Preliminary Exam in Analysis, January, 2025

1. Let K be a compact subset of $\mathbb{R}$, and suppose that $f, f_n : K \to \mathbb{R}$ are continuous for all $n \in \mathbb{N}$. Suppose that, for every $x \in K$ we have $f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$.

   (a) Show that $f_n(x) \to f(x)$ uniformly on K as $n \to \infty$.

   (b) Give a counterexample to the statement of part (a) if K is not assumed to be compact.

2. Assume Ω is a connected, smooth, bounded, open set in $\mathbb{R}^n$ and let $u : \overline{\Omega} \to \mathbb{R}$ be a twice continuously differentiable function that satisfies for some $p > 1$

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \equiv \sum_{\alpha=1}^{n} \partial_\alpha \left(|\nabla u|^{p-2}\partial_\alpha u\right) = 0 \quad \text{in } \Omega.$$

   Assume additionally that $\langle \nu(x), \nabla u(x)\rangle_{\mathbb{R}^n} = 0$ whenever $x \in \partial\Omega$, where $\nu(x)$ denotes the outwards facing unit normal to $x \in \partial\Omega$. Show that u is constant. **Hint:** Consider $\int_\Omega |\nabla u|^p$.

3. Suppose that f is a real-valued smooth function on $\mathbb{R}^2$, such that $f\big(x, \frac{1}{1+x}\big) > 0$ for each $x \in [0,1]$. Show that there exists a constant $c \in \mathbb{R}$, an interval I containing 1, and a continuously differentiable function $g : I \to \mathbb{R}$ such that

$$\int_0^1 \int_0^{\frac{g(x)}{1+u}} f(xu, xv)\, dv\, du = c, \qquad \text{for each } x \in I. \tag{1}$$

4. Let $f : [0.\infty) \to \mathbb{R}$ be continuous. For $n \in \mathbb{N}$ define the continuous function $f_n : [0,\infty) \to \mathbb{R}$ by

$$f_n(x) := f(x^n).$$

   Show that the set $\{f_1, f_2, \cdots\}$ is equicontinuous at $x = 1$ if and only if f is a constant function.

5. Let $p \in (1,\infty)$. Assume that $f_k : \mathbb{R} \to \mathbb{R}$ is a sequence of continuously differentiable functions that satisfy

$$M := \sup_{k\in\mathbb{N}} \left(\int_{-1}^{1} |f_k(z)| dz + \int_{-1}^{1} |f_k'(z)|^p\, dz \right) < \infty. \tag{2}$$

   (a) Prove that for each $k \in \mathbb{N}$ there exists $x_k \in [-1,1]$ such that $|f_k(x_k)| \leq \frac{M}{2}$.

   (b) Show that $(f_k)_{k\in\mathbb{N}}$ has a subsequence which is uniformly convergent on $[-1,1]$.

   Hint: Recall Hölder's inequality $\int_{-1}^1 f(x)g(x)dx \leq \left(\int_{-1}^1 |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_{-1}^1 |g(x)|^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}}$

6. Let $\phi : (0,+\infty) \to \mathbb{R}$, $\phi(x) = \frac{3}{4}x + \frac{1}{x^3}$.

   (a) Show that for any $x \in [\sqrt{2}, 2]$ there exists some $\theta \in [0,1]$ such that

$$\phi(x) = \sqrt{2} + \frac{1}{2!}\frac{12}{(\theta x + (1-\theta)\sqrt{2})^5}\big(x - \sqrt{2}\big)^2. \tag{ϕ}$$

   (b) Consider the recursive sequence

$$x_{n+1} = \frac{3}{4}x_n + \frac{1}{x_n^3}, \qquad \forall n \geq 0.$$

   with the initial value $x_0 = 1.5$. Show that the sequence $\{x_n\}_{n\geq 0}$ converges to $\sqrt{2} \approx 1.4142$, and

$$|\sqrt{2} - x_5| \leq 10^{-32}.$$

   Hint: $0 < x_0 - \sqrt{2} \approx 0.0858 < 10^{-1}$.