EXTENSION OF A SIMPLIFIED BALDWIN-LOMAX MODEL TO NON-EQUILIBRIUM TURBULENCE: MODEL, ANALYSIS AND ALGORITHMS

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Abstract. Complex turbulence not at statistical equilibrium is impossible to simulate using eddy viscosity models due to a backscatter. This research presents the way to correct the Baldwin-Lomax model for non-equilibrium effects and gives an analysis of the energy evolution in the corrected model. Furthermore, a finite element approximation of the corrected eddy model with first-order and second-order time discretization are also presented. A numerical test is given to support the theory.

Key words. Backscatter, Eddy viscosity, Turbulence, Time discretization, Finite element method

1. Introduction.

The most common approach to the prediction of turbulent flow statistics is to add to the Navier-Stokes equations an eddy viscosity term, calibrate the term's coefficients, discretize the result and solve. A well-calibrated model and an effective numerical method have proven to predict reliably turbulent flows at statistical equilibrium. The question considered herein is how to extend such a model to non-equilibrium turbulence and how to adapt algorithms to the extended model. The first work on the approach herein was for the Smagorinsky model in [32]. Herein, we extend the Baldwin-Lomax model, shown below. In the modeling process, several choices must be made. We take a different path through these in developing the non-equilibrium extension than in [32]. Let us begin with the time dependent incompressible Navier-Stokes (NS) equations:

$$\begin{aligned} &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \text{ and } \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\\ &\mathbf{u} = 0 \text{ on } \partial\Omega, \text{ and } \int_\Omega p\, d\mathbf{x} = 0,\\ &\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d(d{=}2{,}3)$ is a bounded polyhedral domain; $\mathbf{u} : \Omega\times[0,T] \to \mathbb{R}^d$ is the fluid velocity; $p : \Omega\times(0,T] \to \mathbb{R}$ is the fluid pressure. The body force $\mathbf{f}$ is known. Re is the Reynolds number and $\nu = \frac{1}{Re}$.

There are many approaches to simulating turbulent flows, see [11, 12, 13, 14, 15]. One of the most commonly used is to model the ensemble-averaged Navier-Stokes equation by eddy viscosity, discretize then solve. In this approach, instantaneous variables are decomposed into the mean (ensemble averaging) and fluctuating components, then reintroduced into the governing equations to obtain the ensemble averaging equations. However, the system is not closed and contains Reynolds stress term that represents the effects of fluctuation. This Reynolds stress must be modeled in order to close the system.

Given the ensemble averaging of fluid velocity and pressure

$$\langle\mathbf{u}\rangle(\mathbf{x},t) = \frac{1}{J}\sum_{j=1}^{J}\mathbf{u}(\mathbf{x},t;\omega_j), \text{ and } \langle p\rangle(\mathbf{x},t) = \frac{1}{J}\sum_{j=1}^{J}p(\mathbf{x},t;\omega_j). \tag{1.2}$$

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Decompose the pressure and velocity into mean and fluctuations:

$$\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}', \text{ and } p = \langle p \rangle + p', \tag{1.3}$$

where $\langle \mathbf{u} \rangle, \langle p \rangle$ are the mean and $\mathbf{u}', p'$ are the fluctuating components. Substituting the ensemble averaging variables into the NS equations (1.1) yields the ensemble averaging equations.

$$\begin{aligned} &\langle \mathbf{u} \rangle_t + \langle \mathbf{u} \rangle \cdot \nabla \langle \mathbf{u} \rangle - \nu \Delta \langle \mathbf{u} \rangle - \nabla \cdot \mathrm{R}(\mathbf{u}, \mathbf{u}) + \nabla \langle p \rangle = \mathbf{f}, \\ &\nabla \cdot \langle \mathbf{u} \rangle = 0, \end{aligned} \tag{1.4}$$

where the Reynolds stress $\mathrm{R}(\mathbf{u}, \mathbf{u}) := \langle \mathbf{u} \rangle \otimes \langle \mathbf{u} \rangle - \langle \mathbf{u} \otimes \mathbf{u} \rangle = -\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$, see, e.g., [14, 16, 23].

By the Boussinesq assumption and eddy viscosity hypothesis ([20, 21, 22, 25]), the Reynolds stress $\mathrm{R}(\mathbf{u}, \mathbf{u})$ is modeled by $(\nu_T(\langle \mathbf{u} \rangle) \nabla \langle \mathbf{u} \rangle)$. Note that the turbulence eddy viscosity $\nu_T(\langle \mathbf{u} \rangle) > 0$. This is the standard eddy viscosity (EV) model:

$$\begin{aligned} &\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} - \nu \Delta \mathbf{w} - \nabla \cdot (\nu_T(\mathbf{w}) \nabla \mathbf{w}) + \nabla q = \mathbf{f}, \\ &\nabla \cdot \mathbf{w} = 0. \end{aligned} \tag{1.5}$$

The solution $(\mathbf{w}, q)$ of (1.5) is an approximation of the mean $(\langle \mathbf{u} \rangle, \langle p \rangle)$.

There have been many techniques to predict the turbulent eddy viscosity ν_T (e.g., [12, 13, 14, 15, 17, 27, 28, 29]). However, EV models have difficulties in simulating backscatter or complex turbulence not at statistical equilibrium, see, e.g., [24, 26, 30]. Since $\nu_T > 0$, the term $-\nabla \cdot (\nu_T(\mathbf{w}) \nabla \mathbf{w})$ in (1.5) can only represent dissipative effects of the Reynolds stress. In order to precisely characterize backscatter, Jiang and Layton [32] presented two approaches to correcting EV models and obtained new models of turbulence not at statistical equilibrium, analyzed the corrected Smagorinsky model and gave algorithms for its discretization.

Like many EV models, the Baldwin-Lomax model (see, e.g., [12, 14, 34]) begins simply and then has evolved substantial complexity to model effects such as separation and wakes not well described by the basic model. We consider herein only its simplest form for which $\nu_T(\langle \mathbf{u} \rangle) = l(\mathbf{x})^2 |\nabla \times \langle \mathbf{u} \rangle|$. Here, $l(\mathbf{x})$ is a mixing length that depends on the distance to the wall. The model extension, analysis and algorithms herein are adapted to more intricate, algebraic $\nu_T(\langle \mathbf{u} \rangle)$ with only notational complexity. The corrected model studied herein is as follows.

$$\begin{aligned} &\mathbf{w}_t + \beta^2 \nabla \times (l^2(\mathbf{x}) \nabla \times \mathbf{w}_t) + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \times (l^2(\mathbf{x}) |\nabla \times \mathbf{w}| \nabla \times \mathbf{w}) - \nu \Delta \mathbf{w} + \nabla q = \mathbf{f}, \\ &\nabla \cdot \mathbf{w} = 0. \end{aligned} \tag{1.6}$$

Here, β is a positive model calibration parameter. The eddy viscosity term is expressed in rotational (curl - curl) form. The model's mixing length $l(\mathbf{x})$ has the property that $0 \le l(\mathbf{x}) \to 0$ as $\mathbf{x} \to \partial\Omega$. The effect of the true Reynolds stress on the mean flow is, on time average, dissipative, see [22, 32, 33]. We prove in Section 3 that the time averaged effect of the terms that modeled the Reynolds stress is also dissipative.

$$\begin{aligned} &\liminf_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \mathbf{w}_t \cdot \nabla \times \mathbf{w} + l^2(\mathbf{x}) |\nabla \times \mathbf{w}|^3) d\mathbf{x} dt \\ &= \liminf_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega l^2(\mathbf{x}) |\nabla \times \mathbf{w}|^3 d\mathbf{x} dt \ge 0. \end{aligned}$$

Our numerical tests in Section 6 show that the term that models pointwise in time, statistical backscatter, $\beta^2 \nabla \times (l^2(\mathbf{x}) \nabla \times \mathbf{w}_t)$, does result in bursts, wherein

$$\int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \mathbf{w}_t \cdot \nabla \times \mathbf{w} + l^2(\mathbf{x}) |\nabla \times \mathbf{w}|^3) d\mathbf{x} < 0.$$

In other words, the eddy viscosity accounts for the persistent effect of the Reynolds stress while the new term accounts for statistical backscatter without artificial negative viscosities. We also present and analyze a numerical method for accurate solution of the new model (1.6) in Section 4. The challenges that occur here are (i) to develop algorithms that are small extensions of the standard methods for the usual Baldwin-Lomax model, and (ii) to perform the analysis for coefficients $l(\mathbf{x}) \to 0$ at walls.

The rest of this paper is organized as follows. Section 2 introduces some notations and preliminaries. In Section 3, we summarized the corrected EV model (1.6) and its properties. In Section 4, a complete analysis are given to demonstrate the stability and convergence of the standard finite element approximation of the model (1.6). Section 5 introduces first and second order time stepping schemes for the model (1.6). A numerical example is given to testify the model (1.6) in Section 6. Finally, Section 7 presents the conclusion.

2. Notations and Preliminaries. This section introduces some widely used notations, inequalities and lemmas. The standard notations $H^k(\Omega)$, $H_0^k(\Omega)$, $W^{k,p}(\Omega)$ denote Sobolev spaces, and $L^p(\Omega)$ denotes L^p spaces, see [1]. The $H^k(\Omega)$ norm and $L^p(\Omega)$ ($p \neq 2$) norm are denoted by $\|\cdot\|_k$ and $\|\cdot\|_{L^p}$, respectively. The $L^2(\Omega)$ norm is denoted by $\|\cdot\|$ and its corresponding inner products by $(\cdot,\cdot)$. Denote the dual space of $H_0^k(\Omega)$ by $H^{-k}(\Omega)$ and its norm by $\|\cdot\|_{-k}$. Furthermore, $\|\cdot\|_{\ell^p}$ is the ℓ^p- norm of vectors in $\mathbb{R}^d$, see [10]. Constants C are different in different places throughout the paper, which do not depend on mesh size and time step but may depend on some known data such as $\Omega, \nu, \mathbf{f}, \cdots$, and so on. We introduce the following spaces and their norms:

$$
\begin{aligned}
&L^p(0,T;L^q(\Omega)) := \{\mathbf{v}(\mathbf{x},t) : (\int_0^T \|\mathbf{v}(\cdot,t)\|_{L^q}^p dt)^{\frac{1}{p}} < \infty\},\\
&L^\infty(0,T;L^q(\Omega)) := \{\mathbf{v}(\mathbf{x},t) : \sup_{0\le t\le T} \|\mathbf{v}(\cdot,t)\|_{L^q} < \infty\},\\
&\|\mathbf{v}\|_{L^p(0,T;L^q)} = (\int_0^T \|\mathbf{v}(\cdot,t)\|_{L^q}^p dt)^{\frac{1}{p}}, \quad \|\mathbf{v}\|_{L^\infty(0,T;L^q)} = \sup_{0\le t\le T} \|\mathbf{v}(\cdot,t)\|_{L^q}.
\end{aligned}
$$

Here, $1 \le p < \infty, 1 \le q \le \infty$. The velocity space X and pressure space Q are defined as follows.

$$
\begin{aligned}
X &:= H_0^1(\Omega)^d = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = 0\},\\
Q &:= L_0^2(\Omega)^d = \{q \in L^2(\Omega) : \int_\Omega q = 0\}.
\end{aligned}
$$

The divergence free space V^2 of $X = W_0^{1,2}(\Omega)$ is given by

$$
V := \{\mathbf{v} \in X : (\nabla\cdot\mathbf{v}, q) = 0 \quad \forall q \in Q)\}.
$$

and the divergence free subspace of $W_0^{1,3}(\Omega)$ is similarly denoted V^3. Define

$$
B(\mathbf{u},\mathbf{v}) := \mathbf{u}\cdot\nabla\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}, \quad b(\mathbf{u},\mathbf{v},\mathbf{w}) := (B(\mathbf{u},\mathbf{v}),\mathbf{w}), \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in X.
$$

Then, we have

$$
\begin{aligned}
&b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}[b(\mathbf{u},\mathbf{v},\mathbf{w}) - b(\mathbf{u},\mathbf{w},\mathbf{v})], \qquad b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0,\\
&b(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}), \quad \text{if} \ \ \mathbf{u} \in V.
\end{aligned}
$$

The weak formulation of (1.6), satisfied by sufficiently smooth solutions, is : Find $(\mathbf{w}, q) \in (X, Q)$ satisfying

$$
\begin{aligned}
&(\mathbf{w}_t,\mathbf{v}) + (l^2(\mathbf{x})\nabla\times\mathbf{w}_t, \nabla\times\mathbf{v}) + b(\mathbf{w},\mathbf{w},\mathbf{v}) + (l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w}, \nabla\times\mathbf{v})\\
&\quad + \nu(\nabla\mathbf{w},\nabla\mathbf{v}) - (q,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}) \qquad \forall\mathbf{v} \in X, \qquad (2.1)\\
&(\nabla\cdot\mathbf{w}, r) = 0 \qquad \forall r \in Q.
\end{aligned}
$$

Let Π_h be a set of triangulations of Ω with $\overline{\Omega} = \bigcup_{K\in\Pi_h} K$ ($h = \sup_{K\in\Pi_h} diam(K)$). It is uniformly regular when $h \to 0$. $X_h \subset X, Q_h \subset Q$ are finite element spaces that satisfy the discrete inf-sup condition: $\inf_{q\in Q_h} \sup_{\mathbf{v}\in X_h} \frac{(q,\nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\|\|q\|} \geq C > 0$. The subspace V_h of X_h is defined by

$$V_h := \{\mathbf{v}_h \in X_h : (\nabla\cdot\mathbf{v}_h, q_h) = 0 \quad \forall q_h \in Q_h\}.$$

Next, we present some inequalities and lemmas which will be used in later sections.

