Linear Algebra Preliminary Exam January 2025

Problem 1 *Find the matrix representation of the derivative operator on the space of polynomials of degree at most 2 in the basis* $\{1, x-1, x^2+2x+1\}$.

Problem 2 *Let* $A, B \in \mathbb{R}^{n\times n}$ *be two matrices. Show that* $p(t) = \det(A + tB)$ *is a polynomial and* $deg(p) \leq rank(B)$.

Problem 3 *Let* S *be a* $n \times n$ *Nilpotent matrix, show that* S *is similar to* $S^{2024} - S$.

Problem 4 *Let* $A : X \to X$ *where* X *is a finite dimensional complex Euclidean space. Let* x *be a unit vector such that*

$$\|Ax\| = \|A\|\,.$$

Show that x *is an eigenvector of* A^*A.

Problem 5 *Let* H, K *be* $n \times n$ *complex matrices and* $A = K + H$. *Suppose* $H^* = H$, $K^* = -K$, *Show that*

$$\sum_{i=1}^{n} |\lambda_i^2| \leq \sum_{i=1}^{n} |\mu_i^2| + \sum_{i=1}^{n} |\sigma_i^2|$$

where $\{\mu_i\}_{i=1}^n$, $\{\sigma_i\}_{i=1}^n$ *and* $\{\lambda_i\}_{i=1}^n$ *are eigenvalues of* K, H *and* A *respectively.*

Problem 6 *Let* $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ *be functions. Prove that the following are equivalent.*

1. *For all sets of* n *distinct points* $x_1, \ldots, x_n \in \mathbb{R}$ *and for all* $y_1, \ldots, y_n$, *there exists a set of coefficients* $a_1, \ldots, a_n$ *such that*

$$\sum_{i=1}^{n} a_i f_i(x_j) = y_j \qquad \forall\ j = 1, \ldots, n.$$

2. *For any set of coefficients* $a_1, \ldots, a_n$, *the function* $x \mapsto \sum_{i=1}^{n} a_i f_i(x)$ *has at most* $n-1$ *zeros in* $\mathbb{R}$ *unless* $a_1 = \cdots = a_n = 0$.