Preliminary Exam September 2020

Problem 1. Let $f : \mathbb{R} \to [0, \infty)$ be continuous and assume that for any $x \in \mathbb{R}$

$$g(x) := \sum_{k=1}^{\infty} (f(x))^k < \infty.$$

Show that g is continuous.

Problem 2.

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Prove that f is uniformly continuous if and only if $|f|$ is uniformly continuous.
(2) Show an example that the above claim is false without assuming continuity of f. That is, find a counterexample to the following claim: $f : \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous if and only if $|f|$ is uniformly continuous.

Problem 3. Show that for any $x, y \in \mathbb{R}^2$ and for any $r, s, t \in [0, 1]$ with $r + s + t = 2$ we have

$$|\det(x, y)| \le |x|^s\, |y|^t\, |x - y|^r.$$

Here (x, y) represents the 2×2 matrix with vectors x and y as columns.
Hint: *If you replace y by $y - x$...*

Problem 4. Assume that $f \in C^\infty(-1, 1)$ is such that the n-th derivative satisfies $|f^{(n)}(x)| \le n$ for all $x \in (-1, 1)$ and all $n = 1, 2, \ldots$ Prove that there is $F \in C^\infty(\mathbb{R})$ such that $F(x) = f(x)$ for all $x \in (-1, 1)$.
Hint: *Taylor.*

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of class C^1 and let $K \subset \mathbb{R}^n$ be a compact set. Assume that

(1) $|f(x) - f(y)| \ge |x - y|^{2020}$ for all $x, y \in K$.
(2) For all $x \in K$,

$$\frac{\partial f_i(x)}{\partial x_i} \ge i, \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and} \quad \frac{\partial f_i(x)}{\partial x_j} = 0, \quad \text{for } 1 \le i < j \le n.$$

Prove that there is an open set U such that $K \subset U$, and f is one-to-one in U.

Problem 6. Let $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| = |y| = 1\}$ and let $F : S \to \mathbb{R}$, $F(x, y) = x \cdot y$.

(a) Use the Lagrange multiplier theorem to find $\sup\limits_{(x,y)\in S} F(x, y)$.
(b) Use the result of part (a) to conclude the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} x_i y_i \le \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}.$$

Remark. You **must** use the Lagrange multiplier theorem in part (a) in order to get credit.