Some Inequalities :

A1: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$, then we have

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|, \tag{2.2}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|. \tag{2.3}$$

A2: Assume $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$Young's\ inequality: \ ab \leq \frac{1}{q\varepsilon^{p/q}}a^p + \frac{\varepsilon}{q}b^q \quad \forall a, b, \varepsilon > 0, \tag{2.4}$$

$$H\ddot{o}lder's\ inequality: \ |(\mathbf{v},\mathbf{w})| \leq \|\mathbf{v}\|_{L^p}\|\mathbf{w}\|_{L^q} \quad \forall \mathbf{v} \in L^p, \mathbf{w} \in L^q, \tag{2.5}$$

$$Poincar\acute{e}\ inequality: \ \|\mathbf{v}\| \leq C\|\nabla\mathbf{v}\| \quad \forall \mathbf{v} \in X. \tag{2.6}$$

Lemma 2.1. **(The discrete Gronwall's inequality)** *Suppose that n and N are nonnegative integers, $n \leq N$. The real numbers $a_n, b_n, c_n, d_n, \triangle t$ are nonnegative and satisfy that*

$$a_N + \Delta t\sum_{n=0}^{N} b_n \leq \Delta t\sum_{n=0}^{N}(c_n a_n + d_n).$$

Then,

$$a_N + \Delta t\sum_{n=0}^{N} b_n \leq exp(\Delta t\sum_{n=0}^{N}\frac{c_n}{1-\Delta t c_n})(\Delta t\sum_{n=0}^{N} d_n),$$

provided that $\Delta t c_n < 1$ for each n.

Lemma 2.2. **(Strong monotonicity and local Lipschitz-continuity)** *There exist positive constants $\underline{C}$ and $\overline{C}$ such that for all $\mathbf{u}', \mathbf{u}'', \mathbf{v} \in (W^{1,3}(\Omega))^d$, we have*

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'', \nabla\times(\mathbf{u}'-\mathbf{u}''))\\ &\geq \underline{C}\|l^{\frac{2}{3}}(x)\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}^3,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'', \nabla\times\mathbf{v})\\ &\leq \overline{C}\gamma\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3},\end{aligned} \tag{2.8}$$

where $l : \mathbf{x} \in \Omega \mapsto \mathbb{R}$ is a non-negative function with $l \in L^\infty(\Omega)$, and $\gamma = \max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}'\|_{L^3}, \|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}''\|_{L^3}\}$.

Remark 1. *Inequalities (A1) and (A2) can be found in [1, 9, 23]. Lemma 2.1 can be found in [31]. The proof of Lemma 2.2 is given in the Appendix.*

3. Analysis of the corrected EV model. This section first recalls the important properties of standard EV models (1.5). Then it presents the derivation of the corrected scheme (1.6) based on (1.5) and shows that the new model (1.6) maintains the time-averaged dissipative effect of the Reynolds stress.

Taking the inner product of the first and second equation in ensemble averaging NS equations (1.4) with $\langle \mathbf{u} \rangle$ and $\langle p \rangle$ respectively, we can obtain the kinetic energy equation for the mean

$$\frac{1}{2}\frac{d}{dt}\|\langle \mathbf{u}\rangle\|^2 + \nu\|\nabla\langle \mathbf{u}\rangle\|^2 + \int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx = \int_\Omega \mathbf{f}\cdot\langle \mathbf{u}\rangle dx. \tag{3.1}$$

In (3.1), the right hand side term $\int_\Omega \mathbf{f}\cdot\langle \mathbf{u}\rangle dx$ is the energy input. The term $\frac{1}{2}\frac{d}{dt}\|\langle \mathbf{u}\rangle\|^2$ is the changing rate of the kinetic energy of the mean. The term $\nu\|\nabla\langle \mathbf{u}\rangle\|^2$ is the energy dissipation of the mean. The term $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx$ indicates the effect of fluctuations on the mean. Moreover, if the term $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx > 0$, the effect is dissipative but if $\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx < 0$, fluctuations transfer energy back to the mean which is called backscatter.

There are two key properties of Reynolds stress (proven in [32]). Time averaged dissipativity:

$$\lim\inf_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dxdt = \lim\inf_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \nu\langle|\nabla \mathbf{u}'|^2\rangle dxdt \geq 0 \tag{3.2}$$

Second, the variance evolution equation:

$$\int_\Omega \mathrm{R}(\mathbf{u},\mathbf{u}) : \nabla\langle \mathbf{u}\rangle dx = \frac{1}{2}\frac{d}{dt}\int_\Omega\langle|\mathbf{u}'|^2\rangle dx + \int_\Omega \nu\langle|\nabla \mathbf{u}'|^2\rangle dx. \tag{3.3}$$

The inequality (3.2) is consistent with the Boussinesq assumption that the effects of turbulent fluctuations are dissipative on the mean in the time averaged case. The interpretation of (3.3) is as follows. For flows at statistical equilibrium $\frac{d}{dt}\int_\Omega\langle|\mathbf{u}'|^2\rangle dx = 0$, while the second term $\int_\Omega \nu\langle|\nabla \mathbf{u}'|^2\rangle dx$ is clearly dissipative. This term is modeled by an eddy viscosity acting on $\langle \mathbf{u}\rangle$ that dissipates energy pointwise in both space and time. Thus the (space averaged) pointwise in time deviation from dissipativity must arise from the first term on the RHS of (3.3). Therefore, the term $\frac{1}{2}\frac{d}{dt}\int_\Omega\langle|\mathbf{u}'|^2\rangle dx$ should be modeled in the corrected EV scheme to represent backscatter.

REMARK 2 (On the Boussinesq hypothesis). *The proof of the Boussinesq hypothesis in [22] is by extracting information on fluctuations from the energy equality for realizations. It was done for strong solutions, the natural setting for turbulence theory since it can be phrased in terms of the Reynolds stresses. This report also gave, Section 2.1 p. 2356, the equation for evolution of space averaged variance and turbulence intensities as well as an extended survey of the long history on the problem. In [5] an extension was given for ensembles of both initial conditions and body forces. In [32] it was shown that (using a result of Duchon and Robert [3]) a similar proof holds for weak solutions and that the spacial localization of backscatter depends only on 4 quantities: the variance of u and ∇u, the skewness of u and the velocity-pressure covariance. The (highly nontrivial) connection between the formulation in terms of the Reynolds stresses and the above proof for weak solutions was recently made by Berselli and Lewandowski [2], which contains the most general formulation of the result. Dissipativity of fluctuations and convergence to statistical equilibrium was extended to space and time discretizations in [33] and to MHD turbulence (through the Elsässer variables) in [7]. Once it was pointed out that dissipativity, while violated at some instants in time, emerges for time averaged quantities, various forms of the Boussinesq hypothesis' proof are implicit in Reynolds transport theory. For example, space and time averaging the trace of the transport equations of the Reynolds stresses, Jovanovic [6] Section 5.1 p. 110, yields the dissipativity of fluctuations. For time, not statistical, averages* *dissipativity of fluctuations result appears already (well hidden) in equation (3.127) p. 75, in Chacón- Rebollo and Lewandowski [29]. This connection was developed further by Lewandowski [8] and in several new directions in [2].*

The key point to describe the Reynolds stress term is how to model $\mathbf{u}'$ via $\langle \mathbf{u} \rangle$. By the Kolmogorov-Prandtl relation for the turbulent viscosity,

$$\nu_T(\langle \mathbf{u} \rangle) = c_l l \sqrt{k'}, \tag{3.4}$$

where $k' = \frac{1}{2}\langle |\mathbf{u}'|^2 \rangle$, c_l is a proportionality constant and l is the mixing length. Consider the simplest case (the inner layer viscosity) of the Baldwin-Lomax model

$$\nu_T(\langle \mathbf{u} \rangle) = l^2 |\nabla \times \langle \mathbf{u} \rangle|, \tag{3.5}$$

where l is the mixing length, such as $l(\mathbf{x}) = 0.41 \times \text{distance}\{x, \partial\Omega\}$. Combine (3.4) with (3.5) to obtain the fluctuation model

$$\text{action}(\mathbf{u}') \simeq \beta l \nabla \times \langle \mathbf{u} \rangle, \tag{3.6}$$

where $\beta > 0$ is a model calibration parameter. Then the Reynolds stresses term $\int_\Omega \mathrm{R}(\mathbf{u}, \mathbf{u}) : \nabla \langle \mathbf{u} \rangle dx$ becomes

$$\int_\Omega \mathrm{R}(\mathbf{u}, \mathbf{u}) : \nabla \langle \mathbf{u} \rangle dx \simeq \frac{1}{2}\frac{d}{dt}\int_\Omega \beta^2 l^2 |\nabla \times \langle \mathbf{u} \rangle|^2 dx + \int_\Omega l^2 |\nabla \times \langle \mathbf{u} \rangle| |\nabla \times \langle \mathbf{u} \rangle|^2 dx, \tag{3.7}$$

Notice the time derivative term comes from the fluctuation model (3.6) and the eddy viscosity term is based on (3.5). Then, (3.7) leads to the closure model

$$-\nabla \cdot \mathrm{R}(\mathbf{u}, \mathbf{u}) \simeq \beta^2 \nabla \times (l^2 \nabla \times \langle \mathbf{u} \rangle_t) + \nabla \times (l^2 |\nabla \times \langle \mathbf{u} \rangle| \nabla \times \langle \mathbf{u} \rangle). \tag{3.8}$$

Combining (3.8) with (1.4) and calling the model's approximation to $\langle \mathbf{u} \rangle$ and $\langle p \rangle$, $\mathbf{w}$ and q yields model (1.6).

$$\mathbf{w}_t + \beta^2 \nabla \times (l^2(\mathbf{x}) \nabla \times \mathbf{w}_t) + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \times (l^2(\mathbf{x}) |\nabla \times \mathbf{w}| \nabla \times \mathbf{w}) - \nu \Delta \mathbf{w} + \nabla q = \mathbf{f},$$
$$\nabla \cdot \mathbf{w} = 0.$$

Remark 3. *Although existence theory for* (1.6) *is not the issue considered in this report, it is useful to note the model's mathematical structure. Compared with the NSE, this model contains two additional terms. The first is* $\beta^2 \nabla \times (l^2 \nabla \times \mathbf{w}_t)$ *which is similar to the commonly used dispersive regularization* $-\Delta \mathbf{w}_t$. *The second is* $\nabla \times (l^2 |\nabla \times \mathbf{w}| \nabla \times \mathbf{w})$. *This second term is strongly monotone and locally Lipschitz being similar to a p-Laplacian term. For* $l > 0$ *it is readily seen that Galerkin approximations in spaces of Stokes eigenfunctions belong to* $L^\infty(0, T; V^2) \cap L^3(0, T; V^3)$ *(and by a limit argument under mild conditions on* $l(x)$*) the solution* $\mathbf{w}$ *satisfies the regularity*

$$\mathbf{w} \in L^\infty(0, T; V^2) \cap L^3(0, T; V^3) \tag{3.9}$$

We therefore conjecture that (again under mild conditions on $l(x)$*) existence and uniqueness of strong solutions hold for the model.*

For our purposes herein, we shall assume the model has a solution in the following sense.

Definition 3.1. *A distributional solution* $\mathbf{w}$ *of model* (1.6) *is a strong solution if* $\mathbf{w}$ *has regularity (3.9),* $\mathbf{w}(x, t) \to \mathbf{w}_0(x)$ *in* $L^2(\Omega)$ *as* $t \to 0$ *and if* $\mathbf{w}$ *satisfies the model's weak form ((2.1) above) for all* $\mathbf{v} \in L^\infty(0, T; V^2) \cap L^3(0, T; V^3)$.

Operationally, the above definition means one may set $\mathbf{v} = \mathbf{w}$ in (2.1). Our numerical tests in Section 6 include ones with different boundary conditions. For these the variational formulation and solution notion must be appropriately adapted. Next, we give a theoretical analysis of model (1.6) and show in Theorem 3.2 that this new model still maintains the property that the effects of turbulent fluctuations on the mean are dissipative in long time averaging sense.

