Preliminary Exam August 2018

Problem 1. For $n \geq 1$ let

$$s_n = \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) \quad \text{and} \quad t_n = \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n+1}\right).$$

(a) Determine, with proof, the limit $\lim_{n\to\infty} s_n$.
(b) Determine, with proof, the limit $\lim_{n\to\infty} t_n$.

Hint. *For part (a) use ideas related to the integral test for convergence of the series.*

Problem 2. Prove that if $f : [a,b] \to \mathbb{R}$ is continuous, then

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b |f(x)|^n \, dx} = \sup_{x\in[a,b]} |f(x)|.$$

Problem 3. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of C^1 functions satisfying the conditions, for each $n \in \mathbb{N}$:

- $f_n\left(\frac{1}{2}\right) = 0,$
- $|f_n'(x)| < \frac{1}{x}$ for all $x \in (0,1)$.

Show that there is a continuous function $f : (0,1] \to \mathbb{R}$ and a subsequence $f_{n_k}(x)$, such that for each $x \in (0,1]$:

$$\lim_{k\to\infty} f_{n_k}(x) = f(x).$$

Problem 4. Let $\mathcal{F} \subset C^\infty[0,1]$ be a uniformly bounded and equicontinuous family of smooth functions on $[0,1]$ such that $f' \in \mathcal{F}$ whenever $f \in \mathcal{F}$. Suppose that

$$\sup_{x\in[0,1]} |f'(x) - g'(x)| \leq (1/2) \sup_{x\in[0,1]} |f(x) - g(x)| \quad \text{for all } f, g \in \mathcal{F}.$$

Show that there exists a sequence f_n of functions in $\mathcal{F}$ that tends uniformly to Ce^x, for some real constant C.
Hint: *Use contraction principle. In order to apply the contraction principle you can use, without proof, the fact that if X is a complete metric space, $A \subset X$ and $T : A \to X$ is uniformly continuous, then T uniquely extends to a continuous map $\overline{T} : \overline{A} \to X$ defined on the closure $\overline{A}$.*

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of class C^1. Prove that there is an open and dense set $\Omega \subset \mathbb{R}^n$ such that the function $R(x) = \operatorname{rank} Df(x)$ is locally constant on Ω, i.e. it is constant in a neighborhood of every point $x \in \Omega$.

Problem 6. Let $\Phi : \mathbb{R}^2 \to \Phi(\mathbb{R}^2) \subset \mathbb{R}^2$ be a diffeomorphism. Prove that

$$\int_{B^2(0,1)} \|D\Phi\| = \int_{\Phi(B^2(0,1))} \|D(\Phi^{-1})\|,$$

where $\|A\| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$ is the Hilbert-Schmidt norm of the matrix.
Hint. *Compare $\|A\|$ and $\|A^{-1}\|$ for a 2×2 matrix.*