Math 0230 **Practice Problems to Midterm Exam 1**

Spring 2021 **S o l u t i o n s**

Sections 5.5, 6.1, and 6.2 **Integration**

1. (10 points) Identify any errors in each solution. Clearly explain the mistakes and how you could correct them.

   (a) (5 points) **Problem Statement:** Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\, dt$.

   Solution: Let $u = \sin t$, then $du = \cos t$.

   Substituting, $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\, dt = \int_0^{\frac{\pi}{4}} u\, du = \frac{u^2}{2} + C.$

   Solution: All corrections are:

   $u = \cos t$ **+1 pt**

   $du = \sin t\, dt$ **+1 pt**

   $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\, dt = \int_{u(0)}^{u(\frac{\pi}{4})} \frac{1}{u}\, du$ **+1 pt**

   $\displaystyle= \int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u}\, du$ **+1 pt**

   $\displaystyle= \ln|u| \Big|_1^{\frac{\sqrt{2}}{2}} = \ln\left(\tfrac{\sqrt{2}}{2}\right)$ **+1 pt**

   (b) (5 points) **Problem Statement:** Evaluate the integral $\displaystyle\int t \ln t\, dt$

   Solution: Letting $u = t$ and $dv = \ln t$, then $du = 1$ and $v = \dfrac{1}{t}$.

   Integration by parts gives, $\displaystyle\int t \ln t\, dt = t\left(\frac{1}{t}\right) + \int \frac{1}{t}\, dt = 1 + \ln t.$

   Solution: All corrections are:

   $u = \ln t,\ \ dv = t\, dt$ **+1 pt**

   $du = \frac{1}{t}\, dt,\ \ v = \frac{1}{2}t^2$ **+1 pt**

   $\displaystyle\int t \ln t\, dt = \tfrac{1}{2}t^2 \ln t - \tfrac{1}{2}\int t\, dt$ **+2 pts**

   $= \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C$ **+1 pt**

2. (10 points) Evaluate the integral $I = \int x\sqrt{4-x^2}\,dx$

Solution: Substitution: $u = 4 - x^2$, **+1 pt**

$du = -2x\,dx,\ \ x\,dx = -\frac{1}{2}\,du.$ **+2 pts**

$x\sqrt{4-x^2}\,dx = \sqrt{4-x^2}\,(x\,dx) = u^{1/2}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\,u^{1/2}\,du.$ **+2 pts**

Then $I = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}$ **+2 pts**

$+C$ **+1 pt**

$= -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}\left(4-x^2\right)^{3/2} + C$ **+2 pts**

3. (10 points) Evaluate the integral $I = \int \sqrt{4-x^2}\,dx$

Solution: Trig. substitution: $x = 2\sin t,\ \ dx = 2\cos t\,dt.$

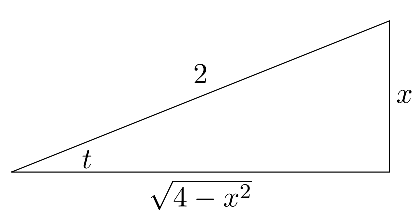
$\sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = 2\cos t.$ **+2 pts**

Then $I = \int 2\cos t\cdot 2\cos t\,dt = 4\int \cos^2 t\,dt = 2\int (1+\cos 2t)\ dt$

$I = 2t + \sin 2t + C = 2t + 2\cos t\sin t + C$ **+2 pts**

We know from the definition of x that $\sin t = \frac{1}{2}x$. Then $t = \sin^{-1}\left(\frac{1}{2}x\right)$. **+1 pt**

To find $\cos t$ we look at the picture below.



Using the fact that $\sin t = \dfrac{x}{2}$ we set the opposite side to the angle t to be x and hypotenuse to be 2. **+1 pt**

The adjacent side is $\sqrt{4-x^2}$ **+1 pt**

and $\cos t = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{\sqrt{4-x^2}}{2}$ **+1 pt**

Therefore

$I = 2\sin^{-1}\left(\frac{1}{2}x\right) - 2\cdot\dfrac{\sqrt{4-x^2}}{2}\cdot\dfrac{x}{2} + C = 2\sin^{-1}\left(\frac{1}{2}x\right) - \frac{1}{2}\,x\sqrt{4-x^2} + C.$ **+2 pts**

4.

$$\text{Find} \quad \int \sin(\theta)\cos(\theta)\ d\theta \qquad (1)$$

(a) You probably solved (1) using the substitution $u = \sin(\theta)$ or $u = \cos(\theta)$. Now find $\int \sin(\theta)\cos(\theta)\ d\theta$ using the <u>other</u> substitution. (i.e. the one you did not use in (1).

(b) There is another way of finding this integral which involves the trig identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Find $\int \sin(\theta)\cos(\theta)\ d\theta$ using one of these identities and the substitution $u = 2\theta$.

(c) You should now have three different expressions for the indefinite integral $\int \sin(\theta)\cos(\theta)\ d\theta$.
Are they really different ?
Are they all correct ?
Explain.

