On the partial orthogonality of faithful characters

by

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ABSTRACT

For conjugacy classes C and D we obtain an expression for $\sum \chi(C)\bar{\chi}(D)$, where the sum extends only over the faithful irreducible characters of a finite group. This expression is shown to be zero whenever the two conjugacy classes are distinct modulo the socle of the group. When C and D are the same class, and the class preserves its cardinality modulo the socle, we express the above sum in terms of certain invariants of the group.

Key words and phrases: Möbius function, socle, simple module, irreducible representation

Proposed running head: Faithful characters

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[2]Funded in part by a Scaife Family Foundation grant

1. The lattice of normal subgroups

All groups considered in this paper are finite. For a subgroup H of G the set of subgroups of H that are normal in G, with inclusion on subgroups as partial order, forms a lattice $L_G(H)$ with meet $T \cap K$ and join TK. We write $L(G)$ for $L_G(G)$. In accordance with the theory of linear representations over a field of coprime characteristic to the order of G, it helps to visualize this lattice as the intersection of kernels of irreducible representations of G; cf. Alperin and Bell (1995, p 150). Making use of well-known results of Weisner (1935), Hall (1933, 1935) and Rota (1964), and the structure of the minimal normal subgroups viewed as irreducible $G-$modules over a prime-order field, results of Kratzer and Thévenaz (1984, 1985) yield an explicit expression for the Möbius function of $L(G)$. Palfy (1979) obtained an explicit formula for the sum of squares of degrees of the faithful irreducible representations. This paper evaluates the inner product of the faithful characters on classes other than the identity, and it highlights an orthogonality modulo the socle that exists for faithful characters; Palfy's result corresponds to the case when the class in question is the identity. Basics on Möbius functions on partially ordered sets are found in Constantine (1987, Chapter 9).

Let μ be the Möbius function of $L(G)$. Denote by S the subgroup of G generated by the set of all minimal normal subgroups of G. The subgroup S is called the *socle* of G. In the context of this paper Hall's result, which states that $\mu(1, K) = 0$ whenever K is not a join of atoms of $L(G)$, informs us that $\mu(1, K) = 0$ unless K is a subgroup of S.

2. Faithful irreducible representations

Denote by χ_i the character of the irreducible representation ρ_i of G, over the field of complex numbers. Fix conjugacy classes C and D of G. Let $f(C, D, N) = \sum \chi(C)\bar{\chi}(D)$, where the sum is over all irreducible characters of G whose kernels (of the corresponding ρs) are equal to subgroup N exactly. We aim to find an explicit form for f in terms of some invariants of G. The *indicator function* $\delta_{AB}(H)$ is by definition equal to 0 if conjugacy classes A and B of group H are distinct in H, and is equal to 1 if they are the same in

H. To simplify notation we write $\delta_{AB}(G/K)$ for $\delta_{\bar{A}\bar{B}}(G/K)$, where A and B are classes of G and $\bar{A}$ and $\bar{B}$ are their epimorphic images in the factor group G/K.

By the column orthogonality of the character table of G, we have

$$\sum_{N \leq T} f(C, D, T) = (|G|/|CN|)\delta_{CD}(G/N),$$

where the sum is over all irreducible representations of G whose kernels contain N. Interpreting this equality on the lattice $L(G)$, we use Möbius inversion to obtain

$$f(C, D, N) = \sum_{N \leq T} (|G|/|CT|)\delta_{CD}(G/T)\mu(N, T).$$

By working in the group G/N it suffices to evaluate $f(C, D, 1)$. Using Hall's result we may further simplify by restricting the sum only over normal subgroups T of G that are in the socle S of G. These observations yield

$$f(C, D, 1) = \sum_{T \leq S} (|G|/|CT|)\delta_{CD}(G/T)\mu(1, T). \tag{1}$$

Expression (1) simplifies to closed form in certain instances. Indeed, it is easy to see that if $\delta_{CD}(G/S) = 0$, then $\delta_{CD}(G/T) = 0$ for all $T \leq S$. We therefore conclude that when classes C and D are distinct modulo the socle S, then every term in (1) is, in fact, zero.

We call two classes A and B of G *faithfully orthogonal* if $\sum \chi(A)\bar{\chi}(B) = 0$, where the sum is taken only over the faithful characters of G. Faithful orthogonality is a symmetric relation; therefore, if A and B are faithfully orthogonal we sometimes say that A is faithfully orthogonal to B. We now summarize our observations as follows

Theorem 1 *If two conjugacy classes are distinct modulo the socle, then they are faithfully orthogonal.*

An immediate consequence is

Corollary 1 *Let G be a finite group and S be the socle of G. Then:*

(a) Any conjugacy class inside the socle is faithfully orthogonal to any conjugacy class outside the socle.

(b) There are at least m conjugacy classes of group G any two of which are faithfully orthogonal, where m is the number of conjugacy classes of G/S.

