+\frac{1}{2}} - u_{2,(k)}^{n+\frac{1}{2}}\big)\Psi^{n+\frac{1}{2}} + u_{2,(\kappa)}^{n+\frac{1}{2}}\big(\big|\Psi^{n+\frac{1}{2}}\big|^{1/2} - \big|\Psi_{(\kappa)}^{n+\frac{1}{2}}\big|^{1/2}\big)\big|\Psi^{n+\frac{1}{2}}\big|^{1/2} \\
&\qquad + u_{2,(\kappa)}^{n+\frac{1}{2}}\big|\Psi_{(\kappa)}^{n+\frac{1}{2}}\big|^{1/2}\big(\big|\Psi^{n+\frac{1}{2}}\big|^{1/2} - \big|\Psi_{(\kappa-1)}^{n+\frac{1}{2}}\big|^{1/2}\big), v_1 \Big\rangle = 0, \\
&\frac{1}{\tau_n/2}\big(u_2^{n+\frac{1}{2}} - u_{2,(k+1)}^{n+\frac{1}{2}}, v_2\big)_{\Omega_2} + \nu_2\big(\nabla(u_2^{n+\frac{1}{2}} - u_{2,(k+1)}^{n+\frac{1}{2}}), \nabla v_2\big)_{\Omega_2} \\
&\quad + \Big\langle \big(u_2^{n+\frac{1}{2}} - u_{2,(\kappa+1)}^{n+\frac{1}{2}}\big)\Psi^{n+\frac{1}{2}} + u_{2,(\kappa+1)}^{n+\frac{1}{2}}\big(\Psi^{n+\frac{1}{2}} - \Psi_{(\kappa)}^{n+\frac{1}{2}}\big) \\
&\qquad - \big(u_1^{n+\frac{1}{2}} - u_{1,(\kappa)}^{n+\frac{1}{2}}\big)\Psi^{n+\frac{1}{2}} + u_{1,(\kappa)}^{n+\frac{1}{2}}\big(\big|\Psi^{n+\frac{1}{2}}\big|^{1/2} - \big|\Psi_{(\kappa)}^{n+\frac{1}{2}}\big|^{1/2}\big)\big|\Psi^{n+\frac{1}{2}}\big|^{1/2} \\
&\qquad + u_{1,(\kappa)}^{n+\frac{1}{2}}\big|\Psi_{(\kappa)}^{n+\frac{1}{2}}\big|^{1/2}\big(\big|\Psi^{n+\frac{1}{2}}\big|^{1/2} - \big|\Psi_{(\kappa-1)}^{n+\frac{1}{2}}\big|^{1/2}\big), v_2 \Big\rangle = 0.
\end{aligned}
$$

Letting $v_i = u_i^{n+\frac{1}{2}} - u_{i,(k+1)}^{n+\frac{1}{2}}$, adding the equations and using the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&\frac{1}{\tau_n/2}\Big(\big\|u_1^{n+\frac{1}{2}} - u_{1,(k+1)}^{n+\frac{1}{2}}\big\|_{\Omega_1}^2 + \big\|u_2^{n+\frac{1}{2}} - u_{2,(k+1)}^{n+\frac{1}{2}}\big\|_{\Omega_2}^2\Big) \\
&\quad + \nu_1\big\|\nabla(u_1^{n+\frac{1}{2}} - u_{1,(k+1)}^{n+\frac{1}{2}})\big\|_{\Omega_1}^2 + \nu_2\big\|\nabla(u_2^{n+\frac{1}{2}} - u_{2,(k+1)}^{n+\frac{1}{2}})\big\|_{\Omega_2}^2 \\
&\leq \frac{1}{2}\int_I \Big(\big|u_{1,(k+1)}^{n+\frac{1}{2}}\big|^2 + \big|u_{2,(\kappa+1)}^{n+\frac{1}{2}}\big|^2\Big)\big|\Psi^{n+\frac{1}{2}} - \Psi_{(k)}^{n+\frac{1}{2}}\big|^2 \\
&\quad + \frac{1}{2}\int_I \Big(\big|u_1^{n+\frac{1}{2}} - u_{1,(\kappa)}^{n+\frac{1}{2}}\big|^2 + \big|u_2^{n+\frac{1}{2}} - u_{2,(k)}^{n+\frac{1}{2}}\big|^2\Big)\Psi^{n+\frac{1}{2}} \\
&\quad + \frac{1}{2}\int_I \Big(\big|u_{1,(\kappa)}^{n+\frac{1}{2}}\big|^2 + \big|u_{2,(\kappa)}^{n+\frac{1}{2}}\big|^2\Big)\big|\Psi^{n+\frac{1}{2}} - \Psi_{(\kappa)}^{n+\frac{1}{2}}\big| \\
&\quad + \frac{1}{2}\int_I \Big(\big|u_{1,(\kappa)}^{n+\frac{1}{2}}\big|^2 + \big|u_{2,(\kappa)}^{n+\frac{1}{2}}\big|^2\Big)\big|\Psi_{(\kappa)}^{n+\frac{1}{2}}\big|\big|\Psi^{n+\frac{1}{2}} - \Psi_{(\kappa-1)}^{n+\frac{1}{2}}\big| \\
&\quad + \int_I \big|u_1^{n+\frac{1}{2}} - u_{1,(k+1)}^{n+\frac{1}{2}}\big|^2 + \int_I \big|u_2^{n+\frac{1}{2}} - u_{2,(\kappa+1)}^{n+\frac{1}{2}}\big|^2.
\end{aligned}
$$

