LOCALLY CONSERVATIVE COUPLING OF STOKES AND DARCY FLOWS

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Abstract. A locally conservative numerical method for solving the coupled Stokes and Darcy flows problem is formulated and analyzed. The approach employs the mixed finite element method for the Darcy region and the discontinuous Galerkin method for the Stokes region. A discrete inf-sup condition and optimal error estimates are derived.

Key words. multiphysics, porous media flow, incompressible fluid flow, discontinuous Galerkin, mixed finite element, error estimates, inf-sup condition

AMS subject classifications. 35Q35, 65N30, 65N15, 76D07, 76S05

1. Introduction.

The numerical modeling of reactive transport necessitates the use of numerical schemes that do not create artificial mass. Mixed finite element (MFE) and discontinuous Galerkin (DG) methods are two examples of locally mass conservative methods, that are used in the geosciences. MFE methods are quite popular for porous media problems [12, 30, 13, 3] and DG methods are attractive for modeling flow on unstructured meshes [29, 27, 26, 28].

Many applications involve different physical processes in different parts of the simulation domain. In this paper we propose a numerical method for approximating the solution to the coupled Darcy-Stokes problem. Such systems arise, for example, in modeling the interaction between surface water (river) and groundwater (aquifer). There are few works in the literature that address the numerical analysis of the coupled Darcy-Stokes problem. In [21], Layton et al. consider a formulation based on the Beavers-Joseph-Saffman interface conditions [4, 31, 20], prove the existence and uniqueness of a weak solution, and analyze a continuous finite element scheme coupled with MFE. A similar formulation is studied by Discacciati et al. [11], where continuous finite elements are used in both regions. An application of this formulation to vugular porous media is studied in [2]. A singularly perturbed Stokes problem, which models Darcy flow as a limiting case, is considered by Mardal et al. [23]. There, a new finite element is proposed which behaves uniformly in the perturbation parameter. Ewing et al. [14] employ finite difference methods for a similar model involving the Navier-Stokes equations with an added Darcy term.

The model we consider, wich is similar to the one in [21], is based on imposing the correct local equations in each region, coupled with appropriate interface conditions. In particular the fluid region is modeled by the Stokes equations and the porous media region is modeled by the Darcy's law. Continuity of flux, balance of forces, and the Beavers-Joseph-Saffman slip with friction condition (see (2.10) below) are imposed on the interface. In this work we emphasize locally mass conservative discretizations. Conserving mass locally is especially important when the flow equations are coupled with the reactive transport of chemical species. In the porous media region, the fluid velocity and pressure are obtained by MFE, and in the incompressible flow region, the fluid velocity and pressure are approximated by DG. An advantage of our approach is the possibility of coupling existing highly optimized MFE-based porous media simu-

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lators with the flexibility and easy implementation of DG methods for incompressible flows. The estimates are derived for two-dimensional problems. The results are also valid in higher dimension, and depend on the existence of approximation operators (See Remark 4.1 below).

The outline of the paper is as follows. In Section 2, the model problem, notation and scheme are presented. Section 3 contains the derivation of the discrete inf-sup condition. In Section 4, approximation results and optimal a priori error estimates are proved. Some concluding remarks follow.

2. Model Problem, Notation, and Scheme. Let Ω be a domain in $\mathbb{R}^d, d = 2$, subdivided into two subdomains Ω_1, Ω_2. Let Γ_{12} be the interface $\partial\Omega_1 \cap \partial\Omega_2$. Define $\Gamma_i = \partial\Omega_i \backslash \Gamma_{12},\ i = 1, 2$. Denote by $\boldsymbol{n}$ the outward normal vector to $\partial\Omega$. Let $\boldsymbol{n}_{12}$ (resp. $\boldsymbol{\tau}_{12}$) be the unit normal (resp. tangential) vector to Γ_{12} outward of Ω_1. Denote by $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ the fluid velocity and by $p = (p_1, p_2)$ the fluid pressure, where $\boldsymbol{u}_i = \boldsymbol{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$. The flow in domain Ω_1 is assumed to be of Stokes type, and therefore the following equations are satisfied:

$$-\nabla \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) = \boldsymbol{f}_1 \quad \text{in } \Omega_1, \tag{2.1}$$
$$\nabla \cdot \boldsymbol{u}_1 = 0 \quad \text{in } \Omega_1, \tag{2.2}$$
$$\boldsymbol{u}_1 = 0 \quad \text{on } \Gamma_1. \tag{2.3}$$

Here $\boldsymbol{T}$ is the stress tensor

$$\boldsymbol{T}(\boldsymbol{u}_1, p_1) = -p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1)$$

which depends on the viscosity $\mu > 0$ and the strain tensor

$$\boldsymbol{D}(\boldsymbol{u}_1) = \frac{1}{2}(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_1^T).$$

In the region Ω_2, the fluid pressure and velocity satisfy the single phase Darcy flow equations:

$$\nabla \cdot \boldsymbol{u}_2 = f_2 \quad \text{in } \Omega_2, \tag{2.4}$$
$$\boldsymbol{u}_2 = -\boldsymbol{K}\nabla p_2 \quad \text{in } \Omega_2, \tag{2.5}$$
$$\boldsymbol{u}_2 \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_2, \tag{2.6}$$

where $\boldsymbol{K}$ is a symmetric and positive definite tensor representing the permeability divided by the viscosity and satisfying, for some $0 < k_0 \le k_1 < \infty$,

$$k_0\xi^T\xi \le \xi^T\boldsymbol{K}(x)\xi \le k_1\xi^T\xi \quad \forall x \in \Omega_2,\ \forall \xi \in \mathbb{R}^d. \tag{2.7}$$

The physical quantities are coupled through appropriate interface conditions

$$\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} = \boldsymbol{u}_2 \cdot \boldsymbol{n}_{12}, \tag{2.8}$$
$$p_1 - 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12}) = p_2, \tag{2.9}$$
$$\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12} = -2G(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \tag{2.10}$$

Note that condition (2.8) represents the mass conservation across the interface, condition (2.9) imposes balance of forces across the interface, and condition (2.10) is the Beavers-Joseph-Saffman law, where $G > 0$ is a friction constant that can be determined experimentally. The reader should refer to [4, 31, 20, 21] for a detailed description and motivation for the choice of these interface conditions.

For $i = 1, 2$, let $\mathcal{E}_h^i$ be a non-degenerate quasi-uniform subdivision of Ω_i, let Γ_h^i be the set of interior edges and let h_i denote the maximum diameter of elements in $\mathcal{E}_h^i$. For $s \geq 0$, $p > 1$, and a domain $E \subset \mathbb{R}^d$, let $W^{s,p}(E)$ be the usual Sobolev spaces [1], let $H^s(E) = W^{s,2}(E)$ be equipped with the usual norm $\|\cdot\|_{s,E}$, and let $L_0^2(E)$ denote the space of L^2 functions with zero average. In the formulation for the Stokes region, we need that both the gradient of $\boldsymbol{u}_1$ and the pressure p_1 have a trace on line segments. For this, it suffices to define the following velocity–pressure spaces for the Stokes region:

$$\begin{aligned} \boldsymbol{X}^1 &= \{\boldsymbol{v}_1 \in (L^2(\Omega_1))^d : \quad \forall E \in \mathcal{E}_h^1, \quad \boldsymbol{v}_1|_E \in (W^{2,4/3}(E))^d\}, \\ M^1 &= \{q_1 \in L^2(\Omega_1) : \quad \forall E \in \mathcal{E}_h^1, \quad q_1|_E \in W^{1,4/3}(E)\}, \end{aligned}$$

with norms

$$\begin{aligned} |||\boldsymbol{v}_1|||_{s,\Omega_1}^2 &= \sum_{E \in \mathcal{E}_h^1} \|\boldsymbol{v}_1\|_{s,E}^2, \\ \|\boldsymbol{v}_1\|_{X^1}^2 &= |||\nabla \boldsymbol{v}_1|||_{0,\Omega_1}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \|[\boldsymbol{v}_1]\|_{0,e}^2 + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \|\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{0,e}^2, \\ \|q_1\|_{M^1} &= \|q_1\|_{0,\Omega_1}. \end{aligned}$$

Here, the parameter $\sigma_e \geq 0$ takes a constant value over each edge e and $|e|$ denotes the measure (or length) of e. Given a fixed normal vector $\boldsymbol{n}_e$ on each edge $e = \partial E_e^1 \cap \partial E_e^1$, directed from E_e^1 to E_e^2, the average and jump of functions in $\boldsymbol{X}^1$ and M^1 can be defined as

$$\begin{aligned} \{w\} &= \tfrac{1}{2}(w|_{E_e^1}) + \tfrac{1}{2}(w|_{E_e^2}), \quad [w] = (w|_{E_e^1}) - (w|_{E_e^2}), \quad \forall e = \partial E_e^1 \cap \partial E_e^2, \\ \{w\} &= w|_{E_e^1}, \qquad\qquad\qquad\quad\; [w] = w|_{E_e^1}, \quad \forall e = \partial E_e^1 \cap \partial \Omega. \end{aligned}$$

