PRELIMINARY EXAMINATION IN LINEAR ALGEBRA
AUGUST 16, 2019

Unless directed otherwise, you may use facts *proved* in the reference textbooks listed in the graduate handbook. Justify all other assertions, including from prior homework or tests.

Problem 1. Let A be an $n \times n$ matrix such that $A_{ij} = 0$ whenever $|i-j| \neq 1$. If n is even, describe $\det(A)$ as a product of ± 1 and certain entries of A. If n is odd, what is $\det(A)$?

Problem 2. For linear operators S and T on a finite-dimensional vector space V, prove:

$$\max\{\text{nullity}(S), \text{nullity}(T)\} \leq \text{nullity}(ST) \leq \text{nullity}(S) + \text{nullity}(T).$$

Problem 3. Let $B \in \mathbb{C}^{n \times n}$ be *positive*: self-adjoint, with $\langle B\mathbf{x}, \mathbf{x}\rangle > 0$ for all $\mathbf{x} \neq \mathbf{0}$, where $\langle \mathbf{x}, \mathbf{y}\rangle := \sum_{i=1}^{n} x_i \bar{y}_i$ is the standard inner product of $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

(a) Define a new inner product $(*, *)$ on $\mathbb{C}^n$ by

$$(\mathbf{x}, \mathbf{y}) := \langle B\mathbf{x}, \mathbf{y}\rangle.$$

For $M \in \mathbb{C}^{n \times n}$, prove that the adjoint of M with respect to the inner product $(*, *)$ is $B^{-1}M^*B$, where $M^* = \bar{M}^t$ is the adjoint of M with respect to the standard inner product $\langle *, *\rangle$. Use this to show that if A is self-adjoint with respect to $\langle *, *\rangle$ then $B^{-1}A$ is self-adjoint with respect to $(*, *)$.

(b) If $A \in \mathbb{C}^{n \times n}$ is self-adjoint, show that the generalized Rayleigh quotient $\dfrac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\langle B\mathbf{x}, \mathbf{x}\rangle}$ attains a maximum $m \in \mathbb{R}$ on $\mathbb{C}^n - \{\mathbf{0}\}$ at a vector $\mathbf{v}$ satisfying $A\mathbf{v} = mB\mathbf{v}$.

Problem 4. For a one-variable polynomial $p(x)$ let p' be the derivative of p, computed in the usual way, and let $E(p)(x) = \frac{1}{2}[p(x) + p(-x)]$ be the *even* part of p. For any fixed $n \in \mathbb{N}$, find the characteristic and minimal polynomials of the operator T defined by

$$T(p)(x) = (1 + x)E(p')(x)$$

on the space V_n of polynomials of degree at most n.

Problem 5. Let V be a finite-dimensional inner product space. Suppose A and B are orthogonal projections of V with the property that for all $\mathbf{x} \in V$,

$$\|A\mathbf{x}\|^2 + \|B\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2.$$

Prove that $A + B$ is also an orthogonal projection.

Problem 6. Suppose that a matrix $A \in \mathbb{R}^{6 \times 6}$ satisfies the following:

- $\text{rank}(A) > 3$;
- A^3 is a projection, but $A\mathbf{x} \neq \mathbf{x}$ for all non-zero vectors $\mathbf{x}$; and
- There is an A-invariant direct sum decomposition $\mathbb{R}^6 = U \oplus V$, such that $\dim U = 2$ and the restriction of A to U is orthogonal, and $\dim V = 4$ and the restriction of A to V is nilpotent.

Describe all possible Jordan forms of A (there is more than one), now regarded as a complex matrix, and prove that these are the only possibilities.