Solution: 1(a).

$$\int \sin(\theta)\cos(\theta)\ d\theta$$

$$\text{Let } u = \sin(\theta) \quad : \quad du = \cos(\theta)\ d\theta$$

$$\int \sin(\theta)\cos(\theta)\ d\theta = \int u\ du = \frac{u^2}{2} = \frac{1}{2}\sin^2(\theta) + C_1$$

OR 1(b).

$$\int \sin(\theta)\cos(\theta)\ d\theta$$

$$\text{Let } u = \cos(\theta) \quad : \quad du = -\sin(\theta)\ d\theta$$

$$\int \sin(\theta)\cos(\theta)\ d\theta = -\int u\ du = -\frac{u^2}{2} = -\frac{1}{2}\cos^2(\theta) + C_2$$

2.

$$\sin(\theta)\cos(\theta) = \frac{\sin(2\theta)}{2}$$

$$\text{Let } u = 2(\theta) \quad : \quad \frac{1}{2}du = d\theta$$

$$\frac{1}{2}\int \sin u\ \frac{1}{2}du = -\frac{1}{4}\cos(2\theta) + C_3$$

3. The functions differ by a constant. [Calculate each of the three results when $\theta = 0$.]

Section 6.3 **Partial fractions**

5. (a) (5 points) Find the partial fraction decomposition of $\dfrac{x+2}{x^2-x}$.

Solution: Partial fraction decomposition:

$$\frac{x+2}{x^2-x} = \frac{1\cdot x+2}{(x-1)x} = \frac{A}{x-1}+\frac{B}{x} = \frac{(A+B)x-B}{x(x-1)}$$ **+2 pts**

A comparison of like terms gives: $A+B=1$, $-B=2 \;\Rightarrow\; A=3$, $B=-2$. **+2 pts**

Hence, $\dfrac{x+2}{x^2-x} = \dfrac{3}{x-1} - \dfrac{2}{x}$. **+1 pt**

(b) (5 points) Evaluate the integral $I = \displaystyle\int_2^3 \frac{x+2}{x^2-x}\,dx$. Simplify your answer.

Solution: $I = \displaystyle\int_2^3 \left(\frac{3}{x-1} - \frac{2}{x}\right) dx$ by using the result from part (a) **+1 pt**

$$I = \Big[3\ln|x-1| - 2\ln|x|\Big]_2^3$$ **+1 pt**

$$I = 3\ln 2 - 2\ln 3 - 3\ln 1 + 2\ln 2 = 5\ln 2 - 2\ln 3$$ **+2 pts**

$$I = \ln 32 - \ln 9 = \ln\left(\tfrac{32}{9}\right).$$ **+1 pt**

Section 6.6 **Improper Integrals**

6. (10 points) Write correct formula for evaluating the improper integral $I = \int_0^\infty \dfrac{3}{e^x\sqrt{x-2}}\,dx$

Do not integrate!

Solution: The integrand $\dfrac{3}{e^x\sqrt{x-2}}$ has discontinuity at $x = 2$. **+2 pts**

Therefore, the integral has to be split into three integrals, the first for $0 \le x \le 2$, the second for $2 \le x \le a$, where a is any number greater than 2, say 3, and the third for $x > a$. **+2 pts**

$$I = \int_0^2 \frac{3}{e^x\sqrt{x-2}}\,dx + \int_2^3 \frac{3}{e^x\sqrt{x-2}}\,dx + \int_3^\infty \frac{3}{e^x\sqrt{x-2}}\,dx$$ **+4 pts**

$$I = \lim_{t\to 2^-}\int_0^t \frac{3}{e^x\sqrt{x-2}}\,dx + \lim_{t\to 2^+}\int_t^{10} \frac{3}{e^x\sqrt{x-2}}\,dx + \lim_{t\to\infty}\int_3^t \frac{3}{e^x\sqrt{x-2}}\,dx$$ **+2 pts**

7. (10 points) Evaluate the improper integral $I = \int_4^5 \dfrac{6x}{\sqrt{x^2-16}}\,dx$

if it is convergent or show that it is divergent.

Solution: The integral is improper since the integrand has discontinuity at $x = 4$. **+2 pts**

Therefore, $I = \lim_{t\to 4^+}\int_t^5 \dfrac{6x}{\sqrt{x^2-16}}\,dx$ **+1 pt**

Subs: $u = x^2 - 16$, $du = 2x\,dx$, $6x\,dx = 3\,du$, $u(t) = t^2 - 16$, **+1 pt**

$u(t) = t^2 - 16$, $u(5) = 9$. **+1 pt**

$$I = \lim_{t\to 4^+} 3 \int_{t^2-16}^{9} u^{-1/2}\,du$$ **+2 pts**

$$I = 6 \lim_{t\to 4^+} u^{1/2}\Big|_{t^2-16}^{9} = 6\lim_{t\to 4^+}\left(3 - (t^2-16)^{1/2}\right)$$ **+2 pts**

$I = 6(3-0) = 18$ **+1 pt**

The value is finite, so the integral is convergent.