The velocity-pressure spaces for the Darcy region are

$$\begin{aligned} \boldsymbol{X}^2 &= \{\boldsymbol{v} \in H(\mathrm{div}; \Omega_2) : \int_{\partial\Omega_2} \boldsymbol{v} \cdot \boldsymbol{n}\, w = 0 \; \forall w \in H^1_{0,\Gamma_{12}}(\Omega_2)\}, \\ M^2 &= L^2(\Omega_2), \end{aligned}$$

where $H(\mathrm{div}; \Omega_2)$ is the space of vectors in $(L^2(\Omega_2))^d$ whose divergence lies in $L^2(\Omega_2)$ and

$$H^1_{0,\Gamma_{12}}(\Omega_2) = \{w \in H^1(\Omega_2) : w = 0 \text{ on } \Gamma_{12}\}.$$

The norms associated with $(\boldsymbol{X}^2, M^2)$ are

$$\|\boldsymbol{v}_2\|_{X^2}^2 = \|\boldsymbol{v}_2\|_{0,\Omega_2}^2 + \|\nabla \cdot \boldsymbol{v}_2\|_{0,\Omega_2}^2, \quad \|q_2\|_{M^2} = \|q_2\|_{0,\Omega_2}. \tag{2.11}$$

We can now define $\boldsymbol{X} = \boldsymbol{X}^1 \times \boldsymbol{X}^2$ and $M = (M^1 \times M^2) \cap L_0^2(\Omega)$, the spaces for the coupled formulation with the usual norms

$$\|\boldsymbol{v}\|_X^2 = \|\boldsymbol{v}_1\|_{X^1}^2 + \|\boldsymbol{v}_2\|_{X^2}^2, \quad \|q\|_M^2 = \|q_1\|_{M^1}^2 + \|q_2\|_{M^2}^2. \tag{2.12}$$

In [21], it was shown that there exists a unique weak solution $(\boldsymbol{u}, p)$ of the coupled problem (2.1)–(2.10), with $\boldsymbol{u}_1 \in (H^1(\Omega_1))^d$, $\boldsymbol{u}_2 \in \boldsymbol{X}^2$, and $p \in M$. We will assume that the solution $(\boldsymbol{u}, p)$ is regular enough, so that it is a strong solution of (2.1)–(2.10). Next, we introduce the bilinear forms $a_1 : \boldsymbol{X}^1 \times \boldsymbol{X}^1 \to \mathbb{R}$ and $b_1 : \boldsymbol{X}^1 \times M^1 \to \mathbb{R}$,

$$a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) = 2\mu \sum_{E \in \mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \int_e [\boldsymbol{u}_1] \cdot [\boldsymbol{v}_1]$$
$$-2\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{u}_1)\} \boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + 2\mu\epsilon \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\} \boldsymbol{n}_e \cdot [\boldsymbol{u}_1]$$
$$+\frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e \boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12} \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}, \tag{2.13}$$

$$b_1(\boldsymbol{v}_1, p_1) = -\sum_{E \in \mathcal{E}_h} \int_E p_1 \nabla \cdot \boldsymbol{v}_1 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{p_1\} [\boldsymbol{v}_1] \cdot \boldsymbol{n}_e. \tag{2.14}$$

Here, ϵ is a constant that takes the value -1 or $+1$, which makes the bilinear form a_1 symmetric or non-symmetric. The bilinear forms corresponding to the Darcy region are $a_2 : \boldsymbol{X}^2 \times \boldsymbol{X}^2 \to \mathbb{R}$ and $b_2 : \boldsymbol{X}^2 \times M^2 \to \mathbb{R}$:

$$a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) = \int_{\Omega_2} \boldsymbol{K}^{-1} \boldsymbol{u}_2 \cdot \boldsymbol{v}_2, \tag{2.15}$$

$$b_2(\boldsymbol{v}_2, q_2) = -\int_{\Omega_2} q_2 \nabla \cdot \boldsymbol{v}_2. \tag{2.16}$$

Let k_1, k_2 and l_2 be positive integers. Let $\boldsymbol{X}_h$ and M_h be finite dimensional subspaces of $\boldsymbol{X}$ and M respectively, such that

$$\boldsymbol{X}_h = \boldsymbol{X}_h^1 \times \boldsymbol{X}_h^2, \quad M_h = M_h^1 \times M_h^2,$$

where $(\boldsymbol{X}_h^1, M_h^1)$ is the pair of discontinuous finite element spaces

$$\boldsymbol{X}_h^1 = \{\boldsymbol{v}_1 \in \boldsymbol{X}^1 : \quad \forall E \in \mathcal{E}_h^1, \quad \boldsymbol{v}_1 \in (\mathbb{P}_{k_1}(E))^d\},$$
$$M_h^1 = \{q_1 \in M^1 : \quad \forall E \in \mathcal{E}_h^1, \quad q_1 \in \mathbb{P}_{k_1-1}(E)\}.$$

The discrete spaces corresponding to the Darcy region, consist of the standard mixed finite element spaces (such as RT spaces [25], BDM spaces [7], BDFM spaces [6] and BDDF spaces [5]). The mixed velocity spaces $\boldsymbol{X}_h^2$ contains polynomials of degree k_2 and the pressure spaces M_h^2 polynomials of degree l_2. Note that for the Raviart-Thomas spaces, the condition $l_2 = k_2 - 1$ holds. We also assume that

$$\forall \boldsymbol{v}_2 \in \boldsymbol{X}_h^2, \quad \boldsymbol{v}_2 \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_2.$$

Let E be a mesh element with diameter h_E. Given $p \in L_0^2(\Omega)$, we denote by $\tilde{p}$ the L^2 projection of p in M_h satisfying

$$\forall q \in \mathbb{P}_{k_1-1}(E), \int_E q(\tilde{p} - p) = 0, \quad \forall E \in \mathcal{E}_h^1, \tag{2.17}$$

$$\forall q \in \mathbb{P}_{l_2}(E), \int_E q(\tilde{p} - p) = 0, \quad \forall E \in \mathcal{E}_h^2, \tag{2.18}$$

and, if $p|_{\Omega_1} \in H^{k_1}(\Omega_1)$ and $p|_{\Omega_2} \in H^{l_2+1}(\Omega_2)$, then

$$\|p - \tilde{p}\|_{m,E} \le C h_E^{k_1 - m} |p|_{k_1,E}, \quad E \subset \Omega_1, \quad m = 0, 1, \tag{2.19}$$

$$\|p - \tilde{p}\|_{m,E} \le C h_E^{l_2+1-m} |p|_{l_2+1,E}, \quad E \subset \Omega_2, \quad m = 0, 1. \tag{2.20}$$

Here and throughout the paper, C denotes a varying constant that is independent of the diameter of the mesh elements. We also make use of the the quasi-local interpolant $\mathbf{\Pi}_h^1 : (H^1(\Omega_1))^d \to \boldsymbol{X}_h^1$ [10, 15, 9, 18] satisfying for all $\boldsymbol{v}_1 \in (H^1(\Omega_1))^d$

$$b_1(\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1, q_1) = 0, \quad \forall\, q_1 \in M_h^1, \tag{2.21}$$

$$\forall\, e \in \Gamma_h^1 \cup \Gamma_1, \ \int_e [\mathbf{\Pi}_h^1 \boldsymbol{v}_1] \cdot q_1 = 0, \quad \forall\, \boldsymbol{v}_1 \in (H^1(\Omega_1))^d : \boldsymbol{v}_1 = 0 \text{ on } \Gamma_1, \ \forall q_1 \in M_h^1, \tag{2.22}$$

$$|\!|\!|\mathbf{\Pi}_h^1 \boldsymbol{v}_1|\!|\!|_{1,\Omega_1} \leq C \|\boldsymbol{v}_1\|_{1,\Omega_1}. \tag{2.23}$$

The operator $\mathbf{\Pi}_h^1$ has the optimal approximation properties

$$|\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1|_{m,E} \leq C h_E^{s-m} |\boldsymbol{v}_1|_{s,\delta(E)} \quad \forall\, 1 \leq s \leq k_1 + 1, \quad \forall\, \boldsymbol{v}_1 \in H^s(\Omega_1), \quad m = 0, 1, \tag{2.24}$$

where $\delta(E)$ is a suitable macro-element containing E. Moreover, it holds that for at least one edge e of every element $E \in \mathcal{E}_h^1$

$$\int_e (\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1) = 0, \quad \forall\, \boldsymbol{v}_1 \in (H^1(\Omega_1))^d. \tag{2.25}$$

We note that (2.25) holds true for all edges in the case $k = 1$ and $k = 2$. For $k = 3$, we can assume without loss of generality, that (2.25) is satisfied for all edges in Γ_{12}. We will make use of the following bounds on $\mathbf{\Pi}_h^1$.

