Linear Algebra Preliminary Exam

May 3, 2024

1. Let A, B be two $n \times n$ complex matrices. Show that there exists two complex numbers c_1, c_2, not all zero, such that
$$\det\left(c_1 A + c_2 B\right) = 0.$$

2. Let X be an n dimensional linear space over $\mathbb{C}$, $S, T \in L(X, X)$. Show that
(a)
$$N_{S-I} \cap N_{T-I} \subset N_{ST-I},$$
(b)
$$\operatorname{rank}(ST - I) \leq \operatorname{rank}(S - I) + \operatorname{rank}(T - I).$$

3. Let A be an $n \times n$ complex matrix. Show that A and A^T have the same JCF and hence are similar to each other.

4. Let A be an $n \times n$ real matrix and r be the spectral radius of $A + A^T$. Show that
$$|\operatorname{Re}\lambda| \leq \frac{r}{2}$$
holds for any eigenvalue λ of A.

5. Let A, B be two self-adjoint positive definite complex matrices such that $A^2 > B^2$, show that $A > B$. (You could use without proof the result that $A > B > 0$ implies $PAP^* > PBP^* > 0$ for any invertible P.)

6. Let A be a given $n \times n$ real matrix. Prove that the following are equivalent:
(a) A and $-A^T$ share no eigenvalue.
(b) For every symmetric real matrix B, there exists a unique symmetric matrix X such that $AX + XA^T = B$. (Hint: (a) implies that $p_A\left(-A^T\right)$ is invertible where p_A is the characteristic polynomial of A.)