Conservation Laws of Turbulence Models

Leo G. Rebholz [1]

Department of Mathematics

University of Pittsburgh, PA 15260

ler6@math.pitt.edu

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Abstract

Conservation of mass, momentum, energy, helicity, and enstrophy in fluid flow are important because these quantities organize a flow, and characterize change in the flow's structure over time. Thus, if a simulation of a turbulent flow is to be qualitatively correct, these quantities should be conserved in the simulation. However, such simulations are typically based on turbulence models whose conservation properties are little explored and might be very different from those of the Navier-Stokes equations.

We explore conservation laws and approximate conservation laws satisfied by LES turbulence models. For the Leray, Leray deconvolution, Bardina, and N^{th}

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order deconvolution models, we give exact or approximate laws for a model mass, momentum, energy, enstrophy and helicity. Comparisons among the models are drawn based on these laws.

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1 Introduction

A major difficulty in turbulence modelling is selecting a model from among the plethora of turbulence models in existence. It is rarely known á priori if a particular model will perform well for a given set flow setting. However, there are other ways to compare turbulence models. For example, determining the physical relevance of a model's solution can give insight into a model's accuracy. It is well known that kinetic energy, ($E = \int_\Omega |u|^2 \, dx$), is critical in the organization of a flow, and hence if a model is to accurately predict turbulent flow, it must also accurately predict the flow's kinetic energy. Enstrophy ($Ens = \int_\Omega |\nabla \times u|^2 \, dx$) and helicity ($H = \int_\Omega u \cdot (\nabla \times u) dx$) are rotational quantities which play critical roles in the organization of two and three dimensional fluid flow, respectively. An accurate turbulence model must also predict these quantities correctly. In this paper, we compare four popular turbulence models based on the analysis of their treatment of kinetic energy, enstrophy, and helicity.

The most fundamental physical property of fluid flow with respect to kinetic energy, enstrophy, and helicity is that each of these quantities is conserved in inviscid flow. If these are not conserved in a turbulence model, non-physical energy or viscosity can be present in a model, which naturally can lead to non-physical solutions. Furthermore, conserving these quantities for inviscid flow is essential for a model to cascade these quantities through the wave numbers for viscous flows.

Conservation of kinetic energy in turbulence models has been extensively studied for many years. Kinetic energy conservation in a model yields stability, is the key step in an existence theory, and is the first step in proving a model's energy cascades from large to small scales. The conservation of enstrophy for two dimensional turbulence has also been extensively studied, and models such as the classical Arakawa scheme have been developed that preserve both energy and enstrophy for inviscid flow. Enstrophy is not conserved in three dimensions because of vortex stretching, a quantity which vanishes in two dimensions but not necessarily in three dimensions.

The most interesting of these conserved quantities is helicity, which is the streamwise vorticity of a flow. Helicity has only recently become a topic of research in fluid mechanics, as its fundamental importance in turbulent flow was unknown until 1961, when Moreau discovered helicity's inviscid invariance. Helicity's fundamental role in the organization of large structure in turbulent fluid flow was recognized in 1969 by Moffatt, who revealed a topological interpretation of helicity in terms of the linkage of vortices. It has been found that a joint cascade of energy and helicity exists for the decay of three dimensional turbulence [DP01],[CCE03],[CCEH03]. Thus the importance of helicity in fluid flow is evident, and most recently, Liu and Wang developed a scheme for three dimensional fluid flow that exactly conserves both energy and helicity[LW04]. Using a vorticity-stream function formulation of the Navier-Stokes equations (NSE), they recast the nonlinear terms as Jacobians, and associate with the Jacobians a trilinear form equipped with a permutation identity. They then devise a scheme to preserve the permutation identities, which leads to preserving energy and helicity. Their computational results found this scheme was able to effectively eliminate numerical viscosity.

Many turbulence models, by their construction, cannot exactly conserve energy, helicity, or enstrophy. Large Eddy Simulation (LES) models of turbulence, for example, solve for approximate averages of flows. These models are often used where fine scale detail is not necessary to estimate quantities of interest accurately.

The approach that an LES turbulence model takes to finding these "in the large" solutions is to average the NSE spacially, which eliminates very fine scale detail in the flow. To further illustrate this development, consider the NSE in an L-periodic box $\Omega \subset \mathbf{R}^3$ or $\mathbf{R}^2$.

$$u_t + \nabla \cdot (uu) + \nabla p - \nu \Delta u = f, \quad \nabla \cdot u = 0, \tag{1}$$

$$u(0,x) = u_0(x), \quad \int_\Omega p \, dx = 0, \quad and \; u(x + Le_i) = u(x). \tag{2}$$

Note that from these equations, in absence of dissipation ($\nu = 0$) and external force ($f = 0$), one can derive for every $t \geq 0$, the conservation of

- mass: $\nabla \cdot u(x,t) = 0 \ \forall x \in \Omega$,
- momentum: $\int_\Omega u(x,t) = \int_\Omega u_0(x)$,
- energy: $E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 = \frac{1}{2}\int_\Omega |u_0(x)|^2 = E(0)$,
- helicity: $H(t) = \int_\Omega u(t) \cdot (\nabla \times u(t)) = \int_\Omega u_0 \cdot (\nabla \times u_0) = H(0)$,
- and enstrophy: $Ens(t) = \frac{1}{2}\int_\Omega |\nabla u(t)|^2 = \frac{1}{2}\int_\Omega |\nabla u_0|^2 = Ens(0)$.