THEOREM 3.2. *Assume* $\mathbf{f} \in L^{\infty}(0,\infty;H^{-1}(\Omega))$*. For the strong solution* $\mathbf{w}$ *of model* (1.6)*, we have*

$$\mathbf{w},\ l\nabla\times\mathbf{w} \in L^{\infty}(0,\infty;L^2(\Omega)), \tag{3.10}$$

$$\lim\inf_{T\to\infty}[\frac{1}{T}\int_0^T \nu\|\nabla\mathbf{w}\|^2dt + \frac{1}{T}\int_0^T\int_{\Omega} l^2|\nabla\times\mathbf{w}|^3d\mathbf{x}dt] = \lim\inf_{T\to\infty}\frac{1}{T}\int_0^T\int_{\Omega}\mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt, \tag{3.11}$$

$$\lim\inf_{T\to\infty}\frac{1}{T}\int_0^T\int_{\Omega}[\beta^2\nabla\times(l^2\nabla\times\mathbf{w}_t)\cdot\mathbf{w} + l^2|\nabla\times\mathbf{w}|^3]d\mathbf{x}dt \geq 0. \tag{3.12}$$

Proof. Taking the inner product of model (1.6) with $\mathbf{w}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2) + \nu\|\nabla\mathbf{w}\|^2 + \int_{\Omega} l^2(\mathbf{x})|\nabla\times\mathbf{w}|^3d\mathbf{x} = \int_{\Omega}\mathbf{f}\cdot\mathbf{w}d\mathbf{x}\\ &\leq \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2 + \frac{\nu}{2}\|\nabla\mathbf{w}\|^2.\end{aligned} \tag{3.13}$$

Rearrange the terms in inequality (3.13) to get

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2) + \nu\|\nabla\mathbf{w}\|^2 + 2\int_{\Omega} l^2(\mathbf{x})|\nabla\times\mathbf{w}|^3d\mathbf{x} \leq \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2. \tag{3.14}$$

For the term $\nu\|\nabla\mathbf{w}\|^2$, we have

$$\begin{aligned}\nu\|\nabla\mathbf{w}\|^2 &\geq \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2}\|\nabla\mathbf{w}\|^2 = \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2}\|\nabla\times\mathbf{w}\|^2\\ &\geq \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2L_{max}}\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2,\end{aligned} \tag{3.15}$$

where we use the Poincaré inequality, and $L_{max} = \sup_{\mathbf{x}\in\Omega}|l(\mathbf{x})|^2$.

Since $2\int_{\Omega} l^2(\mathbf{x})|\nabla\times\mathbf{w}|^3d\mathbf{x} \geq 0$, combining (3.15) and (3.14), we can obtain

$$\frac{d}{dt}(\|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2) + \frac{\nu}{2C_{PF}}\|\mathbf{w}\|^2 + \frac{\nu}{2L_{max}}\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2 \leq \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2. \tag{3.16}$$

Let $g(t) := \|\mathbf{w}\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}\|^2$, we have

$$g'(t) + \alpha g(t) \leq \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2 \leq \frac{1}{\nu}\|\mathbf{f}\|_{L^{\infty}(0,\infty;H^{-1}(\Omega))}^2 < \infty, \tag{3.17}$$

where $\alpha = \max\{\frac{\nu}{2C_{PF}}, \frac{\nu}{2\beta^2L_{max}}\} > 0$. By inequality (3.17), we can deduce (3.10). Then integrate (3.13) on $[0,T]$ and divide it by T to get

$$\begin{aligned}&\frac{1}{2T}(\|\mathbf{w}(T)\|^2 + \beta^2\|l\nabla\times\mathbf{w}(T)\|^2) + \frac{1}{T}\int_0^T\nu\|\nabla\mathbf{w}\|^2dt + \frac{1}{T}\int_0^T\int_{\Omega} l^2|\nabla\times\mathbf{w}|^3d\mathbf{x}dt\\ &= \frac{1}{2T}(\|\mathbf{w}(0)\|^2 + \beta^2\|l\nabla\times\mathbf{w}(0)\|^2) + \frac{1}{T}\int_0^T\int_{\Omega}\mathbf{f}\cdot\mathbf{w}d\mathbf{x}dt.\end{aligned} \tag{3.18}$$

Using (3.10), we have

$$\begin{aligned} &\mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T \nu\|\nabla \mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T \int_\Omega l^2|\nabla \times \mathbf{w}|^3 d\mathbf{x}dt \\ &= \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T \int_\Omega \mathbf{f}\cdot \mathbf{w} d\mathbf{x}dt, \end{aligned} \tag{3.19}$$

which implies (3.11).

Next, consider the Reynolds stress term. Note that (3.18) can be rewritten as

$$\begin{aligned} &\frac{1}{2T}\|\mathbf{w}(T)\|^2 + \frac{\beta^2}{2T}\int_0^T \frac{d}{dt}\|l\nabla \times \mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T \nu\|\nabla \mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T \int_\Omega l^2|\nabla \times \mathbf{w}|^3 d\mathbf{x}dt \\ &= \frac{1}{2T}\|\mathbf{w}(0)\|^2 + \frac{1}{T}\int_0^T \int_\Omega \mathbf{f}\cdot \mathbf{w} d\mathbf{x}dt. \end{aligned} \tag{3.20}$$

Then by (3.19) and (3.20), we have

$$\begin{aligned} &\frac{\beta^2}{2T}\int_0^T \frac{d}{dt}\|l\nabla \times \mathbf{w}\|^2 dt + \frac{1}{T}\int_0^T \int_\Omega l^2|\nabla \times \mathbf{w}|^3 d\mathbf{x}dt \\ &= \mathcal{O}(\frac{1}{T}) - \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T \int_\Omega \mathbf{f}\cdot \mathbf{w} d\mathbf{x}dt - \frac{1}{T}\int_0^T \nu\|\nabla \mathbf{w}\|^2 dt \\ &= \mathcal{O}(\frac{1}{T}) - \mathcal{O}(\frac{1}{T}) + \frac{1}{T}\int_0^T \int_\Omega l^2|\nabla \times \mathbf{w}|^3 d\mathbf{x}dt, \end{aligned} \tag{3.21}$$

which completes the proof of (3.12). ▯

4. The finite element approximation. In this section, we present the finite element approximation for the corrected EV model (1.6) and analyze its stability and convergence. One deviation from the (now standard) numerical analysis of the NS equations is that the balance between the two model terms is not be accounted for. Alternatively speaking, our model has two new terms not in the NS equations. Their relative size, β, is important in the analysis .

The finite element approximation of (2.1) is : Find $(\mathbf{w}_h, q_h) \in (X_h, Q_h)$ satisfying

$$\begin{aligned} &(\mathbf{w}_{h,t}, \mathbf{v}_h) + \beta^2(l^2(\mathbf{x})\nabla \times \mathbf{w}_{h,t}, \nabla \times \mathbf{v}_h) + b(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{w}_h, \nabla \mathbf{v}_h) - (q_h, \nabla\cdot \mathbf{v}_h) \\ &\quad + (l^2(\mathbf{x})|\nabla \times \mathbf{w}_h|\nabla \times \mathbf{w}_h, \nabla \times \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in X_h, \\ &(\nabla\cdot \mathbf{w}_h, r_h) = 0 \qquad \forall r_h \in Q_h. \end{aligned} \tag{4.1}$$

THEOREM 4.1. *Method* (4.1) *is unconditionally energy stable. For all* $0 < t \leq T$,

$$\begin{aligned} &\|\mathbf{w}_h\|^2(t) + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{w}_h\|^2(t) + 2\int_0^t \int_\Omega |l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}_h|^3 d\mathbf{x}ds + \nu\int_0^t \|\nabla \mathbf{w}_h\|^2 ds \\ &\leq \frac{1}{\nu}\int_0^t \|\mathbf{f}\|_{-1}^2 ds + \|\mathbf{w}_h\|^2(0) + \|l(\mathbf{x})\nabla \times \mathbf{w}_h\|^2(0). \end{aligned} \tag{4.2}$$

Proof. Set $\mathbf{v}_h = \mathbf{w}_h, r_h = q_h$ in (4.1) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{w}_h\|^2 + \frac{\beta^2}{2}\frac{d}{dt}\|l(\mathbf{x})\nabla \times \mathbf{w}_h\|^2 + \int_\Omega l^2(\mathbf{x})|\nabla \times \mathbf{w}_h|^3 d\mathbf{x} + \nu\|\nabla \mathbf{w}_h\|^2 \\ &= (\mathbf{f}, \mathbf{w}_h) \leq \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2 + \frac{\nu}{2}\|\nabla \mathbf{w}_h\|^2. \end{aligned} \tag{4.3}$$

Multiplying (4.3) by 2 and integrating it from 0 to t complete the proof.
∎

The next theorem shows the convergence result of method (4.1).

THEOREM 4.2. *Assume* $\mathbf{w}$ *is a strong solution of model* (1.6). *Suppose that* $\nabla \mathbf{w} \in L^4(0,T;L^2(\Omega))$. *Then*

$$\begin{aligned}
&\sup_{0\le t\le T}\|\mathbf{w}-\mathbf{w}_h\|^2(t)+\sup_{0\le t\le T}\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{w}_h)\|^2(t)+\nu\|\nabla(\mathbf{w}-\mathbf{w}_h)\|^2_{L^2(0,T;L^2)}\\
&\quad+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{w}_h)\|^3_{L^3(0,T;L^3)}\\
&\le C[\inf_{\mathbf{v}_h\in\mathbf{X}_h,r_h\in Q_h}\{\sup_{0\le t\le T}\|\mathbf{w}-\mathbf{v}_h\|^2(t)+\sup_{0\le t\le T}\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)\|^2(t)\\
&\quad+\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)_t\|^2_{L^2(0,T;L^2)}+\|\nabla(\mathbf{w}-\mathbf{v}_h)\|^2_{L^4(0,T;L^2)}+\|(\mathbf{w}-\mathbf{v}_h)_t\|^2_{L^2(0,T;L^2)}\\
&\quad+\|\nabla(\mathbf{w}-\mathbf{v}_h)\|^2_{L^2(0,T;L^2)}+\|q-r_h\|^2_{L^2(0,T;L^2)}\\
&\quad+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)\|^{\frac{3}{2}}_{L^3(0,T;L^3)}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{v}_h)\|^3_{L^3(0,T;L^3)}\}\\
&\quad+\|\mathbf{w}-\mathbf{w}_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}-\mathbf{w}_h)\|^2(0)].
\end{aligned} \tag{4.4}$$

REMARK 4. *The convergence proof as a necessarily technical synthesis of two proof ideas. The first is the normal error analysis of Galerkin FEMs for the NSE. The second is the error analysis of Galerkin FEMs for operations that are strongly monotone and locally Lipschitz (such as the p-Laplacian).*

Proof. Let $\widetilde{\mathbf{w}}:[0,T]\to V_h$ be arbitrary. Splitting the error $\mathbf{e}=\mathbf{w}-\mathbf{w}_h$ via $\mathbf{e}=\eta-\phi_h$, where $\eta=\mathbf{w}-\widetilde{\mathbf{w}}$, $\phi_h=\mathbf{w}_h-\widetilde{\mathbf{w}}$. Subtracting (4.1) from (2.1), we obtain

$$\begin{aligned}
&(\mathbf{e}_t,\mathbf{v}_h)+\beta^2(l^2(\mathbf{x})\nabla\times\mathbf{e}_t,\nabla\times\mathbf{v}_h)+b(\mathbf{e},\mathbf{w},\mathbf{v}_h)+b(\mathbf{w}_h,\mathbf{e},\mathbf{v}_h)\\
&\quad+\nu(\nabla\mathbf{e},\nabla\mathbf{v}_h)-(q-q_h,\nabla\cdot\mathbf{v}_h)\\
&\quad+(l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w}-l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\mathbf{v}_h)\\
&\quad+(l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}}-l^2(\mathbf{x})|\nabla\times\mathbf{w}_h|\nabla\times\mathbf{w}_h,\nabla\times\mathbf{v}_h)\\
&=0\qquad\forall\mathbf{v}_h\in X_h,\\
&(\nabla\cdot\mathbf{e},r_h)=0\qquad\forall r_h\in Q_h.
\end{aligned} \tag{4.5}$$

Setting $\mathbf{v}_h=\phi_h$ in (4.5), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2+\frac{\beta^2}{2}\frac{d}{dt}\|l(\mathbf{x})\nabla\times\phi_h\|^2+\nu\|\nabla\phi_h\|^2\\
&\quad+(l^2(\mathbf{x})|\nabla\times\mathbf{w}_h|\nabla\times\mathbf{w}_h-l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h)\\
&=(\eta_t,\phi_h)+(l^2(\mathbf{x})\nabla\times\eta_t,\nabla\times\phi_h)+\nu(\nabla\eta,\nabla\phi_h)-(q-r_h,\nabla\cdot\phi_h)\\
&\quad+b(\eta,\mathbf{w},\phi_h)-b(\phi_h,\mathbf{w},\phi_h)+b(\mathbf{w}_h,\eta,\phi_h)\\
&\quad+(l^2(\mathbf{x})|\nabla\times\mathbf{w}|\nabla\times\mathbf{w}-l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h),
\end{aligned} \tag{4.6}$$

where $r_h\in Q_h$ is arbitrary. Next, we need to bound the terms in (4.6). Using (2.7) in Lemma 2.2, we can obtain

$$(l^2(\mathbf{x})|\nabla\times\mathbf{w}_h|\nabla\times\mathbf{w}_h-l^2(\mathbf{x})|\nabla\times\widetilde{\mathbf{w}}|\nabla\times\widetilde{\mathbf{w}},\nabla\times\phi_h)\ge\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|^3_{L^3}. \tag{4.7}$$

All the terms on the right hand side of (4.6) are bounded as follows. For the term (η_t,ϕ_h),

$$(\eta_t,\phi_h)\le C\|\eta_t\|\|\nabla\phi_h\|\le C\|\eta_t\|^2+\frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{4.8}$$