8. (10 points) Mike was asked to evaluate the integral $I = \int_0^{10} \dfrac{a+1}{x^2-(a-1)x-a}\,dx$

Below is his solution:

$$\frac{a+1}{x^2-(a-1)x-a} = \frac{a+1}{(x-a)(x+1)} = \frac{1}{x-a} - \frac{1}{x+1}$$

$$I = \int_0^{10} \frac{1}{x-a}\,dx - \int_0^{10} \frac{1}{x+1}\,dx = \ln|x-a|\Big|_0^{10} - \ln|x+1|\Big|_0^{10}$$

$$I = \ln|10-a| - \ln|a| - \ln 11 + \ln 1 = \ln\frac{|10-a|}{11|a|}$$

Explain why Mike's solution cannot be considered as a right one. Correct his answer. Do not integrate!

Solution: If $0 \le a \le 10$ the integrand $\dfrac{a+1}{x^2-(a-1)x-a}$ has discontinuity at $x = a$. **+2 pts**

Therefore, the integral is improper and has to be split into two integrals, one for $0 \le x \le a$ and the other for $a \le x \le 10$. **+2 pts**

$$I = \int_0^a \frac{1}{x-a}\,dx + \int_a^{10} \frac{1}{x-a}\,dx + \int_0^{10} \frac{1}{x+1}\,dx$$ **+4 pts**

$$I = \lim_{t\to a^-}\int_0^t \frac{1}{x-a}\,dx + \lim_{t\to a^+}\int_t^{10} \frac{1}{x-a}\,dx + \int_0^{10} \frac{1}{x+1}\,dx$$ **+4 pts**

If $a < 0$ or $a > 10$ Mike's solution is correct. **+2 pts**

Section 7.1 **Areas Between Curves**

9. (10 points) Consider the region R bounded by parabola $y = -x^2 + 4x$ with vertex at $(2, 4)$ and by lines $x = 0$, $x = 3$, $y = ax + 4$, where $a > 0$. The area of R is 12. Find the value of a. Support your answer.

Solution: When $x \geq 0$ all y-values of the line $y = ax + 4$ are greater than 4 because $a \geq 0$. The maximum y-value of the parabola is 4 and is attained at its vertex.

Therefore the line $y = ax + 4$ lies above the parabola. **+2 pts**

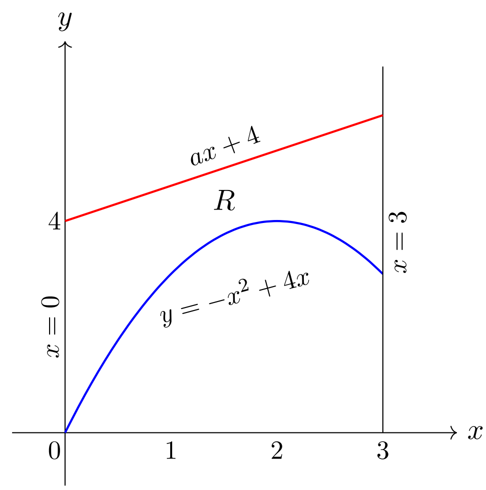
Then the area between the line and the parabola is

$$A = \int_0^3 \left(ax + 4 - (-x^2 + 4x)\right)\, dx$$ **+2 pts**

$$A = \int_0^3 \left(x^2 + ax - 4x + 4\right)\, dx$$ **+1 pt**

$$A = \left[\frac{x^3}{3} + \frac{a}{2}x^2 - 2x^2 + 4x\right]_0^3$$ **+2 pts**

$$A = 9 + \tfrac{9}{2}a - 18 + 12 = \tfrac{9}{2}a + 3 = 12$$ **+2 pts**



Therefore, $\frac{9}{2}a = 9$, $a = 2$. **+1 pt**

Sections 7.2 and 7.3 **Volumes**

10. (10 points) Consider the region R bounded by lines $y = 0$, $x = 1$, and the function $f(x) = a\sqrt{x}$, where $a > 0$. The volume V_1 is generated by rotating the region R about the axis $x = 0$. The volume V_2 is generated by rotating the region R about the axis $y = 0$. Find the value of a for which $V_1 = V_2$. Support your answer.

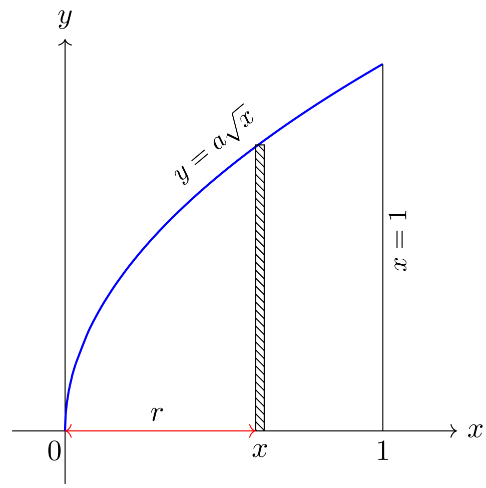
Solution: To evaluate V_1 we use the method of cylindrical shells.