LEMMA 2.1. *Let $1 \leq s \leq k_1 + 1$. For all $\boldsymbol{v}_1 \in (H^s(\Omega_1))^d$,*

$$\|\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1\|_{X^1} \leq C h_1^{s-1} |\boldsymbol{v}_1|_{s,\Omega_1}, \tag{2.26}$$

$$\|\mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{X^1} \leq C \|\boldsymbol{v}_1\|_{1,\Omega_1}. \tag{2.27}$$

Proof. To show (2.26), we note that

$$\|\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{X^1}^2 = |\!|\!|\nabla(\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1)|\!|\!|_{0,\Omega_1}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \|[\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1]\|_{0,e}^2 \tag{2.28}$$

The jump term is bounded as follows:

$$\sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \|[\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1]\|_{0,e}^2 \leq 2 \sum_{E \in \mathcal{E}_h^1} \sum_{e \in \partial E} \frac{1}{|e|} \|\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{0,e}^2.$$

Passing to the reference element $\hat{E}$, and denoting $\hat{\boldsymbol{v}}_1 = \boldsymbol{v}_1 \circ F_E(\hat{x})$, where F_E is the affine mapping that maps $\hat{E}$ onto E, we have

$$\sum_{e \in \partial E} \frac{1}{|e|} \|\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{0,e}^2 \leq C \|\hat{\boldsymbol{v}}_1 - \widehat{\mathbf{\Pi}_h^1 \boldsymbol{v}_1}\|_{0,\partial \hat{E}}^2 \leq C \|\hat{\boldsymbol{v}}_1 - \widehat{\mathbf{\Pi}_h^1 \boldsymbol{v}_1}\|_{1,\hat{E}}^2,$$

owing to the trace theorem. Now (2.25) implies that on one edge $\hat{e}$ of $\hat{E}$, we have $\int_{\hat{e}} (\hat{\boldsymbol{v}}_1 - \widehat{\mathbf{\Pi}_h^1 \boldsymbol{v}_1}) = \mathbf{0}$. Therefore the gradient seminorm is equivalent to the full $H^1(\hat{E})$ norm and

$$\begin{aligned} \sum_{e \in \partial E} \frac{1}{|e|} \|\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{0,e}^2 &\leq C |\hat{\boldsymbol{v}}_1 - \widehat{\mathbf{\Pi}_h^1 \boldsymbol{v}_1}|_{1,\hat{E}}^2 \\ &\leq C |\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1|_{1,E}^2. \end{aligned} \tag{2.29}$$

Similarly, denoting $\mathcal{E}_h^1(\Gamma_{12})$ the set of elements in $\mathcal{E}_h^1$ that share an edge with the interface Γ_{12}, we have

$$\sum_{e\in\Gamma_{12}} \|(\boldsymbol{v}_1 - \boldsymbol{\Pi}_h^1\boldsymbol{v}_1)\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 \le \sum_{E\in\mathcal{E}_h^1(\Gamma_{12})}\sum_{e\in\partial E} \|(\boldsymbol{v}_1 - \boldsymbol{\Pi}_h^1\boldsymbol{v}_1)\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2$$
$$\le C \sum_{E\in\mathcal{E}_h^1(\Gamma_{12})} h|\boldsymbol{v}_1 - \boldsymbol{\Pi}_h^1\boldsymbol{v}_1|_{1,E}^2,$$

which, combined with (2.28) and (2.29), implies

$$\|\boldsymbol{v}_1 - \boldsymbol{\Pi}_h^1\boldsymbol{v}_1\|_{X^1} \le C\|\nabla(\boldsymbol{v}_1 - \boldsymbol{\Pi}_h^1\boldsymbol{v}_1)\|_{0,\Omega_1} \le Ch_1^{s-1}|\boldsymbol{v}_1|_{s,\Omega_1}, \tag{2.30}$$

using (2.24).

The bound (2.27) follows easily from the triangle inequality and (2.26) with $s = 1$, using that $\|\boldsymbol{v}_1\|_{X^1} \le C\|\boldsymbol{v}_1\|_{1,\Omega_1}$ for $\boldsymbol{v}_1 \in (H^1(\Omega_1))^d$. □

We also recall the mixed finite element interpolant $\boldsymbol{\Pi}_h^2 : \boldsymbol{X}^2 \cap (H^\theta(\Omega_2))^d \to \boldsymbol{X}_h^2$ for any $\theta > 0$, satisfying [8], for any $\boldsymbol{v}_2 \in \boldsymbol{X}^2 \cap (H^\theta(\Omega_2))^d$,

$$b_2(\boldsymbol{\Pi}_h^2\boldsymbol{v}_2 - \boldsymbol{v}_2, q_2) = 0,\ \forall\, q_2 \in M_h^2, \tag{2.31}$$

$$\int_e ((\boldsymbol{\Pi}_h^2\boldsymbol{v}_2 - \boldsymbol{v}_2)\cdot\boldsymbol{n}_e)\boldsymbol{w}_2\cdot\boldsymbol{n}_e = 0,\ \forall\, e \in \Gamma_h^2,\ \forall\, \boldsymbol{w}_2 \in \boldsymbol{X}_h^2. \tag{2.32}$$

Moreover, $\boldsymbol{\Pi}_h^2$ satisfies the approximation properties

$$\|\boldsymbol{v}_2 - \boldsymbol{\Pi}_h^2\boldsymbol{v}_2\|_{0,E} \le Ch_E^s|\boldsymbol{v}_2|_{s,E}, \quad 1 \le s \le k_2 + 1, \tag{2.33}$$

$$\|\nabla\cdot(\boldsymbol{v}_2 - \boldsymbol{\Pi}_h^2\boldsymbol{v}_2)\|_{0,E} \le Ch_E^s|\nabla\cdot\boldsymbol{v}_2|_{s,E}, \quad 0 \le s \le l_2 + 1. \tag{2.34}$$

It has been shown by Mathew in [24] for the Raviart-Thomas elements [25] that

$$\|\boldsymbol{\Pi}_h^2\boldsymbol{v}_2\|_{H(\mathrm{div};\Omega_2)} \le C(\|\boldsymbol{v}_2\|_{\theta,\Omega_2} + \|\nabla\cdot\boldsymbol{v}_2\|_{0,\Omega_2}), \tag{2.35}$$

a result that can be trivially extended to the other families of mixed finite element spaces. Recall the basic trace inequalities and Korn's inequality on any mesh element E with diameter h_E

$$\forall\,\phi \in H^1(E), \quad \forall\, e \subset \partial E, \quad \|\phi\|_{0,e}^2 \le C(h_E^{-1}\|\phi\|_{0,E}^2 + h_E|\phi|_{1,E}^2), \tag{2.36}$$

$$\forall\,\phi \in H^2(E), \quad \forall\, e \subset \partial E, \quad \|\nabla\phi\cdot\boldsymbol{n}\|_{0,e}^2 \le C(h_E^{-1}\|\phi\|_{1,E}^2 + h_E|\phi|_{2,E}^2), \tag{2.37}$$

$$\forall\,\phi \in \mathbb{P}_k(E), \quad \forall\, e \subset \partial E, \quad \|\nabla\phi\cdot\boldsymbol{n}\|_{0,e} \le Ch_E^{-1/2}|\phi|_{1,E}, \tag{2.38}$$

$$\forall\,\boldsymbol{\phi} \in (\mathbb{P}_k(E))^d, \quad |\boldsymbol{\phi}|_{1,E} \le \|\boldsymbol{D}(\boldsymbol{\phi})\|_{0,E} \le C|\boldsymbol{\phi}|_{1,E}. \tag{2.39}$$

Define the finite-dimensional space of functions on the interface $\Lambda_h = X_h^2\cdot\boldsymbol{n}_{12}$ and let

$$\boldsymbol{V}_h = \{\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \boldsymbol{X}_h : \sum_{e\in\Gamma_{12}}\int_e \eta(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot\boldsymbol{n}_{12} = 0, \quad \forall\,\eta \in \Lambda_h\}.$$

Defining $a = a_1 + a_2$ and $b = b_1 + b_2$, the numerical scheme is: find $(\boldsymbol{U}, P) \in \boldsymbol{V}_h \times M_h$ such that

$$a(\boldsymbol{U}, \boldsymbol{v}) + b(\boldsymbol{v}, P) = \int_{\Omega_1} \boldsymbol{f}_1\cdot\boldsymbol{v}, \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.40}$$

$$b(\boldsymbol{U}, q) = \int_{\Omega_2} f_2\, q, \quad \forall\, q \in M_h. \tag{2.41}$$

In the rest of the section, we show that the solution of the coupled problem satisfies the scheme up to an interface consistency error. We also prove uniqueness and existence of the discrete solution.