For the term $(l^2(\mathbf{x})\nabla \times \eta_t, \nabla \times \phi_h)$,

$$\begin{aligned}(l^2(\mathbf{x})\nabla \times \eta_t, \nabla \times \phi_h) &\le \|l(\mathbf{x})\nabla \times \eta_t\|\|l(\mathbf{x})\nabla \times \phi_h\| \\ &\le \frac{1}{2}\|l(\mathbf{x})\nabla \times \eta_t\|^2 + \frac{1}{2}\|l(\mathbf{x})\nabla \times \phi_h\|^2.\end{aligned} \tag{4.9}$$

The term $\nu(\nabla\eta, \nabla\phi_h)$ is bounded by

$$\nu(\nabla\eta, \nabla\phi_h) \le C\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{4.10}$$

The term $-(q - r_h, \nabla \cdot \phi_h)$ is bounded by

$$-(q - r_h, \nabla \cdot \phi_h) \le C\|q - r_h\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{4.11}$$

Using (2.2) and (2.3), we have the following estimate.

$$b(\eta, \mathbf{w}, \phi_h) \le C\|\nabla\eta\|\|\nabla\mathbf{w}\|\|\nabla\phi_h\| \le C\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{4.12}$$

$$b(\phi_h, \mathbf{w}, \phi_h) \le C\|\phi_h\|^{\frac{1}{2}}\|\nabla\mathbf{w}\|\|\nabla\phi_h\|^{\frac{3}{2}} \le C\|\nabla\mathbf{w}\|^4\|\phi_h\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2. \tag{4.13}$$

$$\begin{aligned}b(\mathbf{w}_h, \eta, \phi_h) &\le C\|\mathbf{w}_h\|^{\frac{1}{2}}\|\nabla\mathbf{w}_h\|^{\frac{1}{2}}\|\nabla\eta\|\|\nabla\phi_h\| \\ &\le C\|\mathbf{w}_h\|\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2 \\ &\le C\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2 + \frac{\nu}{12}\|\nabla\phi_h\|^2,\end{aligned} \tag{4.14}$$

where we use the stability bound $\sup_{0\le t\le T}\|\mathbf{w}_h\| \le C$ in Theorem 4.1. Lastly, by (2.8) in Lemma 2.2, we have

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla \times \mathbf{w}|\nabla \times \mathbf{w} - l^2(\mathbf{x})|\nabla \times \widetilde{\mathbf{w}}|\nabla \times \widetilde{\mathbf{w}}, \nabla \times \phi_h) \\ &\le \overline{C}\gamma\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \phi_h\|_{L^3},\end{aligned} \tag{4.15}$$

where $\gamma = \max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}\|_{L^3}, \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \widetilde{\mathbf{w}}\|_{L^3}\}$. Since

$$\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \widetilde{\mathbf{w}}\|_{L^3} \le \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}\|_{L^3} + \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}, \tag{4.16}$$

there exists

$$\gamma = \max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}\|_{L^3}, \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \widetilde{\mathbf{w}}\|_{L^3}\} \le \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}\|_{L^3} + \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}. \tag{4.17}$$

Combining (4.17) with (4.15), we obtain

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla \times \mathbf{w}|\nabla \times \mathbf{w} - l^2(\mathbf{x})|\nabla \times \widetilde{\mathbf{w}}|\nabla \times \widetilde{\mathbf{w}}, \nabla \times \phi_h) \\ &\le \overline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \phi_h\|_{L^3} \\ &\quad + \overline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}^2\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \phi_h\|_{L^3} \\ &\le C\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{w}\|_{L^3}^{\frac{3}{2}}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}^{\frac{3}{2}} + C\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \eta\|_{L^3}^3 + \frac{C}{2}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \phi_h\|_{L^3}^3.\end{aligned} \tag{4.18}$$

Combining (4.7)-(4.14), (4.18) with (4.6) gives to

$$\begin{aligned}&\frac{d}{dt}(\|\phi_h\|^2+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2)+\nu\|\nabla\phi_h\|^2+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|^3_{L^3}\\&\leq C\|\nabla\mathbf{w}\|^4\|\phi_h\|^2+\|l(\mathbf{x})\nabla\times\phi_h\|^2\\&\quad+\|l(\mathbf{x})\nabla\times\eta_t\|^2+C\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2+C\|\eta_t\|^2+C\|\nabla\eta\|^2+C\|q-r_h\|^2\\&\quad+C\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2+C\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|^{\frac{3}{2}}_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^{\frac{3}{2}}_{L^3}+C\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}.\end{aligned}\tag{4.19}$$

Integrate (4.19) from 0 to t to obtain

$$\begin{aligned}&\|\phi_h\|^2(t)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t)+\int_0^t[\nu\|\nabla\phi_h\|^2+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|^3_{L^3}]ds\\&\leq\|\phi_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0)\\&\quad+\int_0^t\max\{C\|\nabla\mathbf{w}\|^4,\frac{1}{\beta^2}\}(\|\phi_h\|^2+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2)ds\\&\quad+C\int_0^t[\|l(\mathbf{x})\nabla\times\eta_t\|^2+\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2+\|\eta_t\|^2+\|\nabla\eta\|^2+\|q-r_h\|^2\\&\quad+\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|^{\frac{3}{2}}_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^{\frac{3}{2}}_{L^3}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}]ds.\end{aligned}\tag{4.20}$$

Since $\int_0^T\|\nabla\mathbf{w}\|^4dt\leq C$, using Gronwall's inequality, we have

$$\begin{aligned}&\|\phi_h\|^2(t)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t)+\int_0^t[\nu\|\nabla\phi_h\|^2+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|^3_{L^3}]ds\\&\leq C\{\|\phi_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0)\\&\quad+\int_0^t[\|l(\mathbf{x})\nabla\times\eta_t\|^2+\|\nabla\mathbf{w}\|^2\|\nabla\eta\|^2+\|\eta_t\|^2+\|\nabla\eta\|^2+\|q-r_h\|^2\\&\quad+\|\nabla\mathbf{w}_h\|\|\nabla\eta\|^2+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|^{\frac{3}{2}}_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^{\frac{3}{2}}_{L^3}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}]ds\}\\&\leq C\{\|\phi_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0)+\int_0^t\|l(\mathbf{x})\nabla\times\eta_t\|^2(s)ds\\&\quad+(\int_0^t\|\nabla\mathbf{w}\|^4(s)ds)^{\frac{1}{2}}(\int_0^t\|\nabla\eta\|^4(s)ds)^{\frac{1}{2}}+\int_0^t\|\eta_t\|^2(s)ds+\int_0^t\|\nabla\eta\|^2(s)ds\\&\quad+\int_0^t\|q-r_h\|^2(s)ds+(\int_0^t\|\nabla\mathbf{w}_h\|^2(s)ds)^{\frac{1}{2}}(\int_0^t\|\nabla\eta\|^4(s)ds)^{\frac{1}{2}}\\&\quad+(\int_0^t\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}\|^3_{L^3}(s)ds)^{\frac{1}{2}}(\int_0^t\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}(s)ds)^{\frac{1}{2}}+\int_0^t\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}(s)ds\}.\end{aligned}\tag{4.21}$$

With the stability bound $(\int_0^T\|\nabla\mathbf{w}_h\|^2(s)ds)^{\frac{1}{2}}\leq C$, we can deduce

$$\begin{aligned}&\|\phi_h\|^2(t)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t)+\nu\int_0^t\|\nabla\phi_h\|^2(s)ds+\underline{C}\int_0^t\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|^3_{L^3}(s)ds\\&\leq C\{\|\phi_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0)+\int_0^t\|l(\mathbf{x})\nabla\times\eta_t\|^2(s)ds\\&\quad+(\int_0^t\|\nabla\eta\|^4(s)ds)^{\frac{1}{2}}+\int_0^t\|\eta_t\|^2(s)ds+\int_0^t\|\nabla\eta\|^2(s)ds+\int_0^t\|q-r_h\|^2(s)ds\\&\quad+(\int_0^t\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}(s)ds)^{\frac{1}{2}}+\int_0^t\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3}(s)ds\}.\end{aligned}\tag{4.22}$$

Then we have

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\phi_h\|^2(t)+\sup_{0\le t\le T}\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(t)+\nu\|\nabla\phi_h\|^2_{L^2(0,T;L^2)}+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\phi_h\|^3_{L^3(0,T;L^3)}\\
&\le C\{\|\phi_h\|^2(0)+\beta^2\|l(\mathbf{x})\nabla\times\phi_h\|^2(0)+\|l(\mathbf{x})\nabla\times\eta_t\|^2_{L^2(0,T;L^2)}\\
&\quad+\|\nabla\eta\|^2_{L^4(0,T;L^2)}+\|\eta_t\|^2_{L^2(0,T;L^2)}+\|\nabla\eta\|^2_{L^2(0,T;L^2)}\\
&\quad+\|q-r_h\|^2_{L^2(0,T;L^2)}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^{\frac{3}{2}}_{L^3(0,T;L^3)}+\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3(0,T;L^3)}\}.
\end{aligned}
\tag{4.23}
$$

Adding (4.23) to

$$
\sup_{0\le t\le T}\|\eta\|^2(t)+\sup_{0\le t\le T}\beta^2\|l(\mathbf{x})\nabla\times\eta\|^2(t)+\nu\|\nabla\eta\|^2_{L^2(0,T;L^2)}+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\eta\|^3_{L^3(0,T;L^3)}
\tag{4.24}
$$

and then using the triangle inequality complete the proof.
□

5. Time discretization. In this section, we discuss the time discretization scheme for the corrected EV model (1.6). Divide the time interval $[0,T]$ to m elements (t^n,t^{n+1}). Here, $\Delta t=\frac{T}{m}$, $t^n=n\Delta t$ for $n=0,1,2,\cdots,m$. We denote $\mathbf{w}^n=\mathbf{w}(t^n)$ and similarly for other variables.

Algorithm 1 :(First order Backward-Euler scheme)

Given $(\mathbf{w}^n,q^n)$, find $(\mathbf{w}^{n+1},q^{n+1})$ satisfying

$$
\begin{aligned}
&(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\mathbf{v})+\beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\nabla\times\mathbf{v})+b(\mathbf{w}^{n+1},\mathbf{w}^{n+1},\mathbf{v})+\nu(\nabla\mathbf{w}^{n+1},\nabla\mathbf{v})\\
&\quad+(l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+1}|\nabla\times\mathbf{w}^{n+1},\nabla\times\mathbf{v})-(q^{n+1},\nabla\cdot\mathbf{v})=(\mathbf{f}^{n+1},\mathbf{v})\qquad\forall\mathbf{v}\in X,\\
&(\nabla\cdot\mathbf{w}^{n+1},r)=0\qquad\forall r\in Q.
\end{aligned}
\tag{5.1}
$$

An á priori bound on w^{n+1} is proven in the next theorem. The system (5.1) reduces to a finite dimensional nonlinear system with an á priori bound on any possible solution. Thus, existence of w^{n+1} then follows by a standard fixed point argument similar to the nonlinear system arising in the space and time discretized NSE case. When discretized in time but not in space, existence of w^{n+1} can also be proven by monotonicity techniques.

THEOREM 5.1. *Method* (5.1) *is unconditionally stable.*

$$
\begin{aligned}
&\|\mathbf{w}^m\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{w}^{n+1}-\mathbf{w}^n\|^2+\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}^{n+1}-\mathbf{w}^n)\|^2\\
&\quad+2\Delta t\sum_{n=0}^{m-1}\int_\Omega|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}^{n+1}|^3d\mathbf{x}+\nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+1}\|^2\\
&\le\frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|^2_{-1}+\|\mathbf{w}^0\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned}
\tag{5.2}
$$

Proof. Set $\mathbf{v} = \mathbf{w}^{n+1}, r = q^{n+1}$ in (5.1) to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{w}^{n+1}\|^2 - \|\mathbf{w}^n\|^2 + \|\mathbf{w}^{n+1} - \mathbf{w}^n\|^2) \\
&\quad + \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla \times \mathbf{w}^{n+1}\|^2 - \|l(\mathbf{x})\nabla \times \mathbf{w}^n\|^2 + \|l(\mathbf{x})\nabla \times \mathbf{w}^{n+1} - l(\mathbf{x})\nabla \times \mathbf{w}^n\|^2) \\
&\quad + \int_\Omega l^2(\mathbf{x})|\nabla \times \mathbf{w}^{n+1}|^3 d\mathbf{x} + \nu\|\nabla \mathbf{w}^{n+1}\|^2 \\
&= (\mathbf{f}^{n+1}, \mathbf{w}^{n+1}) \\
&\le \frac{1}{2\nu}\|\mathbf{f}^{n+1}\|_{-1}^2 + \frac{\nu}{2}\|\nabla \mathbf{w}^{n+1}\|^2.
\end{aligned} \tag{5.3}$$

Multiplying (5.3) by $2\Delta t$ and summing it from $n = 0$ to $n = m - 1$ complete the proof.
□

THEOREM 5.2. *Assume* $\mathbf{w}$ *is a strong solution of the model. Assume that the true solution* $\mathbf{w}$ *satisfies the following regularity.*

$$\nabla \mathbf{w} \in L^\infty(0, T; L^2), \quad \mathbf{w}_{tt} \in L^2(0, T; L^2), \quad l(\mathbf{x})\nabla \times \mathbf{w}_{tt} \in L^2(0, T; L^2). \tag{5.4}$$