The shell at x has radius $r = x$, circumference $2\pi x$, and height $a\sqrt{x}$. **+1 pt**

Therefore, $V_1 = \int_0^1 2\pi x \left(a\sqrt{x}\right)\,dx$ **+1 pt**

$= 2\pi a \int_0^1 x^{3/2}\,dx$ **+1 pt**

$= 2\pi a \left[\frac{2}{5}x^{5/2}\right]_0^1 = \frac{4}{5}\pi a$ **+1 pt**



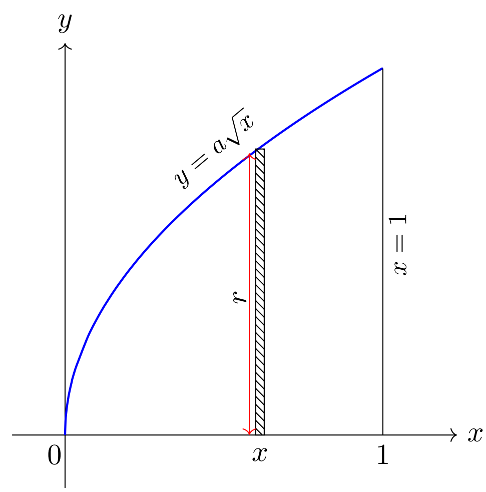
To evaluate V_2 we use the method of washers.

The washer at x has radius $r = a\sqrt{x}$. **+1 pt**

Therefore, $V_2 = \int_0^1 \pi \left(a\sqrt{x}\right)^2\,dx$ **+1 pt**

$= \pi a^2 \int_0^1 x\,dx$ **+1 pt**

$= \pi a^2 \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}\pi a^2$ **+1 pt**



We have $\frac{4}{5}\pi a = \frac{1}{2}\pi a^2$, $\frac{4}{5} = \frac{1}{2}a$, $a = \frac{8}{5} = 1.6$. **+2 pts**

11. (10 points) By using the method of washers find the volume of the solid generated by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ about the line $y = 1$.

Solution: Method of washers (see the picture).

$\sqrt{x} = x^2 \quad \Rightarrow \quad x = 0$ and $x = 1 \quad \Rightarrow \quad 0 \le x \le 1.$ **+1 pt**

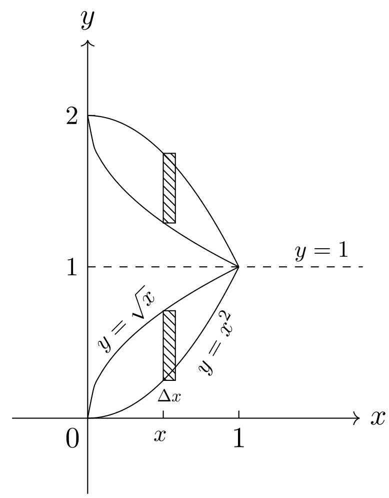
$\Delta V = V_{\text{outer}} - V_{\text{inner}}.$

$V_{\text{outer}} = \pi R^2 \Delta x, \quad V_{\text{inner}} = \pi r^2 \Delta x,$ **+1 pt**

where Δx is the height.

The radius R is the distance between the line $y = 1$ and the curve $y = x^2$. Therefore $R = 1 - x^2$. **+1 pt**

The radius r is the distance between the line $y = 1$ and the curve $y = \sqrt{x}$. Therefore $r = 1 - \sqrt{x}$. **+1 pt**



$\Delta V = \pi \left(\left(1 - x^2\right)^2 - (1 - \sqrt{x})^2 \right) \Delta x.$ **+1 pt**

$\Delta V = \pi \left(1 - 2x^2 + x^4 - 1 + 2\sqrt{x} - x\right) \Delta x.$

$\Delta V = \pi \left(x^4 - 2x^2 - x + 2\sqrt{x}\right) \Delta x.$ **+1 pt**

$dV = \pi \left(x^4 - 2x^2 - x + 2\sqrt{x}\right) \, dx.$ **+1 pt**

$$V = \int_0^1 \pi \left(x^4 - 2x^2 - x + 2\sqrt{x}\right) \, dx = \pi \left[\frac{x^5}{5} - \frac{2x^3}{3} - \frac{x^2}{2} + \tfrac{4}{3} x^{3/2} \right]_0^1$$ **+1 pt**

$= \pi \left(\tfrac{1}{5} - \tfrac{2}{3} - \tfrac{1}{2} + \tfrac{4}{3} \right)$ **+1 pt**

$V = \tfrac{11}{30} \pi.$ **+1 pt**

12. (10 points) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = x^2$, $y = 2x$ about the axis $x = 3$.