LEMMA 2.2. *If $(\boldsymbol{u},p) \in \boldsymbol{X} \times M$ solves the coupled Stokes-Darcy flow problem (2.1)–(2.10), such that $\boldsymbol{u}_i = \boldsymbol{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$, then $(\boldsymbol{u},p)$ satisfies the variational problem:*

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1 - \sum_{e\in\Gamma_{12}} \int_e p_2(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot \boldsymbol{n}_{12}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.42}$$

$$b(\boldsymbol{u},q) = \int_{\Omega_2} f_2 q, \quad \forall q \in M_h. \tag{2.43}$$

Proof. Multiplying the Stokes equation (2.1) by $\boldsymbol{v}_1 \in \boldsymbol{X}_h^1$ and integrating by parts over one element E,

$$\int_E T(\boldsymbol{u}_1,p_1) : \nabla \boldsymbol{v}_1 - \int_{\partial E} T(\boldsymbol{u}_1,p_1)\boldsymbol{n}_E \cdot \boldsymbol{v}_1 = \int_E \boldsymbol{f}_1 \cdot \boldsymbol{v}_1.$$

Summing over all elements E,

$$\begin{aligned}
&\sum_E \int_E (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) : \nabla \boldsymbol{v}_1 - \sum_{e\in\Gamma_h^1} \int_e [(-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))]\boldsymbol{n}_e \cdot \boldsymbol{v}_1 \\
&- \sum_{e\in\Gamma_{12}} \int_e (-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot \boldsymbol{v}_1 - \sum_{e\in\Gamma_1} \int_e (-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}\cdot \boldsymbol{v}_1 \\
&= \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1
\end{aligned}$$

It is easy to show that

$$\boldsymbol{D}(\boldsymbol{u}_1) : \nabla \boldsymbol{v}_1 = \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1), \quad \boldsymbol{I} : \nabla \boldsymbol{v}_1 = \nabla \cdot \boldsymbol{v}_1$$

Thus, the equation becomes

$$\begin{aligned}
&\sum_E \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla\cdot \boldsymbol{v}_1) \\
&- \sum_{e\in\Gamma_h^1} \int_e \{-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] - \sum_{e\in\Gamma_h^1} \int_e [-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1)]\boldsymbol{n}_e \cdot \{\boldsymbol{v}_1\} \\
&- \sum_{e\in\Gamma_{12}} \int_e (-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot \boldsymbol{v}_1 - \sum_{e\in\Gamma_1} \int_e (-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}\cdot \boldsymbol{v}_1 \\
&= \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1
\end{aligned}$$

By regularity of the true solution, we have

$$\sum_E \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla\cdot \boldsymbol{v}_1)$$
$$- \sum_{e\in\Gamma_h^1} \int_e \{-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + \epsilon \sum_{e\in\Gamma_h^1} \int_e \{2\mu \boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_1]$$
$$- \sum_{e\in\Gamma_{12}} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} \cdot \boldsymbol{v}_1$$
$$- \sum_{e\in\Gamma_1} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n} \cdot \boldsymbol{v}_1 + \epsilon \sum_{e\in\Gamma_1} \int_e 2\mu \boldsymbol{D}(\boldsymbol{v}_1)\boldsymbol{n}\cdot \boldsymbol{u}_1$$
$$= \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1$$

Let us now consider the interface term

$$(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} = -p_1\boldsymbol{n}_{12}+(2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}+(2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12})\boldsymbol{n}_{12},$$

which, combined with

$$\boldsymbol{v}_1 = (\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12} + (\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})\boldsymbol{n}_{12},$$

gives

$$(-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot\boldsymbol{v}_1 = -p_1(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})$$
$$+2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12}(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}) + 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12}(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})$$

Thus,

$$-\sum_{e\in\Gamma_{12}}\int_e(-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}\cdot\boldsymbol{v}_1 = -\sum_{e\in\Gamma_{12}}\int_e(-p_1 + 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12})(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})$$
$$-\sum_{e\in\Gamma_{12}}\int_e 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12}(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})$$

With the interface conditions (2.9) and (2.10), we obtain

$$-\sum_{e\in\Gamma_{12}}\int_e(-p_1\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}\cdot\boldsymbol{v}_1 = -\sum_{e\in\Gamma_{12}}\int_e(-p_2)(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})$$
$$+\frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{u}_1\cdot\boldsymbol{\tau}_{12})(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}).$$

Thus

$$\sum_E \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla\cdot \boldsymbol{v}_1)$$
$$- \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{(-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_e\} \cdot [\boldsymbol{v}_1] + \epsilon \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{2\mu \boldsymbol{D}(\boldsymbol{v}_1)\boldsymbol{n}_e\} \cdot [\boldsymbol{u}_1]$$
$$+ \sum_{e\in\Gamma_{12}}\int_e p_2\boldsymbol{v}_1\cdot\boldsymbol{n}_{12} + \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e \boldsymbol{u}_1\cdot\boldsymbol{\tau}_{12}\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12} = \int_{\Omega_1}\boldsymbol{f}_1\cdot\boldsymbol{v}_1$$

which is equivalent to

$$a_1(\boldsymbol{u}_1,\boldsymbol{v}_1)+b_1(\boldsymbol{v}_1,p_1)+\sum_{e\in\Gamma_{12}}\int_e p_2\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}=\int_{\Omega_1}\boldsymbol{f}_1\cdot\boldsymbol{v}_1,\quad \forall \boldsymbol{v}_1\in\boldsymbol{X}_h^1. \tag{2.44}$$

The Darcy's law (2.5) can be rewritten as

$$\boldsymbol{K}^{-1}\boldsymbol{u}_2=-\nabla p_2.$$

As usual, multiplication by $\boldsymbol{v}_2\in\boldsymbol{X}_h^2$ and integration by parts on the Darcy region yields

$$\begin{aligned}\int_{\Omega_2}\boldsymbol{K}^{-1}\boldsymbol{u}_2\cdot\boldsymbol{v}_2&=-\int_{\Omega_2}\nabla p_2\cdot\boldsymbol{v}_2=\int_{\Omega_2}p_2\nabla\cdot\boldsymbol{v}_2-\int_{\partial\Omega_2}p_2\boldsymbol{v}_2\cdot\boldsymbol{n}\\ &=\int_{\Omega_2}p_2\nabla\cdot\boldsymbol{v}_2-\int_{\Gamma_2}p_2\boldsymbol{v}_2\cdot\boldsymbol{n}+\sum_{e\in\Gamma_{12}}\int_e p_2\boldsymbol{v}_2\cdot\boldsymbol{n}_{12}\end{aligned}$$

Or equivalently,

$$a_2(\boldsymbol{u}_2,\boldsymbol{v}_2)+b_2(\boldsymbol{v}_2,p_2)-\sum_{e\in\Gamma_{12}}\int_e p_2\boldsymbol{v}_2\cdot\boldsymbol{n}_{12}=0,\quad\forall\boldsymbol{v}_2\in\boldsymbol{X}_h^2. \tag{2.45}$$

Adding (2.44) and (2.45) yields the equation

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)=\int_{\Omega_1}\boldsymbol{f}_1\cdot\boldsymbol{v}_1-\sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{n}_{12}\quad\forall\boldsymbol{v}\in\boldsymbol{V}_h.$$

Clearly, the equation (2.2) and the regularity of the solution gives

$$b_1(\boldsymbol{u}_1,q)=0,\quad\forall q\in M_h^1.$$

Finally, a simple integration in (2.4) yields

$$b_2(\boldsymbol{u}_2,q)=\int_{\Omega_2}f_2q,\quad\forall q\in M_h^2,$$

and adding to the previous equation gives the result. □

Next, we prove a coercivity lemma that holds true under the following condition.
Hypothesis A: In the definition of the bilinear form $a_1(\cdot,\cdot)$, let us assume that either the condition (a) or (b) holds true.
(a) $\epsilon=1$ and $\sigma_e>0$ for all edges in $\Gamma_h^1\cup\Gamma_1$. For instance, one may choose $\sigma_e=1$.
(b) $\epsilon=-1$ and $\sigma_e\geq\sigma_0>0$ for σ_0 large enough.

LEMMA 2.3. *Assuming Hypothesis A, there exists a positive constant C_0 such that*

$$C_0\|\boldsymbol{v}\|_X^2\leq a(\boldsymbol{v},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\boldsymbol{X}_h:\nabla\cdot\boldsymbol{v}=0\ a.e.\ in\ \Omega_2.$$

Proof. Let $\boldsymbol{v}\in\boldsymbol{X}_h$. Then $\boldsymbol{v}=(\boldsymbol{v}_1,\boldsymbol{v}_2)$ with $\boldsymbol{v}_i\in\boldsymbol{X}_h^i, i=1,2$. Using (2.13) and (2.15),

$$a(\boldsymbol{v},\boldsymbol{v})=2\mu\sum_{E\in\mathcal{E}_h^1}\int_E\boldsymbol{D}(\boldsymbol{v}_1):\boldsymbol{D}(\boldsymbol{v}_1)+\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\int_e[\boldsymbol{v}_1]^2$$

$$-2(1-\epsilon)\mu\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e\cdot[\boldsymbol{v}_1]+\frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})^2+\int_{\Omega_2}\boldsymbol{K}^{-1}\boldsymbol{v}_2\cdot\boldsymbol{v}_2$$

Using Korn's inequality (2.39) and the bound on $\boldsymbol{K}$ (2.7) gives

$$a(\boldsymbol{v},\boldsymbol{v}) \geq 2\mu |\!|\!| \nabla \boldsymbol{v} |\!|\!|^2_{0,\Omega_1} + \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_e}{|e|}\int_e [\boldsymbol{v}_1]^2$$
$$-2(1-\epsilon)\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e\cdot[\boldsymbol{v}_1] + \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e (\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})^2 + \frac{1}{k_1}\|\boldsymbol{v}_2\|^2_{0,\Omega_2}$$