Then,

$$\|\mathbf{e}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla \times \mathbf{e}^m\|^2 + \Delta t\sum_{n=0}^{m-1}\nu\|\nabla \mathbf{e}^{n+1}\|^2 + 2\Delta t\sum_{n=0}^{m-1}\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|_{L^3}^3 \le C\Delta t^2. \tag{5.5}$$

Proof. At time t^{n+1}, the true solution $(\mathbf{w}, q)$ satisfies

$$\begin{aligned}
&(\frac{\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n)}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla \times \frac{\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n)}{\Delta t}, \nabla \times \mathbf{v}) + b(\mathbf{w}(t^{n+1}), \mathbf{w}(t^{n+1}), \mathbf{v}) \\
&\quad + (l^2(\mathbf{x})|\nabla \times \mathbf{w}(t^{n+1})|\nabla \times \mathbf{w}(t^{n+1}), \nabla \times \mathbf{v}) + \nu(\nabla \mathbf{w}(t^{n+1}), \nabla \mathbf{v}) - (p(t^{n+1}), \nabla \cdot \mathbf{v}) \\
&= (\mathbf{f}^{n+1}, \mathbf{v}) + (R^{n+1}, \mathbf{v}) + (l^2(\mathbf{x})\nabla \times R^{n+1}, \nabla \times \mathbf{v}) \qquad \forall \mathbf{v} \in X, \\
&(\nabla \cdot \mathbf{w}(t^{n+1}), r) = 0 \qquad \forall r \in Q,
\end{aligned} \tag{5.6}$$

where $R^{n+1} = \frac{\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n)}{\Delta t} - \mathbf{w}_t(t^{n+1})$. Denote $\mathbf{e}^{n+1} = \mathbf{w}(t^{n+1}) - \mathbf{w}^{n+1}$. Subtracting (5.6) from (5.1), we have

$$\begin{aligned}
&(\frac{\mathbf{e}^{n+1} - \mathbf{e}^n}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla \times \frac{\mathbf{e}^{n+1} - \mathbf{e}^n}{\Delta t}, \nabla \times \mathbf{v}) + b(\mathbf{e}^{n+1}, \mathbf{w}(t^{n+1}), \mathbf{v}) + b(\mathbf{w}^{n+1}, \mathbf{e}^{n+1}, \mathbf{v}) \\
&\quad + (l^2(\mathbf{x})|\nabla \times \mathbf{w}(t^{n+1})|\nabla \times \mathbf{w}(t^{n+1}) - l^2(\mathbf{x})|\nabla \times \mathbf{w}^{n+1}|\nabla \times \mathbf{w}^{n+1}, \nabla \times \mathbf{v}) \\
&\quad + \nu(\nabla \mathbf{e}^{n+1}, \nabla \mathbf{v}) - (p(t^{n+1}) - q^{n+1}, \nabla \cdot \mathbf{v}) \\
&= (R^{n+1}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla \times R^{n+1}, \nabla \times \mathbf{v}) \qquad \forall \mathbf{v} \in X, \\
&(\nabla \cdot \mathbf{e}^{n+1}, r) = 0 \qquad \forall r \in Q.
\end{aligned} \tag{5.7}$$

Setting $\mathbf{v} = \mathbf{e}^{n+1}$ in (5.7), we can obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|^2) + \nu\|\nabla \mathbf{e}^{n+1}\|^2 \\
&\quad + \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1}\|^2 - \|l(\mathbf{x})\nabla \times \mathbf{e}^n\|^2 + \|l(\mathbf{x})\nabla \times \mathbf{e}^{n+1} - l(\mathbf{x})\nabla \times \mathbf{e}^n\|^2) \\
&\quad + (l^2(\mathbf{x})|\nabla \times \mathbf{w}(t^{n+1})|\nabla \times \mathbf{w}(t^{n+1}) - l^2(\mathbf{x})|\nabla \times \mathbf{w}^{n+1}|\nabla \times \mathbf{w}^{n+1}, \nabla \times \mathbf{e}^{n+1}) \\
&= (R^{n+1}, \mathbf{e}^{n+1}) + \beta^2(l^2(\mathbf{x})\nabla \times R^{n+1}, \nabla \times \mathbf{e}^{n+1}) - b(\mathbf{e}^{n+1}, \mathbf{w}(t^{n+1}), \mathbf{e}^{n+1}).
\end{aligned} \tag{5.8}$$

Next, we will bound the terms in (5.8). Using (2.7) in Lemma 2.2, we can obtain

$$\begin{aligned}&(l^2(\mathbf{x})|\nabla\times\mathbf{w}(t^{n+1})|\nabla\times\mathbf{w}(t^{n+1})-l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+1}|\nabla\times\mathbf{w}^{n+1},\nabla\times\mathbf{e}^{n+1})\\&\geq\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|_{L^3}^3.\end{aligned}\tag{5.9}$$

For the term $(R^{n+1},\mathbf{e}^{n+1})$,

$$(R^{n+1},\mathbf{e}^{n+1})\leq C\|R^{n+1}\|\|\nabla\mathbf{e}^{n+1}\|\leq C\|R^{n+1}\|^2+\frac{\nu}{4}\|\nabla\mathbf{e}^{n+1}\|^2.\tag{5.10}$$

For the term $(l^2(\mathbf{x})\nabla\times R^{n+1},\nabla\times\mathbf{e}^{n+1})$,

$$\begin{aligned}\beta^2(l^2(\mathbf{x})\nabla\times R^{n+1},\nabla\times\mathbf{e}^{n+1})&\leq\beta^2\|l(\mathbf{x})\nabla\times R^{n+1}\|\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|\\&\leq\frac{\beta^2}{2}\|l(\mathbf{x})\nabla\times R^{n+1}\|^2+\frac{\beta^2}{2}\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^2.\end{aligned}\tag{5.11}$$

Lastly, the term $-b(\mathbf{e}^{n+1},\mathbf{w}(t^{n+1}),\mathbf{e}^{n+1})$ is bounded by

$$\begin{aligned}-b(\mathbf{e}^{n+1},\mathbf{w}(t^{n+1}),\mathbf{e}^{n+1})&\leq C\|\mathbf{e}^{n+1}\|^{\frac{1}{2}}\|\nabla\mathbf{w}(t^{n+1})\|\|\nabla\mathbf{e}^{n+1}\|^{\frac{3}{2}}\\&\leq C\|\mathbf{e}^{n+1}\|^2\|\nabla\mathbf{w}(t^{n+1})\|^4+\frac{\nu}{4}\|\nabla\mathbf{e}^{n+1}\|^2.\end{aligned}\tag{5.12}$$

Combining (5.9)-(5.12) with (5.8), we have

$$\begin{aligned}&\frac{1}{2\Delta t}(\|\mathbf{e}^{n+1}\|^2-\|\mathbf{e}^n\|^2+\|\mathbf{e}^{n+1}-\mathbf{e}^n\|^2)+\frac{\nu}{2}\|\nabla\mathbf{e}^{n+1}\|^2+\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|_{L^3}^3\\&\quad+\frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^2-\|l(\mathbf{x})\nabla\times\mathbf{e}^n\|^2+\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}-l(\mathbf{x})\nabla\times\mathbf{e}^n\|^2)\\&\leq C\|\nabla\mathbf{w}(t^{n+1})\|^4\|\mathbf{e}^{n+1}\|^2+\frac{\beta^2}{2}\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^2+C\|R^{n+1}\|^2+\frac{\beta^2}{2}\|l(\mathbf{x})\nabla\times R^{n+1}\|^2.\end{aligned}\tag{5.13}$$

Multiplying (5.13) by $2\Delta t$ and summing it from $n=0$ to $n=m-1$, we obtain

$$\begin{aligned}&\|\mathbf{e}^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{e}^{n+1}-\mathbf{e}^n\|^2+\Delta t\sum_{n=0}^{m-1}\nu\|\nabla\mathbf{e}^{n+1}\|^2+2\Delta t\sum_{n=0}^{m-1}\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|_{L^3}^3\\&\quad+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{e}^m\|^2+\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}-l(\mathbf{x})\nabla\times\mathbf{e}^n\|^2\\&\leq\|\mathbf{e}^0\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{e}^0\|^2+C\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}(t^{n+1})\|^4\|\mathbf{e}^{n+1}\|^2\\&\quad+\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|^2+C\Delta t\sum_{n=0}^{m-1}\|R^{n+1}\|^2+\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times R^{n+1}\|^2.\end{aligned}\tag{5.14}$$

Since $\|\nabla\mathbf{w}\|_{L^\infty(0,T;L^2)}\leq C$, using Lemma 2.1, when $\Delta t<\frac{1}{\max\{C\|\nabla\mathbf{w}\|_{L^\infty(0,T;L^2)},1\}}$, we can obtain

$$\begin{aligned}&\|\mathbf{e}^m\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{e}^m\|^2+\Delta t\sum_{n=0}^{m-1}\nu\|\nabla\mathbf{e}^{n+1}\|^2+2\Delta t\sum_{n=0}^{m-1}\underline{C}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{e}^{n+1}\|_{L^3}^3\\&\leq C[\|\mathbf{e}^0\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{e}^0\|^2+\Delta t\sum_{n=0}^{m-1}\|R^{n+1}\|^2+\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times R^{n+1}\|^2].\end{aligned}\tag{5.15}$$

We also have

$$\Delta t\sum_{n=0}^{m-1}\|R^{n+1}\|^2=\Delta t\sum_{n=0}^{m-1}\|\frac{\mathbf{w}(t^{n+1})-\mathbf{w}(t^n)}{\Delta t}-\mathbf{w}_t(t^{n+1})\|^2 \\ \leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\mathbf{w}_{tt}\|^2ds=C\Delta t^2\|\mathbf{w}_{tt}\|_{L^2(0,T;L^2)}, \tag{5.16}$$

and

$$\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times R^{n+1}\|^2=\Delta t\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times(\frac{\mathbf{w}(t^{n+1})-\mathbf{w}(t^n)}{\Delta t}-\mathbf{w}_t(t^{n+1}))\|^2 \\ \leq C\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}_{tt}\|^2ds \\ =C\Delta t^2\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}_{tt}\|_{L^2(0,T;L^2)}. \tag{5.17}$$

Finally, combining (5.16), (5.17), and (5.15) completes the proof.
□

Algorithm 2 :(First order linearly implicit, Backward-Euler scheme)
Given $(\mathbf{w}^n,q^n)$, find $(\mathbf{w}^{n+1},q^{n+1})$ satisfying

$$(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\mathbf{v})+\beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\nabla\times\mathbf{v})+b(\mathbf{w}^n,\mathbf{w}^{n+1},\mathbf{v})+\nu(\nabla\mathbf{w}^{n+1},\nabla\mathbf{v}) \\ +(l^2(\mathbf{x})|\nabla\times\mathbf{w}^n|\nabla\times\mathbf{w}^{n+1},\nabla\times\mathbf{v})-(q^{n+1},\nabla\cdot\mathbf{v})=(\mathbf{f}^{n+1},\mathbf{v})\qquad\forall\mathbf{v}\in X, \\ (\nabla\cdot\mathbf{w}^{n+1},r)=0\qquad\forall r\in Q. \tag{5.18}$$

Algorithm 2 leads to a linear problem for a continuous and coercive operator for w^{n+1}. Existence of w^{n+1} follows from the Lax-Milgram lemma.

THEOREM 5.3. *Method* (5.18) *is unconditionally stable.*

$$\|\mathbf{w}^m\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2+\sum_{n=0}^{m-1}\|\mathbf{w}^{n+1}-\mathbf{w}^n\|^2+\sum_{n=0}^{m-1}\beta^2\|l(\mathbf{x})\nabla\times(\mathbf{w}^{n+1}-\mathbf{w}^n)\|^2 \\ +2\Delta t\sum_{n=0}^{m-1}\int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}^n|\cdot|\nabla\times\mathbf{w}^{n+1}|^2d\mathbf{x}+\nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+1}\|^2 \\ \leq\frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+1}\|_{-1}^2+\|\mathbf{w}^0\|^2+\beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2. \tag{5.19}$$

REMARK 5. *The proof of Theorem* 5.3 *is similar to that of Theorem* 5.1. *We omit it here.*

Algorithm 3 :(Second order Crank-Nicolson scheme)
Given $(\mathbf{w}^n,q^n)$, find $(\mathbf{w}^{n+1},q^{n+1})$ satisfying

$$(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\mathbf{v})+\beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t},\nabla\times\mathbf{v})+b(\mathbf{w}^{n+\frac12},\mathbf{w}^{n+\frac12},\mathbf{v})+\nu(\nabla\mathbf{w}^{n+\frac12},\nabla\mathbf{v}) \\ +(l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+\frac12}|\nabla\times\mathbf{w}^{n+\frac12},\nabla\times\mathbf{v})-(q^{n+\frac12},\nabla\cdot\mathbf{v})=(\mathbf{f}^{n+\frac12},\mathbf{v})\qquad\forall\mathbf{v}\in X, \\ (\nabla\cdot\mathbf{w}^{n+1},r)=0\qquad\forall r\in Q, \tag{5.20}$$

where we denote $\mathbf{w}^{n+\frac12}=\frac{\mathbf{w}^n+\mathbf{w}^{n+1}}{2}$, and similarly for other variables.

