Preliminary Exam in Analysis, August 19, 2024

Problem 1. Let $E \subset \mathbb{R}^n$ be compact and let

$$K = \{x \in \mathbb{R}^n : |x - y| = 1 \text{ for some } y \in E\}.$$

Prove that K is compact.

Problem 2. Let X be a nonempty set, and let $F(X)$ denote the space of all functions $f : X \to \mathbb{R}$. For $f, g \in F(X)$ we define:

$$d(f,g) := \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

It follows that $(F(X), d)$ is a metric space. (`Do not prove this`). Prove that a sequence $(f_n)_n \subset F(X)$ satisfies $d(f_n, f) \to 0$ for some $f \in F(X)$ if and only if $(f_n)_n$ converges uniformly to f.

Problem 3. Let $p > 0$. Assume that $f_k : \mathbb{R} \to \mathbb{R}$ is a sequence of continuously differentiable functions that satisfy

$$M := \sup_{k \in \mathbb{N}} \sup_{x \in [-1,1]} \left(|f_k(x)| + \int_{-1}^{1} |f_k'(z)|^p \, dz \right) < \infty. \tag{1}$$

(a) Show that if $p = 2$, $(f_k)_{k \in \mathbb{N}}$ has a subsequence uniformly convergent on $[-1, 1]$.
(b) If $p = 1$, explicitly construct a sequence $(f_k)_{k \in \mathbb{N}}$ that satisfies (1), but does not have a uniformly convergent subsequence on $[-1, 1]$.

Hint. *(a) Use the Cauchy-Schwarz inequality; (b) Approximate the function $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$. Use piecewise approximation but make sure that the functions are continuously differentiable.*

Problem 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be be infinitely differentiable, $f \in C^\infty$. Assume that there is a vector $v \in \mathbb{R}^n$ and a sequence $0 \neq x_k \to 0$ such that $Df(x_k) = v$ for all k. Prove that $\det D^2 f(0) = 0$.

Problem 5.

(a) Prove that if $a > 1$ and $k \geq 1$, then $\displaystyle\sum_{n=2}^{\infty} \frac{(\log n)^k}{n^a} < \infty$.

(b) Prove that the function $\zeta(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$, $x > 1$, is infinitely differentiable in $(1, \infty)$.

Hint: *You can use part (a) to prove (b), even if you do not know how to prove (a).*

Problem 6. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be of class C^∞. Assume that F has compact support and that

$$\int_{-\infty}^{\infty} f(x,0) g_x(x,0) \, dx = 1.$$

Prove that there is $(x, y) \in \mathbb{R}^2$ such that $\det DF(x,y) \neq 0$. **Hint.** *Apply the divergence theorem to a suitable region.*