We examine next the case in (1) when classes C and D are the same. A few invariants of the socle of G need to be introduced. View an elementary Abelian minimal normal subgroup M of G as a simple $G-$module with conjugation as $G-$module action. Call two such minimal normal subgroups $G-$*isomorphic* if there exists a group isomorphism f between them that preserves the $G-$module action; that is, $f(x^g) = f(x)^g$, for all $g \in G$. Partition the set of elementary Abelian minimal normal subgroups of G into $G-$isomorphism classes. Denote these $G-$classes by $\mathbf{A}_1, \ldots, \mathbf{A}_a$. Let A_i be representatives from the $G-$classes $\mathbf{A}_i$. The group generated by the elements of $\mathbf{A}_i$ is written as S_i. It follows that $S_i \cong A_i^{d_i}$, a direct product of d_i copies of A_i. Denote by $Hom_G(A_i, A_i)$ the field of $G-$homomorphisms from A_i to itself; see Gorenstein (1968, p 79). Let $S_{a+1}, \ldots, S_{a+b}$ be the set of non-Abelian minimal normal subgroups of G. Each S_i, $a+1 \leq i \leq a+b$, is a direct product of isomorphic non-Abelian simple groups. We can therefore express the socle S of G as the following direct product:

$$S = \prod_{i=1}^{a+b} S_i \cong \prod_{i=1}^{a} A_i^{d_i} \prod_{j=a+1}^{a+b} S_j.$$

We now call on properties of the Möbius function, see Kratzer and Thévenaz (1984), to obtain the following generalized version of Lemma 2 of Pálfy (1979):

Theorem 2

(a) The Möbius function on the lattice of normal subgroups of G is $\mu(H,T) = 0$, unless T/H is in the socle S of G/H.

(b) If $S \cong \prod_{i=1}^{a} A_i^{d_i} \prod_{j=a+1}^{a+b} S_j$, and $T/H \cong \prod_{i=1}^{a} A_i^{\alpha_i} \prod_{k=1}^{c} S_{j_k}$, then $\mu(H,T) = (-1)^c \prod_{i=1}^{a} (-1)^{\alpha_i} q_i^{\binom{\alpha_i}{2}}$, where $q_i = |Hom_G(A_i, A_i)|$.

To simplify notation we write $f(C,N)$ for $f(C,C,N)$. When $C = D$ equation (1) becomes

$$f(C,1) = \sum_{T \leq S} (|G|/|CT|)\mu(1,T).$$

Under certain conditions this expression can be written as a product. Assume that the class C is such that $|CS| = |C||S|$, where S is the socle of G. Such classes generally exist; for instance any element of the center of G is such a class. We shall also give an example of a noncentral class in $SL(2,3)$ that satisfies this condition. For any $T \leq S$, T normal in G, we then obviously have $|CT| = |C||T|$, since each coset of S contains at most one element of C, and therefore the same must be true of cosets of T. Using Theorem 2 we thus obtain,

$$f(C,1) = \sum_{T \leq S} (|G|/|C||T|)\mu(1,T)$$

$$= \frac{|G|}{|C|} \sum_{T_1 \leq S_1} \cdots \sum_{T_{a+b} \leq S_{a+b}} (1/\prod T_i|)\mu(1, \prod T_i)$$

$$= \frac{|G|}{|C|} \prod_{i=1}^{a+b} (\sum_{T_i \leq S_i} (1/|T_i|)\mu(1,T_i)).$$

We can turn the above sums over the elements of a lattice into ordinary sums. For $a+1 \leq i \leq a+b$, we have $\sum_{T_i \leq S_i} (1/|T_i|)\mu(1,T_i) = 1 - \frac{1}{|S_i|}$, since in this case S_i is nonabelian minimal normal in G, and therefore T_i is either 1 or S_i. As to the Abelian case, for $1 \leq i \leq a$ we have

$$\sum_{T_i \leq S_i} |T_i|^{-1}\mu(1,T_i) = \sum_{k=0}^{d_i} |A_i|^{-k} \frac{(q_i^{d_i}-1)\cdots(q_i^{d_i}-q_i^{k-1})}{(q_i^k-1)\cdots(q_i^k-q_i^{k-1})} (-1)^k q_i^{\binom{k}{2}}$$

$$= \sum_{k=0}^{d_i} |A_i|^{-k}(-1)^k \frac{(q_i^{d_i}-1)\cdots(q_i^{d_i}-q_i^{k-1})}{(q_i^k-1)\cdots(q_i-1)}.$$