Equations (5.20) in *Algorithm* 3 leads to a nonlinear system of equations for w^{n+1}. These equations can be rearranged into a nonlinear system for $w^{n+1/2}$ with the same structure as the nonlinear system occurring in *Algorithm* 1. By the same argument existence for $w^{n+1/2}$ follows. From this, by subtracting $w^n/2$, existence follows for w^{n+1}.

THEOREM 5.4. *Method* (5.20) *is unconditionally stable.*

$$\begin{aligned}
&\|\mathbf{w}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2 + 2\Delta t\sum_{n=0}^{m-1}\int_\Omega |l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{w}^{n+\frac{1}{2}}|^3 d\mathbf{x} + \nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2\\
&\le \frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+\frac{1}{2}}\|_{-1}^2 + \|\mathbf{w}^0\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned} \tag{5.21}$$

Proof. Set $\mathbf{v} = \mathbf{w}^{n+\frac{1}{2}}$ in (5.20) to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{w}^{n+1}\|^2 - \|\mathbf{w}^n\|^2) + \frac{\beta^2}{2\Delta t}(\|l(\mathbf{x})\nabla\times\mathbf{w}^{n+1}\|^2 - \|l(\mathbf{x})\nabla\times\mathbf{w}^n\|^2)\\
&\quad + \int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}^{n+\frac{1}{2}}|^3 d\mathbf{x} + \nu\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2\\
&= (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{w}^{n+\frac{1}{2}}) \le \frac{1}{2\nu}\|\mathbf{f}^{n+\frac{1}{2}}\|_{-1}^2 + \frac{\nu}{2}\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2.
\end{aligned} \tag{5.22}$$

Multiplying (5.22) by $2\Delta t$ and summing it from $n = 0$ to $n = m-1$ complete the proof.
□

Algorithm 4 :(Second order Crank-Nicolson linearly extrapolation scheme)
Given $\mathbf{w}^{n-1}, \mathbf{w}^n, q^n$, find $(\mathbf{w}^{n+1}, q^{n+1})$ satisfying

$$\begin{aligned}
&(\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}, \mathbf{v}) + \beta^2(l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}, \nabla\times\mathbf{v}) + b(\varphi(\mathbf{w}^n), \mathbf{w}^{n+\frac{1}{2}}, \mathbf{v}) + \nu(\nabla\mathbf{w}^{n+\frac{1}{2}}, \nabla\mathbf{v})\\
&\quad + (l^2(\mathbf{x})|\nabla\times\varphi(\mathbf{w}^n)|\nabla\times\mathbf{w}^{n+\frac{1}{2}}, \nabla\times\mathbf{v}) - (q^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) = (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{v}) \qquad \forall\mathbf{v}\in X,\\
&(\nabla\cdot\mathbf{w}^{n+1}, r) = 0 \qquad \forall r\in Q,
\end{aligned} \tag{5.23}$$

where $\varphi(\mathbf{w}^n) = \frac{3}{2}\mathbf{w}^n - \frac{1}{2}\mathbf{w}^{n-1}$.

Algorithm 4 is linearly implicit. It leads to a system for w^{n+1} that is continuous and coercive. Existence of w^{n+1} then follows, as for the linearly implicit BE method, by the Lax-Milgram lemma.

THEOREM 5.5. *Method* (5.23) *is unconditionally stable.*

$$\begin{aligned}
&\|\mathbf{w}^m\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^m\|^2 + 2\Delta t\sum_{n=0}^{m-1}\int_\Omega l^2(\mathbf{x})|\nabla\times\varphi(\mathbf{w}^n)||\nabla\times\mathbf{w}^{n+\frac{1}{2}}|^2 d\mathbf{x}\\
&\quad + \nu\Delta t\sum_{n=0}^{m-1}\|\nabla\mathbf{w}^{n+\frac{1}{2}}\|^2\\
&\le \frac{\Delta t}{\nu}\sum_{n=0}^{m-1}\|\mathbf{f}^{n+\frac{1}{2}}\|_{-1}^2 + \|\mathbf{w}^0\|^2 + \beta^2\|l(\mathbf{x})\nabla\times\mathbf{w}^0\|^2.
\end{aligned} \tag{5.24}$$

REMARK 6. *The proof of Theorem* 5.5 *is similar to that of Theorem* 5.4. *We just omit it here.*

6. Numerical test. Since we study the simplest realization of an algebraic model, our goal is simply to test if the model correction can exhibit backscatter in the form of a negative total dissipation at some times. We select a 2d test without a global rotational flow (due to the choice of the Baldwin-Lomax as the starting point). We use the first order scheme *Algorithm* 2 and the second order scheme *Algorithm* 4. For both methods, we compute the evolution of four components of dissipation in their approximation to the effect of the Reynolds stress on the mean kinetic energy. The P2-P1 Taylor-Hood mixed finite elements are utilized for discretization in space. FreeFem++ [35] is used for simulation.

Choose a rectangle domain $\Omega = [0,4]\times[0,1]$ with a square obstacle $\Omega = [0.5, 0.6]\times[0.45, 0.55]$ inside it. We compute the problem on a Delaunay-Vornoi generated triangular mesh with more mesh points near the obstacle area and less in other area, shown in Figure 6.1. The degree of freedom is 17625, the shortest triangle edge is 0.0102605, and the longest is 0.097187. The flow passes through this domain from left to right. For boundary conditions on the inflow boundary, we let $\mathbf{u}|_{x-\text{direction}} = 4y(1-y)$, $\mathbf{u}|_{y-\text{direction}} = 0$. **On the right, out-flow, boundary, we impose the "do-nothing" outflow boundary condition, [19] p. 21 eqn. (2.37) and [4] p. 475.** The no-slip condition $\mathbf{u} = 0$ is imposed on other boundaries. We take $\mathbf{f} = 0$, $T = 20$, $\Delta t = 0.01$, and $Re = 10,000$. Let $\bar{y}$ denote the distance of $\mathbf{x}$ to the nearest wall. The mixing length is chosen ([12] Chapter 3 e.g. eqn. (3.99) p. 76) to be

$$l(\mathbf{x}) = \begin{cases} 0.41\cdot\bar{y} & \text{when } 0 < \bar{y} < 0.2\cdot Re^{-\frac{1}{2}}, \\ 0.41\cdot 0.2\cdot Re^{-\frac{1}{2}} & \text{otherwise.} \end{cases}$$

An alternatives is to pick $l(\mathbf{x}) = h$, the local meshwidth, in the spirit of large eddy simulation.

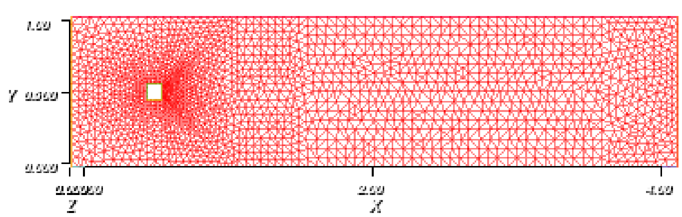

FIG. 6.1. *Mesh used in our computation*

For *Algorithm* 2, [23], since Backward-Euler scheme has substantial numerical dissipation, we select $\beta = 10$ and 100 to compute the following quantities, shown in Figure 6.2 ($\beta = 10$), and Figure 6.3 ($\beta = 100$).

$$\text{Backscatter term (BST)} = \beta^2 \int_\Omega l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}\cdot\nabla\times\mathbf{w}^{n+1}d\mathbf{x}$$

$$\text{Numerical dissipation (ND)} = \frac{\beta^2}{2\Delta t}\|l(\mathbf{x})\nabla\times\mathbf{w}^{n+1} - l(\mathbf{x})\nabla\times\mathbf{w}^n\|^2,$$

$$\text{Fluctuation dissipation (FD)} = \int_\Omega l^2(\mathbf{x})|\nabla\times\mathbf{w}^n||\nabla\times\mathbf{w}^{n+1}|^2 d\mathbf{x},$$

$$\text{Total Dissipation (TD)} = \int_\Omega (\beta^2 l^2(\mathbf{x})\nabla\times\frac{\mathbf{w}^{n+1}-\mathbf{w}^n}{\Delta t}\cdot\nabla\times\mathbf{w}^{n+1} + l^2(\mathbf{x})|\nabla\times\mathbf{w}^n||\nabla\times\mathbf{w}^{n+1}|^2)d\mathbf{x}.$$

For *Algorithm* 4,[23] [36], we compute the following quantities, shown in Figure 6.4 ($\beta =$

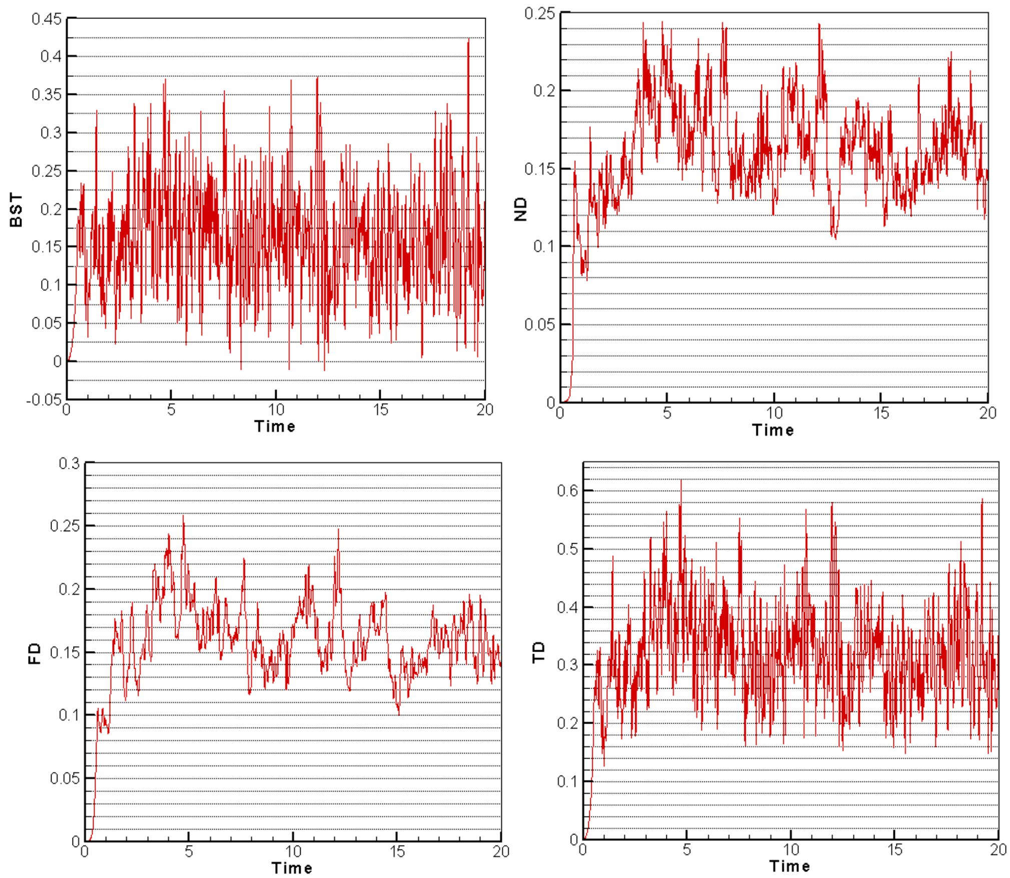

FIG. 6.2. *BST,ND,FD,TD vs Time,* $\beta = 10$*, Algorithm 2*

10).

$$\text{Backscatter term (BST)} = \beta^2 \int_\Omega l^2(\mathbf{x}) \nabla \times \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} \cdot \nabla \times \mathbf{w}^{n+\frac{1}{2}} d\mathbf{x},$$

$$\text{Fluctuation dissipation (FD)} = \int_\Omega l^2(\mathbf{x}) |\nabla \times \varphi(\mathbf{w}^n)| |\nabla \times \mathbf{w}^{n+\frac{1}{2}}|^2 d\mathbf{x},$$

$$\text{Total Dissipation (TD)} = \int_\Omega (\beta^2 l^2(\mathbf{x}) \nabla \times \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} \cdot \nabla \times \mathbf{w}^{n+\frac{1}{2}} + l^2(\mathbf{x}) |\nabla \times \varphi(\mathbf{w}^n)| |\nabla \times \mathbf{w}^{n+\frac{1}{2}}|^2) d\mathbf{x},$$

where $\varphi(\mathbf{w}^n) = \frac{3}{2}\mathbf{w}^n - \frac{1}{2}\mathbf{w}^{n-1}$ and $\mathbf{w}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{w}^{n+1} + \mathbf{w}^n)$.

In both Figure 6.3 and Figure 6.4, TD is negative at some times, indicating that backscatter occurs and energy is transferred from fluctuation to the mean at that moment. However in Figure 6.2, there is no appearance of negative values in TD. The results show that the numerical dissipation in *Algorithm* 2 (the backward Euler time discretization) is compared to the model dissipation and, for $\beta = 10$, dominates the term modeling the flow of energy from fluctuations back to the mean. *Algorithm* 4 does not contain any numerical dissipation. For *Algorithm* 4, $\beta = 10$, the effects of the added term does provide bursts of energy to the mean flow.