If $\epsilon = 1$, then the result is straightforward. If $\epsilon = -1$, we have from trace inequality (2.38)

$$2(1-\epsilon)\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e\cdot[\boldsymbol{v}_1] \leq 4\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} h_1^{-1/2}\|\nabla\boldsymbol{v}_1\|_{0,E_e}(\frac{|e|}{|e|})^{1/2}\|[\boldsymbol{v}_1]\|_{0,e}$$
$$\leq \frac{1}{2}\mu |\!|\!| \nabla\boldsymbol{v}_1 |\!|\!|^2_{0,\Omega_1} + \tilde{C}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{1}{|e|}\int_e[\boldsymbol{v}_1]^2$$

Thus, we obtain if $\epsilon = -1$

$$a(\boldsymbol{v}_1,\boldsymbol{v}_1) \geq \frac{3}{4}\mu |\!|\!| \nabla\boldsymbol{v}_1 |\!|\!|^2_{0,\Omega_1} + \sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e-\tilde{C}}{|e|}\int_e[\boldsymbol{v}_1]^2$$
$$+\frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})^2 + \frac{1}{k_1}\|\boldsymbol{v}_2\|^2_{0,\Omega_2} \geq C_0(\|\boldsymbol{v}_1\|^2_{X^1} + \|\boldsymbol{v}_2\|^2_{0,\Omega_2})$$

with C_0 positive constant, if σ_e is large enough ($\sigma_e - \tilde{C} \geq C > 0$). □

We are now ready to prove that the discrete scheme (2.40)–(2.41)

LEMMA 2.4. *If Hypothesis A holds, then there exists a unique solution to the problem (2.40)–(2.41).*

Proof. Since the problem (2.40)–(2.41) is finite-dimensional, it suffices to show that the solution is unique. Set $f_i = 0$ and choose $\boldsymbol{v} = \boldsymbol{U}$ and $q = P$. Then

$$a(\boldsymbol{U},\boldsymbol{U}) = 0$$

In addition,

$$b(\boldsymbol{U},q) = 0, \quad \forall q\in M_h,$$

which, implies that $\nabla\cdot\boldsymbol{U} = 0$ in Ω_2, since $\nabla\cdot\boldsymbol{X}_h^2 = M_h^2$. Therefore Lemma 2.3 directly implies that $\boldsymbol{U} = 0$. Thus, the pressure satisfies

$$b(\boldsymbol{v},P) = 0, \quad \forall\boldsymbol{v}\in\boldsymbol{V}_h.$$

The inf-sup condition (3.1) proved below implies that $P = 0$. □

3. A Discrete Inf-sup Condition.

In this section, a discrete inf-sup condition is proved.

THEOREM 3.1. *There exists a positive constant β such that*

$$\inf_{q_h\in M_h}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_X\,\|q_h\|_M} \geq \beta. \tag{3.1}$$

Proof. Let $q_h \in M_h$ be given. Then there exists [16, 17] $\boldsymbol{v} \in (H^1(\Omega))^d$ such that

$$\nabla \cdot \boldsymbol{v} = -q_h, \quad \text{in } \Omega, \ \boldsymbol{v} = 0 \ \text{ on } \partial\Omega,$$

satisfying

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C\|q_h\|_{0,\Omega}.$$

Note that

$$b(v, q_h) = -\int_\Omega (\nabla \cdot \boldsymbol{v}) q_h = \|q_h\|_M^2,$$

which, together with the above á priori bound, implies

$$b(\boldsymbol{v}, q_h) \geq \frac{1}{C} \ \|\boldsymbol{v}\|_{1,\Omega} \ \|q_h\|_M.$$

Next, we need to construct an operator $\pi_h : \boldsymbol{X}^1 \times (\boldsymbol{X}^2 \cap (H^1(\Omega_2))^d) \rightarrow \boldsymbol{V}_h$ satisfying

$$b(\pi_h \boldsymbol{v} - \boldsymbol{v}, q_h) = 0, \ \forall \ q_h \in M_h, \ \text{and } \|\pi_h \boldsymbol{v}\|_X \leq C\|\boldsymbol{v}\|_{1,\Omega}. \tag{3.2}$$

Let $\pi_h \boldsymbol{v} = (\pi_h^1 \boldsymbol{v}, \pi_h^2 \boldsymbol{v}) \in \boldsymbol{X}_h^1 \times \boldsymbol{X}_h^2$. We take $\pi_h^1 \boldsymbol{v} = \boldsymbol{\Pi}_h^1 \boldsymbol{v}_1$ where $\boldsymbol{\Pi}_h^1 : \boldsymbol{X}^1 \rightarrow \boldsymbol{X}_h^1$ is the quasi-local interpolant defined in (2.21). Clearly, due to (2.27),

$$\|\pi_h^1 \boldsymbol{v}\|_{X^1} \leq C\|\boldsymbol{v}\|_{1,\Omega_1}. \tag{3.3}$$

To define $\pi_h^2 \boldsymbol{v}$, consider the auxiliary problem

$$\nabla \cdot \nabla \varphi = 0, \ \text{in } \Omega_2, \tag{3.4}$$
$$\nabla \varphi \cdot \boldsymbol{n} = 0, \ \text{on } \Gamma_2, \tag{3.5}$$
$$\nabla \varphi \cdot \boldsymbol{n}_{12} = (\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12}, \ \text{on } \Gamma_{12}. \tag{3.6}$$

The problem is well posed, since

$$\int_{\Gamma_{12}} (\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12} = 0,$$

due to (2.25). Let $\boldsymbol{z} = \nabla \varphi$. We note that the piecewise smooth function $\pi_h^1 \boldsymbol{v} \cdot \boldsymbol{n}_{12} \in H^\theta(\Gamma_{12})$ for any $0 < \theta < 1/2$. By elliptic regularity [22],

$$\|\boldsymbol{z}\|_{\theta,\Omega_2} \leq C\|(\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12}\|_{\theta-1/2,\Gamma_{12}}, \quad 0 \leq \theta \leq 1/2. \tag{3.7}$$

Let $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{z}$. Clearly $\nabla \cdot \boldsymbol{w} = \nabla \cdot \boldsymbol{v}$ in Ω_2 and $\boldsymbol{w} \cdot \boldsymbol{n}_{12} = \pi_h^1 \boldsymbol{v} \cdot \boldsymbol{n}_{12}$ on Γ_{12}. We now define $\pi_h^2 \boldsymbol{v} := \boldsymbol{\Pi}_h^2 \boldsymbol{w}$, where $\boldsymbol{\Pi}_h^2 : \boldsymbol{X}^2 \cap (H^\theta(\Omega_2))^d \rightarrow \boldsymbol{X}_h^2$ is the mixed finite element interpolant defined in (2.31). Note that, using (2.31),

$$\begin{aligned} b_2(\pi_h^2 \boldsymbol{v}, q_h) &= b_2(\boldsymbol{\Pi}_h^2 \boldsymbol{w}, q_h) = b_2(\boldsymbol{w}, q_h) \\ &= -\int_{\Omega_2} (\nabla \cdot \boldsymbol{w}) q_h = -\int_{\Omega_2} (\nabla \cdot \boldsymbol{v}) q_h = b_2(\boldsymbol{v}, q_h), \quad \forall q_h \in M_h^2, \end{aligned}$$

thus the so constructed $\pi_h \boldsymbol{v} = (\pi_h^1 \boldsymbol{v}, \pi_h^2 \boldsymbol{v})$ satisfies

$$b(\pi_h \boldsymbol{v} - \boldsymbol{v}, q_h) = 0, \ \forall \ q_h \in M_h.$$

It is easy to see that $\pi_h \boldsymbol{v} \in \boldsymbol{V}_h$. Indeed, for every $e \in \Gamma_h^{12}$ and $\eta \in \Lambda_h$, using (2.32) and the fact that $\Lambda_h = X_h^2 \cdot \boldsymbol{n}_{12}$,

$$\int_e \pi_h^2 \boldsymbol{v} \cdot \boldsymbol{n}_{12}\eta = \int_e \boldsymbol{\Pi}_h^2 \boldsymbol{w} \cdot \boldsymbol{n}_{12}\eta = \int_e \boldsymbol{w} \cdot \boldsymbol{n}_{12}\eta = \int_e \pi_h^1 \boldsymbol{v} \cdot \boldsymbol{n}_{12}\eta,$$

It remains to show the bound in (3.2). Using (2.33), (2.34) and (3.7),

$$\begin{aligned}
\|\pi_h^2 \boldsymbol{v}\|_{X^2} &= \|\boldsymbol{\Pi}_h^2 \boldsymbol{w}\|_{X^2} \\
&\le \|\boldsymbol{\Pi}_h^2 \boldsymbol{v}\|_{X^2} + \|\boldsymbol{\Pi}_h^2 \boldsymbol{z}\|_{X^2} \\
&\le C(\|\boldsymbol{v}\|_{1,\Omega_2} + \|\boldsymbol{z}\|_{\theta,\Omega_2}) \\
&\le C(\|\boldsymbol{v}\|_{1,\Omega_1} + \|(\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}\|_{\Gamma_{12}})
\end{aligned}$$

