Linear Algebra Preliminary Exam
August 25, 2021

1. Let A, B be two $n \times n$ matrices over $\mathbb{C}$. Suppose that B is nilpotent and $AB = BA$. Prove that
$$\det(A + B) = \det(A).$$

2. Let A be an $n \times n$ matrix over $\mathbb{R}$. Suppose that $A^2 = AA^T$. Prove that A is symmetric. **Note:** *A is not assumed to be invertible. No points will be awarded if your argument relies on assuming the invertibility of A or something equivalent.*

3. Suppose that an $n \times n$ complex matrix $A = (a_{ij})_{n\times n}$ satisfies
$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji} = 0$$
for $1 \leq i \leq n$. Prove that all n^2 cofactors of A are equal.

4. Let P_1, P_2 be two projections of a vector space X over $\mathbb{C}$ into itself. If $\frac{P_1+P_2}{2}$ is also a projection, show that $\operatorname{rank} P_1 = \operatorname{rank} P_2$.

5. Let $n \geq 1$, $P = (p_{ij})_{n\times n}$ be a positive definite real symmetric matrix and $Q = P^{-1}$. Show that for any nonzero real column vector $x \in \mathbb{R}^n$,
$$\frac{\left(\sum_{i=1}^{n} x_i\right)^2}{x^T Q x} \leq \sum_{i,j=1}^{n} p_{ij}.$$

6. Let A, B be two square matrices over $\mathbb{C}$. Show that $\operatorname{rank}(AB - BA) = 1$ implies A and B have at least one common eigenvector.