Solution: $x^2 = 2x \quad \Rightarrow \quad x = 0, \, x = 2 \quad \Rightarrow \quad 0 \le x \le 2$ **+2 pts**

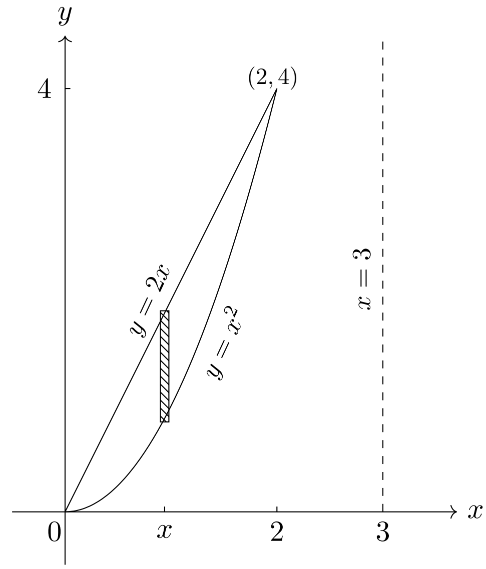
The shell has radius $3 - x$, circumference $2\pi(3 - x)$, and height $2x - x^2$. **+2 pts**

Therefore, $V = \int_0^2 2\pi(3 - x)\left(2x - x^2\right)\,dx$ **+2 pts**

$= 2\pi \int_0^2 (6x - 5x^2 + x^3)\,dx$ **+2 pts**

$= 2\pi \left[3x^2 - \frac{5x^3}{3} + \frac{x^4}{4}\right]_0^2$ **+1 pt**

$= 2\pi \left[12 - \frac{40}{3} + 4\right] = \frac{16}{3}\pi$ **+1 pt**



Section 7.4 **Arc Length**

13. (10 points) Mary was asked to evaluate arc length of the curve $y = \sqrt{|x|}$ when x changes from -2 to 3. She found that the arc length L can be evaluated by using the integral $L = \int_{-2}^{3} \sqrt{1 + \frac{1}{2x}}\,dx$

What is wrong in Mary's formula? Correct her answer. Do not integrate!

Solution: The integrand $\sqrt{1 + \frac{1}{2x}}$, if it is correct, has discontinuity at $x = 0$.

Therefore, the integral is improper and has to be split into two integrals, one for $-2 \le x \le 0$ and the other for $0 \le x \le 3$. **+2 pts**

Let's find the correct integrand:

$$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases} \qquad y = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$ **+2 pts**

$$\frac{dy}{dx} = \begin{cases} \frac{1}{2\sqrt{x}}, & x \ge 0 \\ -\frac{1}{2\sqrt{-x}}, & x < 0 \end{cases} \qquad \left(\frac{dy}{dx}\right)^2 = \begin{cases} \frac{1}{4x}, & x \ge 0 \\ -\frac{1}{4x}, & x < 0 \end{cases}$$ **+2 pts**

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \begin{cases} \sqrt{1 + \frac{1}{4x}}, & x \ge 0 \\ \sqrt{1 - \frac{1}{4x}}, & x < 0 \end{cases}$$ **+2 pts**

Therefore, $L = \int_{-2}^{0} \sqrt{1 - \frac{1}{4x}}\, dx \; + \; \int_{0}^{3} \sqrt{1 + \frac{1}{4x}}\, dx$ **+2 pts**

14. (10 points) Find the exact length L of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ when $1 \le x \le 2$.

Hint: $1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \left(\frac{x}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x}\right)^2 = \left(\frac{x}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$

Solution: $\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$ **+2 pts**

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2, \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{x}{2} + \frac{1}{2x}.$ **+2 pts**

$L = \frac{1}{2} \int_{1}^{2} \left(x + \frac{1}{x}\right) dx$ **+2 pts**

$L = \frac{1}{2} \left[\frac{1}{2}x^2 + \ln x\right]_1^2$ **+2 pts**

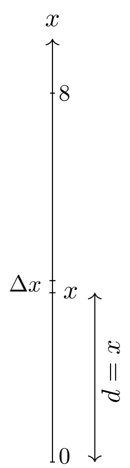
$L = \frac{1}{2} \left[2 + \ln 2 - \frac{1}{2} + 0\right] = \frac{3}{4} + \frac{\ln 2}{2}$ **+2 pts**

Section 7.6 **Applications to physics and engineering (work, force)**

15. (10 points) A chain laying on the ground is 12 m long and its mass is 72 kg. How much work is required to raise one end of the chain to the height of 8 m?

Solution: Consider the end of the chain of the length 8 m. It is lifted vertically. Draw the x-axis vertically, directed up.

Consider a small piece of the chain of the length Δx that was lifted to the height x, $0 \le x \le 8$, where $x = 0$ corresponds to the ground level. **+2 pts**



If, for example, we lift the right end of the chain then only its right part of the length of 8 m is being lifted. The left part of the length 4 m lays on the ground. The point denoted by x is $4 + x$ m away from the most left end. It is lifted to the height of x m.

Therefore its displacement is $d = x$. **+1 pt**

The mass of the small piece is $\Delta m = \dfrac{72}{12}\Delta x = 6\Delta x$. **+2 pts**

The work needed to lift it to the height x is

$\Delta W = \Delta m \cdot g \cdot d = (6\Delta x)\, gx = 6gx\Delta x.$ **+2 pts**

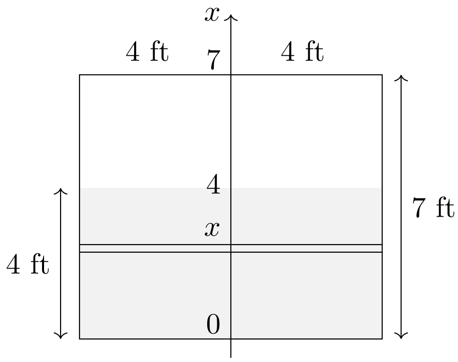
Therefore $W = \displaystyle\int_0^8 6gx\,dx$ **+1 pt**

$W = \Big[3gx^2\Big]_0^8 = 3g \cdot 64 = 192g$ Joules. **+2 pts**

16. (10 points) A cylindrical aquarium has radius of 4 feet and height of 7 feet. The depth of water is 4 feet. How much work is required to pump all of the water out over the side?