The last term can be bounded as follows. For every $e \in \Gamma_{12}$, and edge (face) of $E \in \mathcal{E}_h^1$, using (2.36) and (2.24),

$$\|(\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12}\|_e \le C(h_E^{-1/2}\|\pi_h^1 \boldsymbol{v} - \boldsymbol{v}\|_{0,E} + h_E^{1/2}|\pi_h^1 \boldsymbol{v} - \boldsymbol{v}|_{1,E}) \le C h_E^{1/2}|\boldsymbol{v}|_{1,\delta(E)}. \quad (3.8)$$

Therefore

$$\|\pi_h^2 \boldsymbol{v}\|_{X^2} \le C\|\boldsymbol{v}\|_{1,\Omega},$$

which, combined with (3.3), implies the bound in (3.2). Now using (3.2),

$$\frac{1}{C}\|q_h\|_M \le \frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{1,\Omega}} = \frac{b(\pi_h \boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{1,\Omega}} \le \frac{b(\pi_h \boldsymbol{v}, q_h)}{\frac{1}{C}\|\pi_h \boldsymbol{v}\|_X}, \text{ for all } q_h \in M_h,$$

which proves (3.1). □

4. A Priori Error Estimates. In this section, optimal error estimates in the energy norm are obtained for the velocity field. Also, optimal error estimates in the L^2 norm of the error for the pressure are obtained. We start with an approximation result for the weakly normal-continuous velocity space $\boldsymbol{V}_h$.

LEMMA 4.1. *For $\boldsymbol{v} \in (H^1(\Omega))^d$ such that $\boldsymbol{v}|_{\Omega_1} \in (H^{k_1+1}(\Omega_1))^d$ and $\boldsymbol{v}|_{\Omega_2} \in (H^{k_2+1}(\Omega_2))^d$, and $\nabla \cdot \boldsymbol{v}|_{\Omega_2} \in (H^{l_2+1}(\Omega_2))^d$, there exists $\tilde{\boldsymbol{v}} \in \boldsymbol{V}_h$ such that*

$$b(\boldsymbol{v} - \tilde{\boldsymbol{v}}, q) = 0, \quad \forall q \in M_h, \quad (4.1)$$

$$\forall e \in \Gamma_h^1 \cup \Gamma_1, \quad \int_e [\tilde{v}] \cdot \boldsymbol{q} = 0, \quad \forall \boldsymbol{q} \in (I\!P_{k_1-1}(e))^d, \quad (4.2)$$

$$\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_X \le C\left\{h_1^{k_1}|\boldsymbol{v}|_{k_1+1,\Omega_1} + h_2^{k_2+1}|\boldsymbol{v}|_{k_2+1,\Omega_1} + h_2^{l_2+1}|\nabla \cdot \boldsymbol{v}|_{l_2+1,\Omega_1}\right\}. \quad (4.3)$$

Proof. We will show that the interpolant $\pi_h \boldsymbol{v}$ constructed in Theorem 3.1 satisfies the above conditions. Indeed, (4.1) and (4.2) follow directly from the construction of $\pi_h \boldsymbol{v}$. To show (4.3), we first note that (2.26) implies that

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{X^1} \le C h_1^{k_1}|\boldsymbol{v}|_{k_1+1,\Omega_1}. \quad (4.4)$$

Next,

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{X^2} = \|\boldsymbol{v} - \boldsymbol{\Pi}_h^2 \boldsymbol{w}\|_{X^2} \le \|\boldsymbol{v} - \boldsymbol{\Pi}_h^2 \boldsymbol{v}\|_{X^2} + \|\boldsymbol{\Pi}_h^2(\boldsymbol{w} - \boldsymbol{v})\|_{X^2} \quad (4.5)$$

For the first term on the right in (4.5), using (2.33) and (2.34),

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_h^2 \boldsymbol{v}\|_{X^2} \leq C h_2^{k_2+1} |\boldsymbol{v}|_{k_2+1,\Omega_1} + h_2^{l_2+1} |\nabla \cdot \boldsymbol{v}|_{l_2+1,\Omega_1}. \tag{4.6}$$

The last term in (4.5) can be bounded as follows, using (2.35), (3.7), (3.8), and (2.24),

$$\begin{aligned}\|\boldsymbol{\Pi}_h^2(\boldsymbol{w}-\boldsymbol{v})\|_{X^2} &= \|\boldsymbol{\Pi}_h^2 \boldsymbol{z}\|_{X^2} \leq \|\boldsymbol{z}\|_{\theta,\Omega_2} \\ &\leq C\|(\pi_h^1 \boldsymbol{v} - \boldsymbol{v})\cdot \boldsymbol{n}_{12}\|_{0,\Gamma_{12}} \leq C h_1^{k_1+1/2} |\boldsymbol{v}|_{k_1+1,\Omega_1}. \end{aligned} \tag{4.7}$$

A combination of (4.4)–(4.7) completes the proof. □

THEOREM 4.2. *Let $(\boldsymbol{u},p) \in \boldsymbol{X} \times M$ be the solution of the coupled problem (2.1)-(2.10). Assume that $\boldsymbol{u}|_{\Omega_i} \in H^{k_i+1}(\Omega_i)$ for $i=1,2$. Assume that $p|_{\Omega_1} \in H^{k_1}(\Omega_1)$ and that $p|_{\Omega_2} \in H^{l_2+1}(\Omega_2)$. Assume that hypothesis A holds. Let $(\boldsymbol{U},P)$ be the discrete solution of (2.40)-(2.41) Then, the following estimate holds:*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{U}\|_X \leq{}& C h_1^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) + C h_2^{k_2+1}|\boldsymbol{u}|_{k_2+1,\Omega_2} \\ &+ C(h_2^{l_2+1} + h_2^{l_2+1/2} h_1^{1/2})|p|_{l_2+1,\Omega_2}.\end{aligned}$$

Proof. Let $\tilde{\boldsymbol{u}}$ be the interpolant of $\boldsymbol{u}$ defined in Lemma 4.1 and let $\tilde{p}$ be the interpolants of p, satisfying (2.17)-(2.20). From (2.42), (2.43) and (2.40), (2.41), the error equation is

$$a(\boldsymbol{U}-\tilde{\boldsymbol{u}},\boldsymbol{v}) + b(\boldsymbol{v},P-\tilde{p}) = a(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{v}) + b(\boldsymbol{v},p-\tilde{p}) \tag{4.8}$$

$$-\sum_{e\in\Gamma_{12}} \int_e p_2(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot \boldsymbol{n}_{12}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{4.9}$$

$$b(\boldsymbol{U}-\tilde{\boldsymbol{u}},q) = b(\boldsymbol{u}-\tilde{\boldsymbol{u}},q), \quad \forall q \in M_h. \tag{4.10}$$

Note that (4.1) implies that $b(\boldsymbol{U}-\tilde{\boldsymbol{u}},q)=0$, $\forall q \in M_h$, which implies that

$$\nabla\cdot(\boldsymbol{U}-\tilde{\boldsymbol{u}}) = 0 \text{ in } \Omega_2,$$

since $\nabla \cdot \boldsymbol{X}_h^2 = M_h^2$. Define $\boldsymbol{\chi} = \boldsymbol{U} - \tilde{\boldsymbol{u}}$ and $\xi = P - \tilde{p}$. Choose $\boldsymbol{v} = \boldsymbol{\chi}$ and $q = \xi$. Then,

$$a(\boldsymbol{\chi},\boldsymbol{\chi}) + b(\boldsymbol{\chi},\xi) = a(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi}) + b(\boldsymbol{\chi},p-\tilde{p}) - \sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12},$$

$$b(\boldsymbol{\chi},\xi) = 0.$$

Equivalently,

$$a(\boldsymbol{\chi},\boldsymbol{\chi}) = a(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi}) + b(\boldsymbol{\chi},p-\tilde{p}) - \sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12}. \tag{4.11}$$

The first term on the right can be estimated as follows:

$$a_1(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi}) = 2\mu \sum_{E\in\mathcal{E}_h^1}\int_E \boldsymbol{D}(\boldsymbol{u}-\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{\chi}) \tag{4.12}$$

$$-2\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\boldsymbol{D}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}] + 2\mu\epsilon \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\boldsymbol{D}(\boldsymbol{\chi})\}\boldsymbol{n}_e\cdot[\boldsymbol{u}-\tilde{\boldsymbol{u}}] \tag{4.13}$$

$$+\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\int_e [\boldsymbol{u}-\tilde{\boldsymbol{u}}]\cdot[\boldsymbol{\chi}] + \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e (\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\boldsymbol{\tau}_{12}\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12} \tag{4.14}$$

$$= T_1 + \cdots + T_5 \tag{4.15}$$

Using Cauchy-Schwarz inequality, and the approximation result (4.3), we have

$$T_1 \le 2\mu \sum_{E\in\mathcal{E}_h^1} \|\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}})\|_{0,E}\|\nabla\boldsymbol{\chi}\|_{0,E} \le \frac{1}{8}|\!|\!|\nabla\boldsymbol{\chi}|\!|\!|_{0,\Omega_1}^2 + C|\!|\!|\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}})|\!|\!|_{0,\Omega_1}^2$$
$$\le \frac{1}{8}|\!|\!|\nabla\boldsymbol{\chi}|\!|\!|_{0,\Omega_1}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

