Preliminary Exam in Analysis, August 2019

Problem 1. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a norm, that is for all $x, y \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ we have

(1) $F(x) \geq 0$ and $F(x) = 0$ if and only if $x = 0$,
(2) $F(x + y) \leq F(x) + F(y)$,
(3) $F(tx) = |t|F(x)$.

Prove the following statements:

(a) F is bounded on the unit sphere.
(b) F is continuous.
(c) There are constants $A, B > 0$ such that $A\|x\| \leq F(x) \leq B\|x\|$, for all $x \in \mathbb{R}^n$.

Hint. *It is useful to represent $v \in \mathbb{R}^n$ in a standard basis. For part (c) think what happens on the unit sphere.*

Problem 2. Prove that if the partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ of a function $f : \mathbb{R}^2 \to \mathbb{R}$ exist at every point of $\mathbb{R}^2$, and the partial derivative $\partial f/\partial x_1$ is continuous on $\mathbb{R}^2$, then f is differentiable at every point of $\mathbb{R}^2$.

Problem 3. Let S be the subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, $n \geq 2$, consisting of vectors (v_1, v_2) such that $\|v_1\| = \|v_2\| = 1$ and $v_1 \cdot v_2 = 0$. Prove that S is a smooth k-dimensional surface in $\mathbb{R}^{2n}$ for some $1 \leq k < 2n$. That is, in a neighborhood of any point of S, the set S is a graph of a smooth function of k-variables. Find the dimension k of S.

Problem 4. Assume that $f : [0, 1] \to [0, 1]$ is a continuous function such that the set $\{x \in [0, 1] : f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0 \, .$$

Problem 5. Prove that:

(a) There is a unique continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = 1 + \int_0^x t^2 f(t) \, dt \quad \text{for all } x \in [0, 1].$$

(b) The function from (a) is of class $f \in C^\infty(0, 1)$.

Problem 6. Suppose that $K \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$K(x) = \frac{K(x/\|x\|)}{\|x\|} \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

(a) Prove that $\nabla K(tx) = t^{-2}\nabla K(x)$ for $x \neq 0$ and $t > 0$.
(b) Use the divergence theorem to prove that (on both sides we integrate vector valued functions)

$$\int_{\{1\leq\|x\|\leq 2019\}} \nabla K(x) \, dx = \int_{\partial\{1\leq\|x\|\leq 2019\}} K(x)\vec{\mathbf{n}}(x) \, d\sigma(x).$$

(c) Prove that

$$\int_{\{1\leq\|x\|\leq 2019\}} \nabla K(x) \, dx = 0.$$

Hint. *Show first that $K(tx) = t^{-1}K(x)$ for $x \neq 0$, $t > 0$. In (a) differentiate $K(tx)$. Part (a) is not needed for parts (b) and (c).*