(You may use the following equality $62.5 = \dfrac{1000}{16}$ lb/ft^2)

Solution: We place the x-axis vertically in the middle of the aquarium with the origin is in the bottom of the cylinder. A horizontal cylindrical slice of water Δx is placed at the coordinate x, where $0 \le x \le 4$. See the picture. **+1 pt**



The slice Δx thick has a volume of

$\Delta V = \pi r^2 h = \pi \cdot 4^2 \cdot \Delta x = 16\pi\Delta x$ ft^3 **+1 pt**

and weighs $\frac{1000}{16} \cdot 16\pi\Delta x = 1000\pi\Delta x$ lb **+1 pt**

The distance between the slice and the

top of the aquarium is $d = 7 - x$ **+1 pt**

The work needed to pump water out is

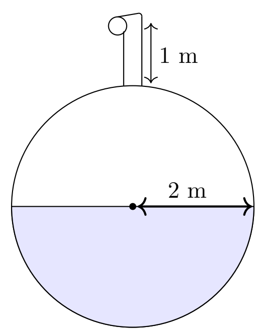
$$W = \int_0^4 1000\pi(7 - x)\,dx$$ **+2 pts**

$$W = 1000\pi\left[7x - \frac{x^2}{2}\right]_0^4$$ **+2 pts**

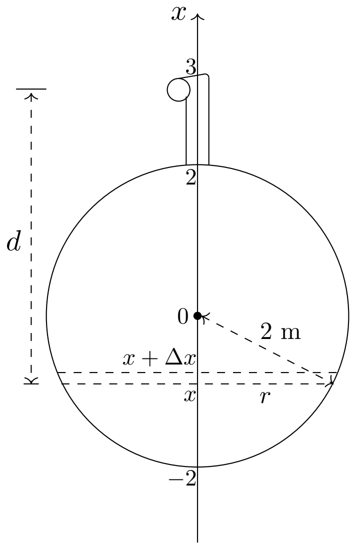
$$= 1000\pi\,[28 - 8] = 20{,}000\pi$$ **+2 pts**

17. (10 points) A water tank has the shape of a sphere of radius 2 m with a spout at the top of length of 1 m, as in the picture below. The spherical part of the tank is half full of water and the spout is empty.

Set up **but do not evaluate** an integral to find the work required to empty the tank by pumping all of the water out of the spout. The density of water is $\rho = 1000\frac{\text{kg}}{\text{m}^3}$ and the acceleration due to gravity is $g = 9.8\frac{\text{m}}{\text{s}^2}$.



Solution:



We use the direction up as a positive one for the x-axis.

The origin is in the center of the sphere.

Let x be an arbitrary number on the x-axis in the lower half of the spherical tank. Then $-2 \le x \le 0$. **+1 pt**

The small volume of the slice that passes through the point x is $\Delta V = \pi r^2 \Delta x$. **+2 pts**

We have $r^2 + x^2 = 2^2, \quad r^2 = 4 - x^2$ **+1 pt**

$\Delta V = \pi(4 - x^2)\Delta x$ **+1 pt**

$\Delta F = \rho g \Delta V = 9800\pi(4 - x^2)\Delta x$ **+1 pt**

$d = 2 + 1 - x = 3 - x$. (Note that $-x \ge 0$) **+1 pt**

$\Delta W = \Delta F \cdot d = 9800\pi(4 - x^2)(3 - x)\Delta x$ **+1 pt**

$$W = \int_{-2}^{0} 9800\pi(4 - x^2)(3 - x)\, dx$$ **+2 pts**

Section 7.7 **Differential Equations and Applications of DEs**

18. Consider the initial value problem:

$$y' - (\tan x)\, y = 1, \qquad y(0) = -3.$$

(a) (2 points) Determine the type of the differential equation.

Solution: It is a **first order linear** differential equation. **+2 pts**

(b) (6 points) Find the general solution of the problem.

Solution: The integrating factor is $u = e^{-\int \tan x\, dx} = e^{\ln(\cos x)} = \cos x$. **+2 pts**

(Note: $-\displaystyle\int \tan x\, dx = -\int \frac{\sin x}{\cos x}\, dx$ and use the substitution $v = \cos x$ to evaluate the integral).

Then

$(\cos x)\, y' - (\sin x)\, y = \cos x, \quad ((\cos x)\, y)' = \cos x, \quad (\cos x)\, y = \sin x + c.$ **+2 pts**

So, the general solution is $y(x) = \tan x + c \sec x$. **+2 pts**

(c) (2 points) Find the particular solution of this problem that satisfies given initial condition.