Let $L_h(\boldsymbol{u})$ denote the standard Lagrange interpolant of degree k_1 defined in Ω_1 and let us insert it in the second integral term. Note that $L_h(\boldsymbol{u})$ satisfies the optimal error estimates

$$|L_h(\boldsymbol{u})-\boldsymbol{u}|_{m,E} \le Ch_E^{s-m}|\boldsymbol{u}|_{s,E} \quad \forall\, 2\le s\le k_1+1, \quad m=0,1,2. \tag{4.16}$$

For e a segment of $\Gamma_h^1\cup\Gamma_1$, we have

$$\int_e \{\boldsymbol{D}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}] = \int_e \{\boldsymbol{D}(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}] + \int_e \{\boldsymbol{D}(L_h(\boldsymbol{u})-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}].$$

Expanding the first integral, we obtain from the trace inequality (2.37) and from the fact that the Lagrange interpolant satisfies (4.16)

$$\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\boldsymbol{D}(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}]$$
$$\le \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_e^{1/2}}{|e|^{1/2}}\|[\boldsymbol{\chi}]\|_{0,e}\frac{|e|^{1/2}}{\sigma_e^{1/2}}\|\{\boldsymbol{D}(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\|_{0.e}$$
$$\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + C\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{|e|}{\sigma_e}(h_e^{-1}|\boldsymbol{u}-L_h(\boldsymbol{u})|_{1,E_e^{12}}^2 + h_e|\boldsymbol{u}-L_h(\boldsymbol{u})|_{2,E_e^{12}}^2)$$
$$\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

Similarly, using the trace inequality (2.38), triangle inequality and (4.3)

$$\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\boldsymbol{D}(L_h(\boldsymbol{u})-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}] \le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2$$
$$+C\sum_{e\in\Gamma_h^1\cup\Gamma_1}|\tilde{\boldsymbol{u}}-L_h(\boldsymbol{u})|_{1,E_e^{12}}^2$$
$$\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

Therefore,

$$T_2 \le \frac{1}{4}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

The third term vanishes because of the continuity of $\boldsymbol{u}$ and property (4.2) of $\tilde{\boldsymbol{u}}$:

$$T_3 = 0. \tag{4.17}$$

Using Cauchy-Schwarz inequality, the jump term is bounded by virtue of (2.24) and (2.36):

$$\begin{aligned}
T_4 &\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + C\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\|_{0,e}^2\\
&\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k+1,\Omega_1}^2.
\end{aligned}$$

The last term is bounded as follows, from the trace inequality (2.36)

$$\begin{aligned}
T_5 &\le \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,e}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}\\
&\le \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + C\sum_{e\in\Gamma_1}(h_e^{-1}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,E}^2 + h_e|\boldsymbol{u}-\tilde{\boldsymbol{u}}|_{1,E}^2)\\
&\le \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2
\end{aligned}$$

Let us now estimate $a_2(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi})$, using the result (4.3)

$$a_2(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi}) = \int_{\Omega_2}\boldsymbol{K}^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\boldsymbol{\chi} \le \frac{1}{8}\|\boldsymbol{K}^{-1/2}\boldsymbol{\chi}\|_{0,\Omega_2}^2 + h_2^{2k_2+2}|\boldsymbol{u}|_{k_2+1,\Omega_2}^2$$

Let us now estimate $b_1(\boldsymbol{\chi},p-\tilde{p})$. By property (2.17) and (2.19) and the trace estimate (2.36)

$$b_1(\boldsymbol{\chi},p-\tilde{p}) = -\sum_{E\in\mathcal{E}_h}\int_E(p-\tilde{p})\nabla\cdot\boldsymbol{\chi} + \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{p-\tilde{p}\}[\boldsymbol{\chi}]\cdot\boldsymbol{n}_e \tag{4.18}$$

$$= \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{p-\tilde{p}\}[\boldsymbol{\chi}]\cdot\boldsymbol{n}_e \tag{4.19}$$

$$\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\int_e[\boldsymbol{\chi}]^2 + Ch_1^{2k_1}|p|_{k_1,\Omega_1}^2 \tag{4.20}$$

Now estimate $b_2(\boldsymbol{\chi},p-\tilde{p})$ using Cauchy-Schwarz's inequality and approximation result (2.20)

$$b_2(\boldsymbol{\chi},p-\tilde{p}) = -\int_{\Omega_2}(p-\tilde{p})\nabla\cdot\boldsymbol{\chi} \le \frac{1}{8}\|\nabla\boldsymbol{\chi}\|_{0,\Omega_2}^2 + Ch_2^{2l_2+2}|p|_{l_2+1,\Omega_2}^2$$

It remains to bound the last term in (4.11). Since $\boldsymbol{\chi}$ belongs to $\boldsymbol{V}_h$, we have

$$\sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12} = \sum_{e\in\Gamma_{12}}\int_e(p_2-\tilde{p}_2^e)(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12},$$

where $\tilde{p}_2^e\in\Lambda_h$ is the L^2 projection of p_2 with respect to the L^2 inner product on the edge e. Therefore, by definition of the projection and since $\Lambda_h = \boldsymbol{X}_h^2\cdot\boldsymbol{n}_{12}$, we have

$$\sum_{e\in\Gamma_{12}}\int_e(p_2-\tilde{p}_2^e)\boldsymbol{\chi}_2\cdot\boldsymbol{n}_{12} = 0.$$

We also note that for any edge e and any constant vector $\boldsymbol{c}_e$, we have

$$\sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12} = \sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)(\boldsymbol{\chi}_1-\boldsymbol{c}_e)\cdot\boldsymbol{n}_{12}$$
$$\leq \sum_{e\in\Gamma_{12}} \|p_2-\tilde{p}_2^e\|_{0,e}\|\boldsymbol{\chi}_1-\boldsymbol{c}_e\|_{0,e}$$

Assume that each edge e of Γ_{12} is shared by the elements $E_e^1\in\mathcal{E}_h^1$ and $E_e^2\in\mathcal{E}_h^2$. Then, from the approximation properties and the trace inequality (2.36), we obtain

$$\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12} \leq h_2^{l_2+1/2}\|p_2\|_{l_2+1,E_e^2}(h_1^{-1/2}\|\boldsymbol{\chi}_1-\boldsymbol{c}_e\|_{0,E_e^1}+h_1^{1/2}\|\nabla\boldsymbol{\chi}_1\|_{0,E_e^1}),$$

thus

$$\sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12} \leq \sum_{e\in\Gamma_{12}} h_2^{l_2+1/2}|p_2|_{l_2+1,E_e^2}(h_1^{1/2}\|\nabla\boldsymbol{\chi}_1\|_{0,E_e^1}+h_1^{1/2}\|\nabla\boldsymbol{\chi}_1\|_{0,E_e^1})$$
$$\leq \frac{1}{8}\|\!|\nabla\boldsymbol{\chi}\|\!|_{0,\Omega_1}^2 + Ch_2^{2l_2+1}h_1|p_2|_{l_2+1,\Omega_2}^2.$$

Combining all bounds above yield

$$a(\boldsymbol{\chi},\boldsymbol{\chi}) \leq \frac{1}{4}\|\!|\nabla\boldsymbol{\chi}\|\!|_{0,\Omega_1}^2 + \frac{3}{4}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2$$
$$+\frac{1}{4}\|\boldsymbol{K}^{-1/2}\boldsymbol{\chi}\|_{0,\Omega_2}^2 + Ch_2^{2k_2+2}|\boldsymbol{u}|_{k_2+1,\Omega_2} + C(h_2^{2l_2+2}+h_2^{2l_2+1}h_1)|p|_{l_2+1,\Omega_2}^2$$
$$+Ch_2^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2 + Ch_1^{2k_1}|p|_{k_1,\Omega_1}^2.$$

Equivalently,

$$a(\boldsymbol{\chi},\boldsymbol{\chi}) \leq Ch_2^{2k_2+2}|\boldsymbol{u}|_{k_2+1,\Omega_2}^2 + C(h_2^{2l_2+2}+h_2^{2l_2+1}h_1)|p|_{l_2+1,\Omega_2}^2$$
$$+Ch_1^{2k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1}^2+|p|_{k_1,\Omega_1}^2)$$

Now, since $\nabla\cdot\boldsymbol{\chi}=0$ in Ω_2, the coercivity Lemma 2.3 implies

$$\|\boldsymbol{u}-\boldsymbol{U}\|_X \leq \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X + \|\boldsymbol{U}-\tilde{\boldsymbol{u}}\|_X$$
$$\leq \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X + \frac{1}{\sqrt{C_0}}a(\boldsymbol{\chi},\boldsymbol{\chi})^{1/2}$$

which concludes the proof, using (4.3) □

THEOREM 4.3. *Under the assumptions and notation of Theorem 4.2, we have*

$$\|p-P\|_{0,\Omega} \leq Ch_1^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1}+|p|_{k_1,\Omega_1}) + Ch_2^{k_2+1}|\boldsymbol{u}|_{k_2+1,\Omega_2}$$
$$+C(h_2^{l_2+1}+h_2^{l_2+1/2}h_1^{1/2})|p|_{l_2+1,\Omega_2}.$$

where C is a constant independent of h_1,h_2.