In order to test the convergence, we reduce both mesh size and time step, then recompute

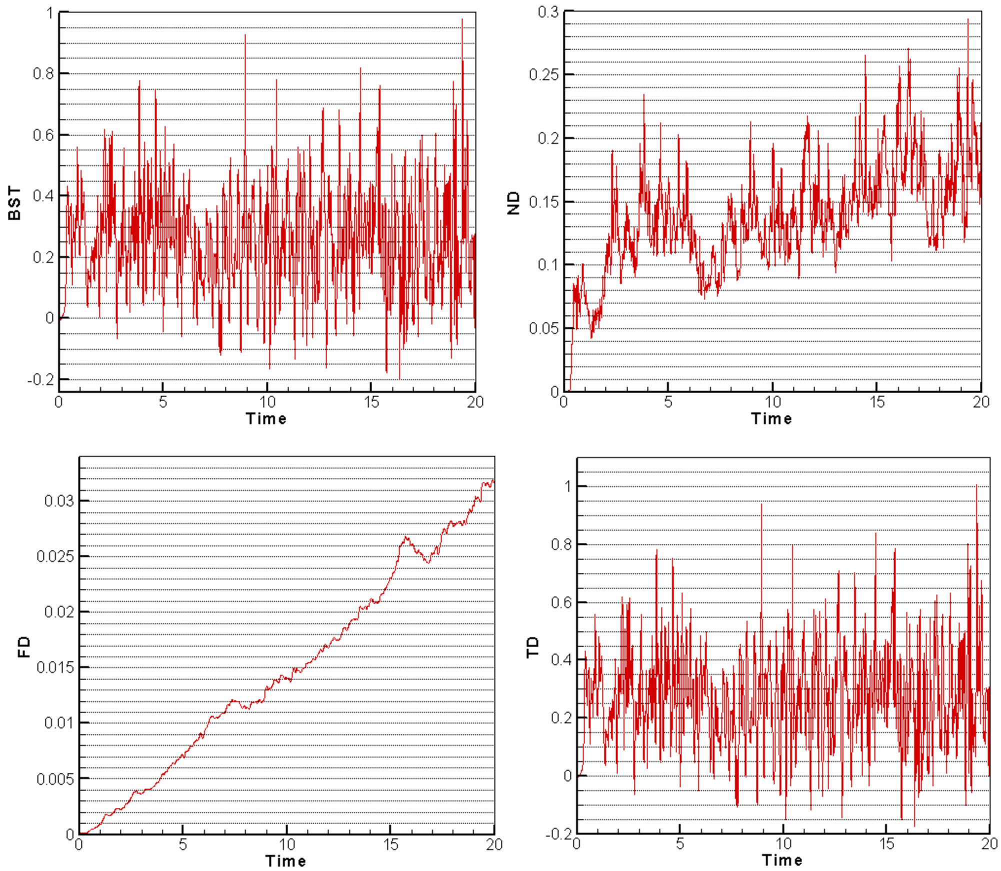

FIG. 6.3. *BST,ND,FD,TD vs Time,* $\beta = 100$*, Algorithm 2*

the backscatter term and dissipation terms. Since *Algorithm* 4 does not contain any numerical dissipation, we use *Algorithm* 4 to do this test.

Firstly, we double the mesh points of all edges. The degree of freedom is 69657, the shortest triangle edge is 0.00503187, and the longest is 0.0495673. We also halve the time step and take $\Delta t = 0.005$. Figure 6.5 shows the result.

We furthermore double the mesh points of all edges. The degree of freedom is 276438, the shortest triangle edge is 0.0024586, and the longest is 0.0270444. The time step $\Delta t = 0.0025$. Figure 6.6 shows the result.

Both Figure 6.5 and Figure 6.6 indicate the occurrence of backscatter at the same moments on successively refined meshes.

7. Conclusion. We have considered the simplest form of the Baldwin-Lomax eddy viscosity model, making the simplest choices within the approach of [32] to adapt it to incorporate the effects of energy flow from fluctuations back to means (a form of statistical backscatter). For internal flow with no-slip boundary condition in both 2d and 3d, the effects of fluctuations on means are dissipative on time average but can have bursts (with time average zero) for which energy flow reverses. We have shown that the corrected Baldwin-Lomax model shares this property. We have given a stability analysis of two numerical methods for numerical approximation of the resulting model: one with substantial numerical dissipation and one without. Using this two methods, numerical tests confirm backscatter does occur and that the results

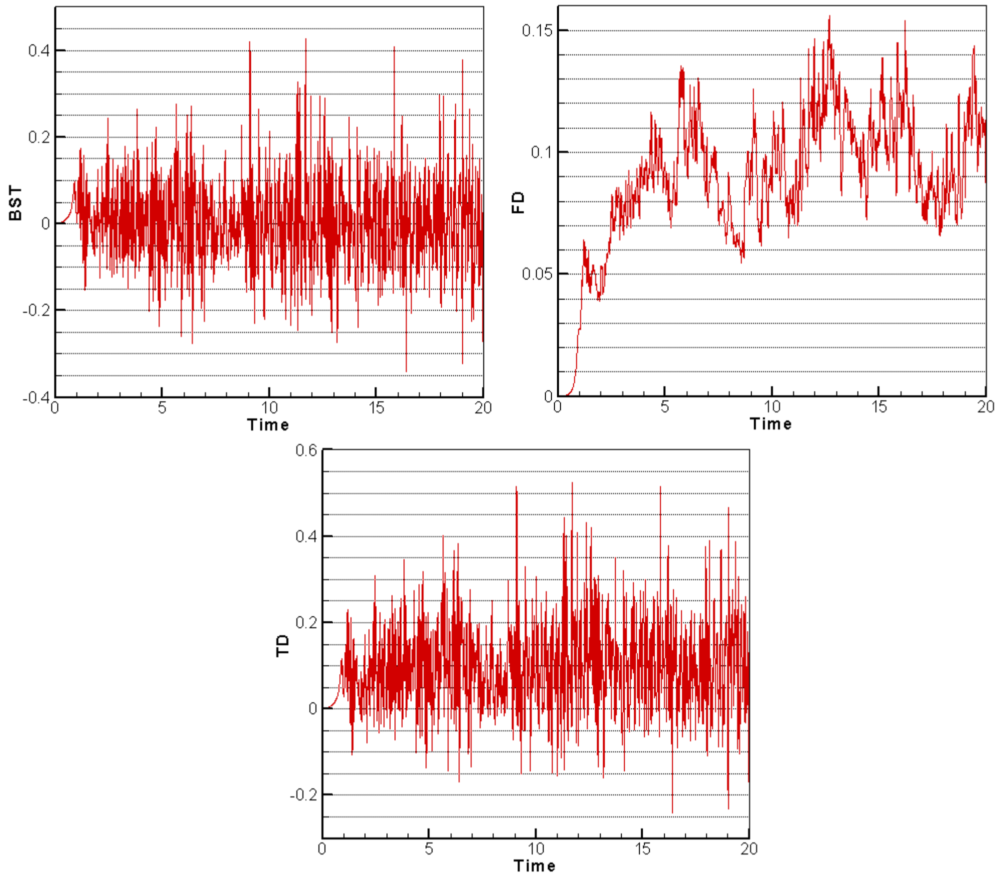

FIG. 6.4. *BST,FD,TD vs Time,* $\beta = 10$*, Algorithm 4*

obtained depend upon the numerical dissipation in the algorithms used and the single model calibration parameter β.

8. Appendix. REMARK 7. *The proof of monotonicity and local Lipschitz continuity are similar to the analogous ones for the Smagorinsky model. Since the present work involves the rotational form, we include both for completeness. Our proofs are adapted from Lemma 8.5 in [18].*

Proof. (Proof of Lemma 2.2) First, we prove the strong monotonicity (2.7). Define an operator $\mathbf{F} : (L^3(\Omega))^d \to (L^{\frac{3}{2}}(\Omega))^d$ by

$$\mathbf{F}(\nabla \times \mathbf{u}) = l^2(\mathbf{x})|\nabla \times \mathbf{u}|\nabla \times \mathbf{u}, \tag{8.1}$$

where $\mathbf{u} \in (W^{1,3}(\Omega))^d$, $l : \mathbf{x} \in \Omega \mapsto \mathbb{R}$ is a non-negative function and $l \in L^\infty(\Omega)$. We further define $\mathbf{u} = \tau\mathbf{u}' + (1-\tau)\mathbf{u}''$, $\tau \in [0,1]$. Then, we have

$$\mathbf{F}(\nabla \times \mathbf{u}') - \mathbf{F}(\nabla \times \mathbf{u}'') = \int_0^1 \frac{d}{d\tau}\mathbf{F}(\nabla \times \mathbf{u})d\tau. \tag{8.2}$$

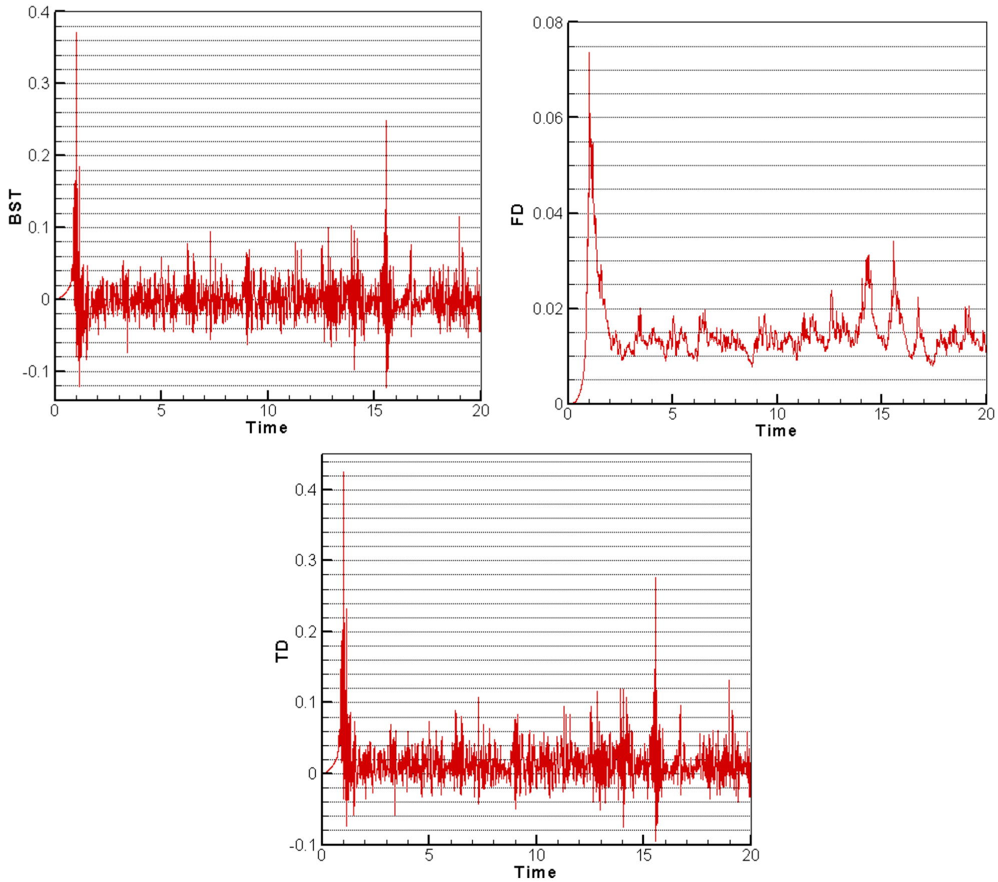

Fig. 6.5. *BST,FD,TD vs Time,* $\beta = 10$, $\Delta t = 0.005$,*Algorithm 4*

Combine (8.1) and (8.2) to get

$$\begin{aligned}
&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'')\cdot(\nabla\times(\mathbf{u}'-\mathbf{u}''))\\
&= (\mathbf{F}(\nabla\times\mathbf{u}') - \mathbf{F}(\nabla\times\mathbf{u}''))\cdot(\nabla\times(\mathbf{u}'-\mathbf{u}''))\\
&= \sum_{i=1}^{3}(\int_0^1 \frac{d}{d\tau}\mathbf{F}_i(\nabla\times\mathbf{u})d\tau)[(\frac{\partial\mathbf{u}'_k}{\partial x_j} - \frac{\partial\mathbf{u}'_j}{\partial x_k}) - (\frac{\partial\mathbf{u}''_k}{\partial x_j} - \frac{\partial\mathbf{u}''_j}{\partial x_k})].
\end{aligned} \tag{8.3}$$