Solution: $y(0) = \tan 0 + c \sec 0 = c = -3$. **+1 pt**

The particular solution is $y(x) = \tan x - 3 \sec x$. **+1 pt**

19. Determine the type of the given differential equation. Solve the initial-value problem. Show all work.

(a) (10 points) $xy' = (1 + x)y, \qquad y(1) = e.$

Solution: It is a separable equation **+2 pts**

$x\dfrac{dy}{dx} = (1+x)y, \quad \dfrac{dy}{y} = \dfrac{1+x}{x}\, dx, \quad \dfrac{dy}{y} = \left(\dfrac{1}{x} + 1\right) dx$ **+2 pts**

$\displaystyle\int \frac{dy}{y} = \int \left(\frac{1}{x} + 1\right) dx, \quad \ln|y| = \ln|x| + x + C$ **+2 pts**

$|y| = e^C\, |x| e^x, \quad y = Axe^x$ **+2 pts**

$y(1) = Ae = e, \quad A = 1$ **+1 pts**

$y = xe^x$ **+1 pts**

Alternative solution: $y' - \dfrac{1+x}{x}y = 0$. First order linear equation. **+2 pts**

The integrating factor is $I(x) = e^{\int -\left(\frac{1}{x}+1\right)dx} = e^{-\ln x - x} = x^{-1}e^{-x}$. **+2 pts**

Then

$\left(x^{-1}e^{-x}y\right)' = 0$ **+2 pts**

$x^{-1}e^{-x}y = C, \quad y = Cxe^{x}$ **+2 pts**

$y(1) = Ce = e, \quad C = 1$ **+1 pt**

$y = xe^{x}$ **+1 pt**

(b) (10 points) $y' + \dfrac{2x}{x^2+1}y = \dfrac{1}{x^2+1}, \qquad y(0) = 3.$

Solution: It is a first order linear differential equation. **+2 pts**

The integrating factor is $I(x) = e^{\int \frac{2x}{x^2+1}\,dx} = e^{\ln(x^2+1)} = x^2+1$. **+3 pts**

Then

$(x^2+1)y' + 2xy = 1, \quad \left((x^2+1)y\right)' = 1, \quad (x^2+1)y = \int dx = x + C$ **+2 pts**

$y = \dfrac{x+C}{x^2+1}$. **+1 pt**

$y(0) = C = 3, \quad C = 3$ **+1 pt**

$y(x) = \dfrac{x+3}{x^2+1}$ **+1 pt**

20. A large tank initially contains 10 liters of salty water with a concentration of 60 grams of salt per liter of water. At time zero, a solution with concentration of 10 grams of salt per liter begins flowing into the tank at a rate of 2 liters per minute. Simultaneously, the well mixed solution drains out at the rate of 2 liters per minute.

(a) (1 point) Find the initial condition.

Solution: Let $x(t)$ be the amount of salt in pounds in the tank at time t measured in minutes.

Then the **initial condition** is $x(0) = 10$ liters $\times$ 60 gram/liter $=$ 600 grams. **+1 pt**

(b) (2 points) Write down the balance equation that models the process.

Solution: **Rate in** $=$ 10 gram/liter $\times$ 2 liter/minute $=$ 20 gram/minute.

Rate out $= \dfrac{x}{10}$ gram/liter $\times$ 2 liter/minute $= \dfrac{x}{5}$ gram/minute. **+1 pt**

Balance equation: $x' = 20 - \dfrac{x}{5} \quad \Leftrightarrow \quad x' = -\dfrac{x-100}{5}.$ **+1 pt**

(c) (6 points) Solve the balance equation to find a formula of the amount of salt in the tank after t minutes?

Solution: The balance equation is a **separable** equation. **+1 pt**

$$\frac{dx}{dt} = -\frac{x-100}{5}, \qquad \frac{dx}{x-100} = -\tfrac{1}{5}\,dt, \qquad \int \frac{dx}{x-100} = -\tfrac{1}{5}\int dt.$$ **+2 pts**

$\ln|x-100| = -\frac{1}{5}t + C, \quad |x-100| = e^{-t/5}\cdot e^{C}, \quad x - 100 = Ae^{-t/5},$ where $A = \pm e^{C}$.

$x(t) = Ae^{-t/5} + 100$ is the general solution. **+1 pt**

To find A we use the initial condition found before:

$x(0) = 600 = A + 100 \quad \Leftrightarrow \quad A = 500.$ **+1 pt**

Therefore the number of pounds of salt in the tank in t minutes is

$x(t) = 500e^{-t/5} + 100.$ **+1 pt**

(d) (1 points) How much salt is in the tank after 10 minutes? Leave answer in exact form.

Solution: $x(10) = 500e^{-10/5} + 100 = 500e^{-2} + 100 = \dfrac{500}{e^2} + 100$ gram. **+1 pt**

Section **Inhomogeneous Second Order Differential Equations**

21. (10 points) Consider the second-order ordinary differential equation for $y = y(x)$:

$$3y'' - 6y' + ky = G(x),$$

where k is a constant, and $G(x)$ is a function of x.

