Preliminary Exam in Analysis, May 1, 2024

Problem 1. Suppose that $f, g : [0,1] \to \mathbb{R}$ are continuous functions and such that $f(x) < g(x)$ for all $x \in [0,1]$. Prove that there is a polynomial $p(x)$ such that

$$f(x) < p(x) < g(x) \qquad \text{for all } x \in [0,1].$$

Problem 2. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{|x^2 y + \cos(n)|^{1/4}}{\sqrt{n}(n + x^2 + y^2)}$ is uniformly convergent for $(x,y) \in \mathbb{R}^2$.

Problem 3. Recall that a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one and onto map such that both f and f^{-1} are continuous. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, then $\lim_{|x|\to\infty} |f(x)| = \infty$.

Problem 4. Let $D \subset \mathbb{R}^2$ be closed and bounded, and suppose that $f, g \in C^1(\mathbb{R}^2)$ are such that

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \neq 0 \qquad \text{in } D.$$

Prove that there are at most finitely many points $(x,y) \in D$ such that $f(x,y) = g(x,y) = 0$.

Problem 5. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $f : D \to \mathbb{R}$ be a continuous function. Prove that there exists a point $p_0 \in D$ such that

$$\iint_D (x^4 + y^4) f(x,y)\, dA = \frac{\pi f(p_0)}{4}\,.$$

Problem 6. Let $\Phi : \mathbb{R}^2 \to \Phi(\mathbb{R}^2) \subset \mathbb{R}^2$ be a diffeomorphism. Prove that

$$\iint_{B^2(0,1)} \|D\Phi\|\, dA = \iint_{\Phi(B^2(0,1))} \|D(\Phi^{-1})\|\, dA,$$

where $\|A\| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$ is the Hilbert-Schmidt norm of the matrix.
Hint. *Compare* $\|A\|$ *and* $\|A^{-1}\|$ *for a* 2×2 *matrix.*

$\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$, $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.