Here, consider 3D case ($d = 3$). The notations $i, j, k = 1, 2, 3$, or $i, j, k = 2, 3, 1$, or $i, j, k = 3, 1, 2$. We have

$$\frac{d}{d\tau}\mathbf{F}_i(\nabla\times\mathbf{u}) = (l^2(\mathbf{x})\frac{d}{d\tau}|\nabla\times\mathbf{u}|)(\frac{\partial\mathbf{u}_k}{\partial x_j} - \frac{\partial\mathbf{u}_j}{\partial x_k}) + l^2(\mathbf{x})|\nabla\times\mathbf{u}|[(\frac{\partial\mathbf{u}'_k}{\partial x_j} - \frac{\partial\mathbf{u}'_j}{\partial x_k}) - (\frac{\partial\mathbf{u}''_k}{\partial x_j} - \frac{\partial\mathbf{u}''_j}{\partial x_k})], \tag{8.4}$$

and

$$\begin{aligned}
\frac{d}{d\tau}|\nabla\times\mathbf{u}| &= \frac{d}{d\tau}(\sum_{l=1}^{3}[\tau(\frac{\partial\mathbf{u}'_n}{\partial x_m} - \frac{\partial\mathbf{u}'_m}{\partial x_n}) + (1-\tau)(\frac{\partial\mathbf{u}''_n}{\partial x_m} - \frac{\partial\mathbf{u}''_m}{\partial x_n})]^2)^{\frac{1}{2}}\\
&= \frac{1}{|\nabla\times\mathbf{u}|}\sum_{l=1}^{3}(\frac{\partial\mathbf{u}_n}{\partial x_m} - \frac{\partial\mathbf{u}_m}{\partial x_n})[(\frac{\partial\mathbf{u}'_n}{\partial x_m} - \frac{\partial\mathbf{u}'_m}{\partial x_n}) - (\frac{\partial\mathbf{u}''_n}{\partial x_m} - \frac{\partial\mathbf{u}''_m}{\partial x_n})],
\end{aligned} \tag{8.5}$$

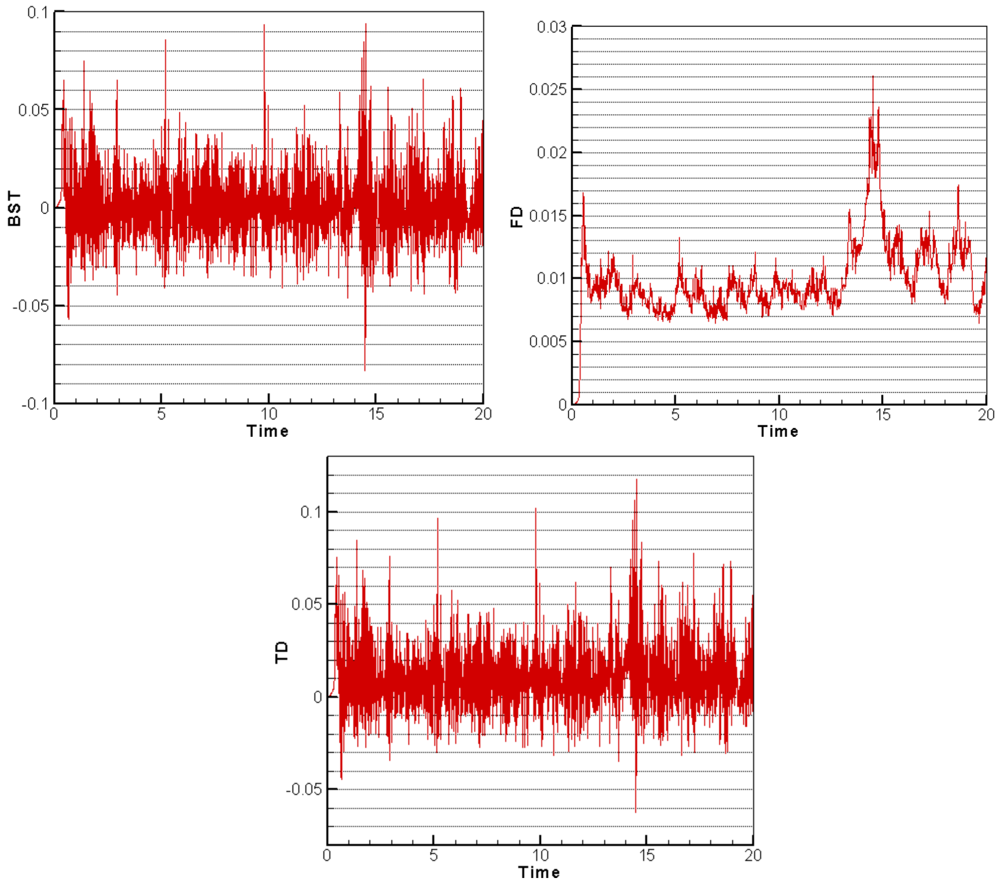

FIG. 6.6. *BST,FD,TD vs Time,* $\beta = 10$*,* $\Delta t = 0.0025$*,Algorithm 4*

where the notations $l, m, n = 1, 2, 3$, or $l, m, n = 2, 3, 1$, or $l, m, n = 3, 1, 2$.

By (8.4) and (8.5), we obtain

$$
\begin{aligned}
\frac{d}{d\tau}\mathbf{F}_i(\nabla\times\mathbf{u}) =& \frac{l^2(\mathbf{x})}{|\nabla\times\mathbf{u}|}\sum_{l=1}^{3}(\frac{\partial\mathbf{u}_n}{\partial x_m}-\frac{\partial\mathbf{u}_m}{\partial x_n})[(\frac{\partial\mathbf{u}'_n}{\partial x_m}-\frac{\partial\mathbf{u}'_m}{\partial x_n})-(\frac{\partial\mathbf{u}''_n}{\partial x_m}-\frac{\partial\mathbf{u}''_m}{\partial x_n})](\frac{\partial\mathbf{u}_k}{\partial x_j}-\frac{\partial\mathbf{u}_j}{\partial x_k}) \\
&+ l^2(\mathbf{x})|\nabla\times\mathbf{u}|[(\frac{\partial\mathbf{u}'_k}{\partial x_j}-\frac{\partial\mathbf{u}'_j}{\partial x_k})-(\frac{\partial\mathbf{u}''_k}{\partial x_j}-\frac{\partial\mathbf{u}''_j}{\partial x_k})].
\end{aligned}
\tag{8.6}
$$

Combining (8.6) and (8.3), we have

$$
(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}' - l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'')\cdot(\nabla\times(\mathbf{u}'-\mathbf{u}'')) = Q_1 + Q_2. \tag{8.7}
$$

Here,

$$
\begin{aligned}
Q_1 =& \int_0^1 \frac{l^2(\mathbf{x})}{|\nabla\times\mathbf{u}|}\sum_{i,l=1}^{3}(\frac{\partial\mathbf{u}_n}{\partial x_m}-\frac{\partial\mathbf{u}_m}{\partial x_n})(\frac{\partial\mathbf{u}_k}{\partial x_j}-\frac{\partial\mathbf{u}_j}{\partial x_k}) \\
&\cdot[(\frac{\partial\mathbf{u}'_n}{\partial x_m}-\frac{\partial\mathbf{u}'_m}{\partial x_n})-(\frac{\partial\mathbf{u}''_n}{\partial x_m}-\frac{\partial\mathbf{u}''_m}{\partial x_n})][(\frac{\partial\mathbf{u}'_k}{\partial x_j}-\frac{\partial\mathbf{u}'_j}{\partial x_k})-(\frac{\partial\mathbf{u}''_k}{\partial x_j}-\frac{\partial\mathbf{u}''_j}{\partial x_k})]d\tau \\
\geq& 0.
\end{aligned}
\tag{8.8}
$$

Furthermore, we have

$$
\begin{aligned}
Q_2 &= \int_0^1 l^2(\mathbf{x})|\nabla\times\mathbf{u}|\sum_{i=1}^{3}[(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 d\tau \\
&\geq C_d\int_0^1 l^2(\mathbf{x})\sum_{i=1}^{3}(\sum_{l=1}^{3}|\frac{\partial \mathbf{u}_n}{\partial x_m}-\frac{\partial \mathbf{u}_m}{\partial x_n}|)[(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 d\tau \\
&\geq C_d\sum_{i=1}^{3}\sum_{l=1}^{3} l^2(\mathbf{x})[\int_0^1|\tau(\frac{\partial \mathbf{u}'_n}{\partial x_m}-\frac{\partial \mathbf{u}'_m}{\partial x_n})+(1-\tau)(\frac{\partial \mathbf{u}''_n}{\partial x_m}-\frac{\partial \mathbf{u}''_m}{\partial x_n})|d\tau] \\
&\qquad\qquad \cdot[(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 \\
&\geq C_d\sum_{i=1}^{3}\sum_{l=1}^{3} l^2(\mathbf{x})[\frac{1}{4}|(\frac{\partial \mathbf{u}'_n}{\partial x_m}-\frac{\partial \mathbf{u}'_m}{\partial x_n})-(\frac{\partial \mathbf{u}''_n}{\partial x_m}-\frac{\partial \mathbf{u}''_m}{\partial x_n})|][(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})]^2 \\
&\geq \underline{C}\sum_{i=1}^{3} l^2(\mathbf{x})|(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})|^3,
\end{aligned}
\tag{8.9}
$$

where we use those inequalities $\sqrt{a^2+b^2+c^2}\geq C_d(|a|+|b|+|c|)$ and $\int_0^1|\tau a+(1-\tau)b|d\tau\geq \frac{1}{4}|a-b|$, $\forall\, a,b,c\in\mathbb{R}$.

By (8.7), (8.8) and (8.9), we have

$$
\begin{aligned}
&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}''),\nabla\times(\mathbf{u}'-\mathbf{u}''))=\int_\Omega(Q_1+Q_2)d\mathbf{x} \\
&\geq \underline{C}\int_\Omega\sum_{i=1}^{3} l^2(\mathbf{x})|(\frac{\partial \mathbf{u}'_k}{\partial x_j}-\frac{\partial \mathbf{u}'_j}{\partial x_k})-(\frac{\partial \mathbf{u}''_k}{\partial x_j}-\frac{\partial \mathbf{u}''_j}{\partial x_k})|^3 d\mathbf{x} \\
&= \underline{C}\|l^{\frac{2}{3}}(x)\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}^3,
\end{aligned}
\tag{8.10}
$$

which completes the proof of strong monotonicity (2.7).

Next, we prove the local Lipschitz-continuity (2.8). Using triangle inequality, we have

$$
\begin{aligned}
&(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'',\nabla\times\mathbf{v}) \\
&\leq (l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'-l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}'',\nabla\times\mathbf{v}) \\
&\quad +(l^2(\mathbf{x})|\nabla\times\mathbf{u}'|\nabla\times\mathbf{u}''-l^2(\mathbf{x})|\nabla\times\mathbf{u}''|\nabla\times\mathbf{u}'',\nabla\times\mathbf{v}) \\
&\leq \|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'|\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times(\mathbf{u}'-\mathbf{u}'')\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3} \\
&\quad +\|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'|-l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}''|\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}''\|_{L^3}\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{v}\|_{L^3}.
\end{aligned}
\tag{8.11}
$$

Since

$$
\begin{aligned}
\|l^{\frac{2}{3}}(\mathbf{x})|\nabla\times\mathbf{u}'|\|_{L^3} &= \int_\Omega(\sum_{i=1}^{3}|l^{\frac{2}{3}}(\mathbf{x})(\nabla\times\mathbf{u}')_i|^2)^{\frac{3}{2}}d\mathbf{x} \\
&\leq \tilde{C}_d\int_\Omega(\sum_{i=1}^{3}|l^{\frac{2}{3}}(\mathbf{x})(\nabla\times\mathbf{u}')_i|^3)d\mathbf{x} \\
&= \tilde{C}_d\|l^{\frac{2}{3}}(\mathbf{x})\nabla\times\mathbf{u}'\|_{L^3},
\end{aligned}
\tag{8.12}
$$

where we use the inequality $\|\mathbf{x}\|_{\ell^2} \leq \tilde{C}_d \|\mathbf{x}\|_{\ell^3}$, $\forall\, \mathbf{x} \in \mathbb{R}^d$. We also have

$$\begin{aligned} \|l^{\frac{2}{3}}(\mathbf{x})|\nabla \times \mathbf{u}'| - l^{\frac{2}{3}}(\mathbf{x})|\nabla \times \mathbf{u}''|\|_{L^3} &\leq \|l^{\frac{2}{3}}(\mathbf{x})|\nabla \times \mathbf{u}' - \nabla \times \mathbf{u}''|\|_{L^3} \\ &\leq \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times (\mathbf{u}' - \mathbf{u}'')\|_{L^3}. \end{aligned} \tag{8.13}$$

Finally, by (8.11), (8.12), and (8.13), we can obtain the local Lipschitz-continuity (2.8).

$$\begin{aligned} &(l^2(\mathbf{x})|\nabla \times \mathbf{u}'|\nabla \times \mathbf{u}' - l^2(\mathbf{x})|\nabla \times \mathbf{u}''|\nabla \times \mathbf{u}'', \nabla \times \mathbf{v}) \\ &\leq \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{u}'\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times (\mathbf{u}' - \mathbf{u}'')\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{v}\|_{L^3} \\ &\quad + \tilde{C}_d \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{u}''\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times (\mathbf{u}' - \mathbf{u}'')\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{v}\|_{L^3} \\ &= \overline{C}\gamma \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times (\mathbf{u}' - \mathbf{u}'')\|_{L^3} \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{v}\|_{L^3}, \end{aligned} \tag{8.14}$$

where $\gamma = \max\{\|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{u}'\|_{L^3}, \|l^{\frac{2}{3}}(\mathbf{x})\nabla \times \mathbf{u}''\|_{L^3}\}$.
□

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