We note that for $s \in [0,1]$ we have

$$
|\Psi^{n+\frac{1}{2}} - \Psi_{(k)}^{n+\frac{1}{2}}| \leq \alpha 2^{2(s-1)}\big(\big|u_1^{n+\frac{1}{2}} - u_{1,(\kappa)}^{n+\frac{1}{2}}\big|^s + \big|u_{2,(\kappa)}^{n+\frac{1}{2}} - u_2^{n+\frac{1}{2}}\big|^s\big),
$$

and therefore

$$
\begin{aligned}
&\frac{1}{\tau_n/2}\Big(\big\|u_1^{n+\frac12}-u_{1,(k+1)}^{n+\frac12}\big\|_{\Omega_1}^2+\big\|u_2^{n+\frac12}-u_{2,(k+1)}^{n+\frac12}\big\|_{\Omega_2}^2\Big) \qquad (3.10)\\
&\quad+\nu_1\big\|\nabla(u_1^{n+\frac12}-u_{1,(k+1)}^{n+\frac12})\big\|_{\Omega_1}^2+\nu_2\big\|\nabla(u_2^{n+\frac12}-u_{2,(k+1)}^{n+\frac12})\big\|_{\Omega_2}^2\\
&\le\int_I\Big(\big|u_1^{n+\frac12}-u_{1,(k+1)}^{n+\frac12}\big|^2+\big|u_2^{n+\frac12}-u_{2,(\kappa+1)}^{n+\frac12}\big|^2\Big)\\
&\quad+\alpha^2 2^{4(s-1)}\int_I\Big(\big|u_{1,(k+1)}^{n+\frac12}\big|^2+\big|u_{2,(\kappa+1)}^{n+\frac12}\big|^2\Big)\big(\big|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big|^{2s}+\big|u_{2,(\kappa)}^{n+\frac12}-u_2^{n+\frac12}\big|^{2s}\big)\\
&\quad+\frac12\int_I\Big(\big|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big|^2+\big|u_2^{n+\frac12}-u_{2,(k)}^{n+\frac12}\big|^2\Big)\Psi^{n+\frac12}\\
&\quad+\alpha 2^{2s-3}\int_I\Big(\big|u_{1,(\kappa)}^{n+\frac12}\big|^2+\big|u_{2,(\kappa)}^{n+\frac12}\big|^2\Big)\big(\big|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big|^{s}+\big|u_{2,(\kappa)}^{n+\frac12}-u_2^{n+\frac12}\big|^{s}\big)\\
&\quad+\alpha 2^{2s-3}\int_I\Big(\big|u_{1,(\kappa)}^{n+\frac12}\big|^2+\big|u_{2,(\kappa)}^{n+\frac12}\big|^2\Big)\big|\Psi_{(\kappa)}^{n+\frac12}\big|\big(\big|u_1^{n+\frac12}-u_{1,(\kappa-1)}^{n+\frac12}\big|^{s}+\big|u_{2,(\kappa-1)}^{n+\frac12}-u_2^{n+\frac12}\big|^{s}\big).
\end{aligned}
$$

Using the continuity of the trace operator [10,12,14] and the inverse inequality [3] we have that

$$
\begin{aligned}
M\|u_i^{n+\frac12}-u_{i,(k+1)}^{n+\frac12}\|_{L^4(I)}^2 &\le \frac{M^2}{4\nu_i}\frac{C^2(\Omega_i)}{h_i^2}\|u_i^{n+\frac12}-u_{i,(k+1)}^{n+\frac12}\|_{L^2(\Omega_i)}^2 \qquad (3.11)\\
&\quad+\nu_i\|\nabla(u_i^{n+\frac12}-u_{i,(k+1)}^{n+\frac12}))\|_{L^2(\Omega_i)}^2,
\end{aligned}
$$

for any arbitrary constant M, to be chosen later. Under the small time step assumption (3.12)

$$
\tau_n\le\frac{8}{M^2}\max_{i=1,2}\frac{\nu_i h_i^2}{C^2(\Omega_i)}, \qquad (3.12)
$$

relation (3.11) yields

$$
\begin{aligned}
&M\|u_i^{n+\frac12}-u_{i,(k+1)}^{n+\frac12}\|_{L^4(I)}^2 \qquad (3.13)\\
&\le\frac{2}{\tau_n}\big\|u_i^{n+\frac12}-u_{i,(k+1)}^{n+\frac12}\big\|_{\Omega_i}^2+\nu_i\big\|\nabla(u_i^{n+\frac12}-u_{i,(k+1)}^{n+\frac12})\big\|_{\Omega_i}^2.
\end{aligned}
$$