Proof. The error equation (4.9) can be written as

$$\forall\boldsymbol{v}\in\boldsymbol{V}_h,\quad a(\boldsymbol{U}-\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},P-\tilde{p}) = b(\boldsymbol{v},p-\tilde{p}) - \sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{n}_{12}. \quad (4.21)$$

From the discrete inf-sup condition (3.1),

$$\|P-\tilde{p}\|_{0,\Omega} \leq \frac{1}{\beta} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, P-\tilde{p})}{\|\boldsymbol{v}_h\|_X}. \tag{4.22}$$

Using (4.21), for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$b(\boldsymbol{v}_h, P-\tilde{p}) = -a(\boldsymbol{U}-\boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p-\tilde{p}) - \sum_{e\in\Gamma_{12}} \int_e p_2(\boldsymbol{v}_{h1} - \boldsymbol{v}_{h2}) \cdot \boldsymbol{n}_{12}.$$

For the first term on the right,

$$\begin{aligned}
a(\boldsymbol{U}-\boldsymbol{u}, \boldsymbol{v}_h) &= 2\mu \sum_{E\in\mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{U}-\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}_h) + \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_e}{|e|} \int_e [\boldsymbol{U}-\boldsymbol{u}] \cdot [\boldsymbol{v}_h] \\
-2\mu &\sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{U}-\boldsymbol{u})\boldsymbol{n}_e\} \cdot [\boldsymbol{v}_h] + 2\mu\epsilon \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\} \cdot [\boldsymbol{U}-\boldsymbol{u}] \\
&+ \frac{\mu}{G} \sum_{e\in\Gamma_{12}} \int_e (\boldsymbol{U}-\boldsymbol{u}) \cdot \boldsymbol{\tau}_{12} \boldsymbol{v}_h \cdot \boldsymbol{\tau}_{12} + \int_{\Omega_2} \boldsymbol{K}^{-1}(\boldsymbol{U}-\boldsymbol{u}) \cdot \boldsymbol{v}_h \\
&= Q_1 + \cdots + Q_6
\end{aligned}$$

We now bound each Q_i term. From Cauchy-Schwarz's inequality, the terms Q_1, Q_2, Q_5 and Q_6 are easily bounded

$$Q_1 + Q_2 + Q_5 + Q_6 \leq C\|\boldsymbol{v}_h\|_X \|\boldsymbol{U}-\boldsymbol{u}\|_X.$$

We now bound Q_3.

$$\begin{aligned}
Q_3 &\leq C \sum_{e\in\Gamma_h^1\cup\Gamma_1} (\frac{|e|}{\sigma_e})^{1/2} \|\nabla(\boldsymbol{U}-\boldsymbol{u})\|_{0,e} (\frac{\sigma_e}{|e|})^{1/2} \|[\boldsymbol{v}_h]\|_{0,e} \\
Q_3 &\leq C\|\boldsymbol{v}_h\|_X (\sum_{e\in\Gamma_h^1\cup\Gamma_1} (h_1\|\nabla(\boldsymbol{U}-\tilde{\boldsymbol{u}})\|_{0,e}^2 + h_1\|\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}})\|_{0,e}^2))^{1/2} \\
Q_3 &\leq C\|\boldsymbol{v}_h\|_X (\|U-\tilde{\boldsymbol{u}}\|_X^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2)^{1/2}
\end{aligned}$$

Now, Q_4 is bounded similarly, from trace inequality (2.38)

$$\begin{aligned}
Q_4 &\leq C \sum_{e\in\Gamma_h^1\cup\Gamma_1} \|\{\boldsymbol{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\}\|_{0,e} \|[\boldsymbol{U}-\boldsymbol{u}]\|_{0,e} \\
Q_4 &\leq C \sum_{e\in\Gamma_h^1\cup\Gamma_1} h^{-1/2}\|\nabla\boldsymbol{v}_h\|_{0,E_e^{12}} (\frac{\sigma_e}{|e|})^{1/2-1/2} \|[\boldsymbol{U}-\boldsymbol{u}]\|_{0,e} \\
Q_4 &\leq C\|\boldsymbol{v}_h\|_X \|\boldsymbol{U}-\boldsymbol{u}\|_X
\end{aligned}$$

Let us now estimate $b(\boldsymbol{v}_h, p-\tilde{p})$. From the property (2.17), it is reduced to

$$\begin{aligned}
b(\boldsymbol{v}_h, p-\tilde{p}) &= \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{p-\tilde{p}\}[\boldsymbol{v}_h] \cdot \boldsymbol{n}_e \\
&\leq \sum_{e\in\Gamma_h^1\cup\Gamma_1} (\frac{\sigma_e}{|e|})^{1/2} \|[\boldsymbol{v}_h]\|_{0,e} (\frac{|e|}{\sigma_e})^{1/2} \|\{p-\tilde{p}\}\|_{0,e} \\
&\leq \|\boldsymbol{v}_h\|_X Ch_1^{k_1}|p|_{k_1,\Omega_1}
\end{aligned}$$

Finally, following the same approach as in the proof of Theorem 4.2, we bound the interface integral

$$\sum_{e\in\Gamma_{12}} \int_e p_2(\boldsymbol{v}_{h1} - \boldsymbol{v}_{h2}) \cdot \boldsymbol{n}_{12} = \sum_{e\in\Gamma_{12}} \int_e (p_2 - \tilde{p}_2^e)\boldsymbol{v}_{h1} \cdot \boldsymbol{n}_{12}$$
$$\leq C\|\boldsymbol{v}_h\|_X h_2^{l_2+1/2} h_1^{1/2} |p_2|_{l_2+1,\Omega_2}$$

Combining all the bounds with (4.22) yields

$$\|P - \tilde{p}\|_{0,\Omega} \leq C(\|\boldsymbol{U} - \boldsymbol{u}\|_X + h_1^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) + h_2^{l_2+1/2} h_1^{1/2} \|p\|_{l_2+1,\Omega_2}).$$

Using Theorem 4.2 concludes the proof. □

REMARK 4.1. The results proven in this section, are valid and unchanged in three-dimensional domains, assuming there exist interpolants $\boldsymbol{\Pi}_h^1$ and $\boldsymbol{\Pi}_h^2$ defined in (2.21) and (2.31). The existence of $\boldsymbol{\Pi}_h^1$ for $k = 1$ in three dimensions is given in [10]. The existence of $\boldsymbol{\Pi}_h^2$ in any dimension is a well-known fact [8].

5. Implementation Issues and Conclusions. In this paper, the convergence of a numerical scheme for solving the coupled Darcy-Stokes problem is proved. In order to parallelize the implementation of the scheme, a Lagrange multiplier $\lambda \in \Lambda_h$ can be introduced. Defining the bilinear form on the interface,

$$\Lambda(\eta, \boldsymbol{v}) = \sum_{e\in\Gamma_{12}} \int_e \eta(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{n}_{12}, \quad \forall \eta \in \Lambda_h, \quad \forall \boldsymbol{v} \in \boldsymbol{X}_h,$$

the scheme can be rewritten as: find $(\boldsymbol{U}, P, \lambda) \in \boldsymbol{X}_h \times M_h \times \Lambda_h$ such that $\boldsymbol{U}_i = \boldsymbol{U}|_{\Omega_i}$ and $P_i = P|_{\Omega_i}$ satisfy

$$a_1(\boldsymbol{U}_1, \boldsymbol{v}_1) + b_1(\boldsymbol{v}_1, P_1) + \Lambda(\lambda, \boldsymbol{v}_1) = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1, \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h^1, \tag{5.1}$$
$$b_1(\boldsymbol{U}_1, q_1) = 0, \quad \forall q_1 \in M_h^1, \tag{5.2}$$
$$a_2(\boldsymbol{U}_2, \boldsymbol{v}_2) + b_2(\boldsymbol{v}_2, P_2) - \Lambda(\lambda, \boldsymbol{v}_2) = 0, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{X}_h^2, \tag{5.3}$$
$$b_2(\boldsymbol{U}_2, q_2) = \int_{\Omega_2} f_2 q_2, \quad \forall q_2 \in M_h^2, \tag{5.4}$$
$$\Lambda(\eta, \boldsymbol{U}_1 - \boldsymbol{U}_2) = 0, \quad \forall \eta \in \Lambda_h. \tag{5.5}$$

It can easily be shown that the two discrete formulations are equivalent. Formulation (5.1)–(5.5) is suitable for a parallel implementation. In particular, using an approach from [19], a non-overlapping domain decomposition algorithm can be formulated that reduces the coupled system to a symmetric and positive definite interface problem for λ. In addition to its parallel eficiency, this approach allows for existing codes solving the Stokes or the Darcy equations to be utilized.

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