(a) (5 points) Find the value of k such that the corresponding characteristic equation has two repeated real roots, and then for this value of k find the general solution to the corresponding homogeneous differential equation.

Solution: Homogeneous equation is $3y'' - 6y' + ky = 0$. **+1 pt**

Char. eq. is $r^2 - 6r + k = 0$.

It has two repeated real roots when $D = 0$. **+1 pt**

Then $D = 36 - 4\cdot 3\cdot k = 36 - 12k = 0, \; k = 3.$ **+1 pt**

The repeated real roots are $r_1 = r_2 = -\dfrac{-6}{2\cdot 3} = 1.$ **+1 pt**

Solution to homogeneous DE is $y_h(x) = e^{x}\,(c_1 + c_2 x)$. **+1 pt**

(b) (5 points) If $G(x) = P_m(x)\cos x$ where $P_m(x)$ is a polynomial of degree $m > 0$, what is the trial form of a particular solution to the inhomogeneous differential equation? For $m = 1$ pick an example of $P_m(x)$ and find a form of a particular solution to the inhomogeneous equation.

Solution: $y_p(x) = Q_m(x)\cos x + R_m(x)\sin x$, where $Q_m(x)$ and $R_m(x)$ are polynomials of degree m written in general form. **+2 pts**

For $m = 1$ let $P_1(x) = 3x - 11$. Then $G(x) = (3x - 11)\cos x$ **+1 pt**

and $y_p(x) = (ax + b)\cos x + (cx + d)\sin x$. **+2 pts**

22. (10 points) For the equation $y'' - 4y' - 5y = 12e^{-t}$.

(a) (3 points) Find the solution of the corresponding homogeneous equation.

Solution: Homogeneous equation is $y'' - 4y' - 5y = 0$. **+1 pt**

Char. eq. is $\lambda^2 - 4\lambda - 5 = 0$.

It has distinct real roots $\lambda_1 = -1$ and $\lambda_2 = 5$. **+1 pt**

$y_h(t) = c_1e^{-t} + c_2e^{5t}$. **+1 pt**

(b) (5 points) Find a particular solution by using the method of undetermined coefficients.

Solution: Since the function e^{-t} in one of solutions of the homogeneous equation we are looking for a particular solution in the form

$y_p(t) = ate^{-t}$. **+2 pts**

Then $y_p' = a(1 - t)e^{-t}$ and $y_p'' = a(-2 + t)e^{-t}$. **+1 pt**

$y_p'' - 4y_p' - 5y_p = a(-2 + t - 4 + 4t - 5t)e^{-t} = -6ae^{-t} = 12e^{-t} \Rightarrow a = -2$. **+1 pt**

Hence $y_p(t) = -2te^{-t}$. **+1 pt**

(c) (2 points) Find the general solution.

Solution: $y(t) = y_h + y_p = c_1e^{-t} + c_2e^{5t} - 2te^{-t} = (c_1 - 2t)\,e^{-t} + c_2e^{5t}$ **+2 pts**

23. (10 points) A forced mass spring system with an external driving force is modeled by

$$x'' + 4x' + 5x = 10\sin 3t,$$

where t is measured in seconds and x in meters.

(a) (3 points) Find the transient state, i.e. the solution to the associated homogeneous equation.

Solution: This is the underdamped case because $c = 1 < w_0 = \sqrt{5}$.

Homogeneous equation is $x'' + 4x' + 5x = 0$. **+1 pt**

Char. eq. is $r^2 + 4r + 5 = 0$.

It has complex root $r = -2 + i$ (and its conjugate $\bar{r} = -2 - i$). **+1 pt**

The transient state is $x(t) = e^{-2t}\,(c_1 \cos t + c_2 \sin t)$. **+1 pt**

(b) (5 points) By using the method of undetermined coefficients find the steady-state, i.e. a particular solution, to the forced equation.

Solution: $x_p(t) = a\cos 3t + b\sin 3t$,

$x_p'(t) = -3a\sin 3t + 3b\cos 3t$, $x_p''(t) = -9x_p$. **+1 pt**

Then

$x_p'' + 4x_p' + 5x_p = -9x_p + 4x_p' + 5x_p = -4x_p + 4x_p'$

$= -4a\cos 3t - 4b\sin 3t - 12a\sin 3t + 12b\cos 3t = 10\sin 3t$ **+2 pts**

$-4a + 12b = 0,\ \ -4b - 12a = 10 \quad \Leftrightarrow \quad a - 3b = 0,\ 2b + 6a = -5 \quad \Leftrightarrow$

$a = 3b,\ 20b = -5 \quad \Leftrightarrow \quad a = -\frac{3}{4},\ b = -\frac{1}{4}$. **+1 pt**

Therefore, $x_p(t) = -\frac{3}{4}\cos 3t - \frac{1}{4}\sin 3t$. **+1 pt**

(c) (2 points) Write down the general solution.

Solution: $x(t) = e^{-2t}\,(c_1 \cos t + c_2 \sin t) - \frac{3}{4}\cos 3t - \frac{1}{4}\sin 3t$. **+2 pts**