Linear Algebra Preliminary Exam August 2024

Problem 1 *Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices. Show that either $A + tB$ is either invertible for all except for a finite number of values $t \in \mathbb{R}$ or for* no *value of t.*

Problem 2 *If P_1, P_2 and $\frac{P_1+P_2}{2}$ are all projections, then* $\operatorname{rank} P_1 = \operatorname{rank} P_2$.

Problem 3 *Let $m, n \geq 1$ and $A_{m\times m}, B_{n\times n}$ be two square matrices. Define the $(m+n)\times(m+n)$ matrix*

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Show that the standard operator norm satisfies $\|C\| = \max(\|A\|, \|B\|)$.

Problem 4 *Let A, B symmetric real matrices. Show that $\lambda_{min}(A + B) \geq \lambda_{min}(A) + \lambda_{min}(B)$.*

Problem 5 *Let A be an 7×7 nilpotent matrix over $\mathbb{C}$, i.e., $A^k = 0$ for some positive integer k. Suppose $rank(A) = 4$ and $rank(A^2) = 2$ and $rank(A^3) = 1$. Find the Jordan Canonical Form of A.*

Problem 6 *For $n \in \mathbb{N}_0$, define the function $p_n : \mathbb{R} \to \mathbb{R}$,*

$$p_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right)$$

Show that the sets of functions $\{1, x, \ldots, x^n\}$ and $\{p_0, \ldots, p_n\}$ span the same subspace in the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.