Using the $q-$identity

$$\frac{(q^m-1)\cdots(q^m-q^{k-1})}{(q^k-1)\cdots(q-1)} + q^m \frac{(q^m-1)\cdots(q^m-q^{k-2})}{(q^{k-1}-1)\cdots(q-1)}$$

$$= \frac{(q^m-1)\cdots(q^m-q^{k-2})}{(q^{k-1}-1)\cdots(q-1)} (\frac{q^m-q^{k-1}}{q^k-1} + \frac{q^{m+k}-q^m}{q^k-1})$$

$$= \frac{(q^{m+1}-1)\cdots(q^{m+1}-q^{k-1})}{(q^k-1)\cdots(q-1)},$$

we obtain

$$(\sum_{k=0}^{m} |A_i|^{-k}(-1)^k \frac{(q_i^m-1)\cdots(q_i^m-q_i^{k-1})}{(q_i^k-1)\cdots(q_i-1)})(1-\frac{q_i^m}{|A_i|})$$

$$= \sum_{k=0}^{m+1} |A_i|^{-k}(-1)^k \frac{(q_i^{m+1}-1)\cdots(q_i^{m+1}-q_i^{k-1})}{(q_i^k-1)\cdots(q_i-1)},$$

and hence, by induction,

$$\sum_{k=0}^{d_i} |A_i|^{-k}(-1)^k \frac{(q_i^{d_i}-1)\cdots(q_i^{d_i}-q_i^{k-1})}{(q_i^k-1)\cdots(q_i-1)} = \prod_{j=0}^{d_i-1}(1-\frac{q_i^j}{|A_i|}).$$

Therefore,

$$f(C,1) = \frac{|G|}{|C|}\prod_{i=1}^{a}\prod_{j=0}^{d_i-1}(1-\frac{q_i^j}{|A_i|})\prod_{k=a+1}^{a+b}(1-\frac{1}{|S_k|}).$$

We proved the following:

Theorem 3 *Let C be a conjugacy class of group G. Consider $\sum_f \chi(C)\bar{\chi}(C)$, with the sum extending over the faithful irreducible characters of G. We have*

(a)

$$\sum_f \chi(C)\bar{\chi}(C) = |G| \sum_{T_1 \leq S_1} \cdots \sum_{T_{a+b} \leq S_{a+b}} (1/|C \prod T_i|) \prod \mu(1, T_i)$$

(b) If class C is such that $|CS| = |C||S|$, where S is the socle of G, then

$$\sum_f \chi(C)\bar{\chi}(C) = \frac{|G|}{|C|}\prod_{i=1}^{a}\prod_{j=0}^{d_i-1}(1-\frac{q_i^j}{|A_i|})\prod_{k=a+1}^{a+b}(1-\frac{1}{|S_k|}).$$

Theorem 1 and part (b) of Theorem 3 may be viewed as a form of column orthogonality for the character table of any finite group, in which the inner product of columns is only taken over the faithful characters. The right hand side of the equation can be interpreted as a correction factor to the usual index of the centralizer when the inner product extends over all irreducible characters. To examine a few special cases, let C be a class in the center of G. Then $|C| = 1$, and part (b) of Theorem 3 informs us that, irrespective of C, $\sum_f \chi(C)\bar{\chi}(C)$ is equal to $\sum_f \chi(1)^2$, which represents the sum of squares of the degrees of the faithful irreducible representations of G. With $C = 1$, this is the main result of Pálfy (1979) who further explores the conditions under which G has a faithful irreducible representation.

Corollary 2 (Pálfy)

The sum of squares of degrees of the faithful irreducible representations is

$$\sum_f \chi(1)^2 = |G|\prod_{i=1}^{a}\prod_{j=0}^{d_i-1}(1-\frac{q_i^j}{|A_i|})\prod_{k=a+1}^{a+b}(1-\frac{1}{|S_k|}).$$

Interestingly, $\sum_f \chi(1)^2$ is divisible by $|C|$, where C is any conjugacy class of G such that $|CS| = |C||S|$. Indeed, Theorem 3 (b) can be writen as $\sum_f \chi(C)\bar{\chi}(C) = \frac{1}{|C|}\sum_f \chi(1)^2$. The right hand side is a rational number, whereas the left hand side is an algebraic integer. It follows that the right hand side is in fact an integer, possibly zero. We summarize:

Corollary 3

The sum $\sum_f \chi(1)^2$ is divisible by the cardinality of any conjugacy class C of G that satisfies $|CS| = |C||S|$, where S is the socle of G.

As an example, we examine the group $SL(2,3)$. There is a single minimal normal subgroup, namely $S = < \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} >$, which is the socle of $SL(2,3)$. Corollary 3 yields $\sum_f \chi(1)^2 = 24(1-\frac{1}{2}) = 12$, since in this case $a = 1$, $b = 0$, and $d_1 = 1$. Consider the conjugacy class C of the element $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Class C contains 4 elements, and it satisfies $|CS| = |C||S|$. Indeed, $SL(2,3)$ has three faithful irreducible representations of degree 2, and $12 = \sum_f \chi(1)^2 = 2^2 + 2^2 + 2^2$ is divisible by $4 = |C|$, as Corollary 3 intimates.

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