Substituting (3.13) into (3.10) gives

$$
\begin{aligned}
&(M-1)\Big(\big\|u_1^{n+\frac12}-u_{1,(k+1)}^{n+\frac12}\big\|_{L^4(I)}^2+\big\|u_2^{n+\frac12}-u_{2,(\kappa+1)}^{n+\frac12}\big\|_{L^4(I)}^2\Big)\\
&\le\alpha^2 2^{4(s-1)}\int_I\Big(\big|u_{1,(k+1)}^{n+\frac12}\big|^2+\big|u_{2,(\kappa+1)}^{n+\frac12}\big|^2\Big)\big(\big|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big|^{2s}+\big|u_{2,(\kappa)}^{n+\frac12}-u_2^{n+\frac12}\big|^{2s}\big)\\
&\quad+\frac12\int_I\Big(\big|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big|^2+\big|u_2^{n+\frac12}-u_{2,(k)}^{n+\frac12}\big|^2\Big)\Psi^{n+\frac12}\\
&\quad+\alpha 2^{2s-3}\int_I\Big(\big|u_{1,(\kappa)}^{n+\frac12}\big|^2+\big|u_{2,(\kappa)}^{n+\frac12}\big|^2\Big)\big(\big|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big|^{s}+\big|u_{2,(\kappa)}^{n+\frac12}-u_2^{n+\frac12}\big|^{s}\big)
\end{aligned}
$$

$$+\alpha 2^{2s-3}\int_I \Big(\big|u_{1,(\kappa)}^{n+\frac12}\big|^2+\big|u_{2,(\kappa)}^{n+\frac12}\big|^2\Big)\big|\Psi_{(\kappa)}^{n+\frac12}\big|\big(\big|u_1^{n+\frac12}-u_{1,(\kappa-1)}^{n+\frac12}\big|^s+\big|u_{2,(\kappa-1)}^{n+\frac12}-u_2^{n+\frac12}\big|^s\big),$$

which by the Hölder inequality further yields

$$\begin{aligned}
&(M-1)\Big(\|u_1^{n+\frac12}-u_{1,(k+1)}^{n+\frac12}\|_{L^4(I)}^2+\|u_2^{n+\frac12}-u_{2,(\kappa+1)}^{n+\frac12}\|_{L^4(I)}^2\Big)\\
&\le \alpha^2 2^{4(s-1)}\big(\|u_{1,(k+1)}^{n+\frac12}\|^2_{L^{\frac{4}{2-s}}(I)}+\|u_{2,(k+1)}^{n+\frac12}\|^2_{L^{\frac{4}{2-s}}(I)}\big)\\
&\qquad\qquad\times\big(\|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\|^{\frac{s}{2}}_{L^4(I)}+\|u_2^{n+\frac12}-u_{2,(\kappa)}^{n+\frac12}\|^{\frac{s}{2}}_{L^4(I)}\big)\\
&\quad+\frac{\alpha}{2}\Big(\|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\|^2_{L^4(I)}+\|u_2^{n+\frac12}-u_{2,(\kappa)}^{n+\frac12}\|^2_{L^4(I)}\Big)\|u_1^{n+\frac12}-u_2^{n+\frac12}\|^s_{L^{2s}(I)}\\
&\quad+\alpha 2^{2s-3}\Big(\big\|u_{1,(k)}^{n+\frac12}\big\|^2_{L^{\frac{8}{4-s}}(I)}+\big\|u_{2,(k)}^{n+\frac12}\big\|^2_{L^{\frac{8}{4-s}}(I)}\Big)\\
&\qquad\qquad\times\Big(\big\|u_1^{n+\frac12}-u_{1,(\kappa)}^{n+\frac12}\big\|^s_{L^4(I)}+\big\|u_2^{n+\frac12}-u_{2,(\kappa)}^{n+\frac12}\big\|^s_{L^4(I)}\Big)\\
&\quad+\alpha^2 2^{3(s-1)}\Big(\big\|u_{1,(k)}^{n+\frac12}\big\|^{s+2}_{L^{\frac{4(s+2)}{4-s}}(I)}+\big\|u_{2,(k)}^{n+\frac12}\big\|^{s+2}_{L^{\frac{4(s+2)}{4-s}}(I)}\Big)\\
&\qquad\qquad\times\Big(\big\|u_1^{n+\frac12}-u_{1,(\kappa-1)}^{n+\frac12}\big\|^s_{L^4(I)}+\big\|u_2^{n+\frac12}-u_{2,(\kappa-1)}^{n+\frac12}\big\|^s_{L^4(I)}\Big).
\end{aligned}$$

□

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