See, for example, [Fr95] or [GS98].

An LES model can be derived from the NSE as follows. Let $\overline{\phi}$ denote a spacial average of ϕ where the operator $(\overline{\cdot})$ is a differential filter (defined precisely in Section 2). Then the spacially filtered NSE (SFNSE) are

$$\overline{u}_t + \nabla \cdot \overline{uu} + \nabla \overline{p} - \nu \Delta \overline{u} = \overline{f}, \quad \nabla \cdot \overline{u} = 0, \tag{3}$$

$$\overline{u}(0,x) = \overline{u_0}(x), \quad \int_\Omega \overline{p}\ dx = 0, \quad and\ \overline{u}(x + Le_i) = \overline{u}(x) \tag{4}$$

A closure problem arises in the SFNSE; the $\overline{uu}$ term must be modeled, and each different way of modelling this term leads to a different LES model. Since $\overline{uu} \neq \bar{u}\bar{u}$, not every LES model will conserve energy, helicity or enstophy. However, LES models can conserve naturally arising model quantities analogous to energy, helicity, or enstrophy. In the Navier-Stokes-alpha ($NS\alpha$) model studied by Foias, Holm and Titi in [FHT01], a model energy and a model helicity were found:

$$E_{NS\alpha} = \int_\Omega v \cdot \bar{v}, \quad H_{NS\alpha} = \int_\Omega v \cdot (\nabla \times v),$$

where v is the model's velocity solution and $\bar{v}$ is a spacial average of the solution. Both $E_{NS\alpha}$ and $H_{NS\alpha}$ are conserved under periodic boundary conditions for inviscid flow [FHT01]. In the N^{th} order Stolz-Adams approximate deconvolution model (ADM) studied in [DE04], a model energy E_{ADM}, defined in Section 2, was found to be conserved for inviscid flow under periodic boundary conditions.

The work of [FHT01] motivated this paper, as it shows LES models can conserve a model helicity as well as a model energy. For a turbulence model, conservation of quantities analogous to the five conserved in the NSE is highly desirable; it can provide a diagnostic check for stability and accuracy of a model, and in practice, the presence of conserved quantities in a model allows solutions to be monitorred for physical relevance of solutions. Furthermore, as LES models are often used for modelling large scale rotational flows, such as in geophysics or oceanic modelling, they should exhibit the conservation of rotational quantities. Hence in this report we present a study of conservation laws in four popular LES models to see if they also conserve quantities analogous to those conserved in the NSE. The models we study are: the ADM, the Leray model, the Bardina model, and a new alteration of the Leray model proposed by A. Dunca and studied by Layton and Lewandowski [LL05] which we will refer to as the Leray deconvolution model. Formal definitions of these models will be given in Section 2.

The rest of this paper is arranged as follows. Section 2 will give notation and preliminaries, Section 3 will present the conservation laws of the models, and Section 4 presents comparisons and conclusions.

2 Notation and Preliminaries

The domain Ω used throughout this article will be a box: $\Omega = (0, L)^d$, $d = 2$ or 3, with periodic boundary conditions. All results except for that of enstrophy will hold for either $d = 2$ or $d = 3$, but conservation of enstrophy (as explained above) is restricted in these models, as well as in the NSE, to only two dimensions.

We shall assume that solutions are smooth enough to justify each manipulation used.

The usual L^2 norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively:

$$(v, w) = \int_\Omega v \cdot w, \quad \|v\| = (v, v)^{\frac{1}{2}}$$

Definition 2.1. *(The differential filter $\overline{\ \cdot\ }$) Given $\phi \in L^2(\Omega)$ and a filtering radius δ, define the average of ϕ, $\overline{\phi}$, to be the unique L-periodic solution of*

$$-\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi$$

This filtering operation will also be denoted by $\overline{\phi} = A^{-1}\phi$ for ease of notation. Note $A = (-\delta^2 \Delta + I)$ is self adjoint.

Definition 2.2. *(The approximate deconvolution operator G_N) For a fixed finite N, define the N^{th} approximate deconvolution operator by*

$$G_N \phi = \sum_{n=0}^{N} (I - A^{-1})^n \phi$$

Note that since A is self adjoint, G_N is also. G_N was shown to be an $O(\delta^{2N+2})$ approximate inverse to the filter operator in [DE05].

Corollary 2.3. *G_N is compact, positive, and is an asymptotic inverse to the filter A^{-1}: for very smooth ϕ and as $\delta \to 0$,*

$$\phi = G_N\overline{\phi} + (-1)^{(N+1)}\delta^{2N+2}\Delta^{N+1}A^{-(N+1)}\phi$$

The proof of Corollary 2.3 is found in ([DE05], Lemma 2.1).

Lemma 2.4. *$\|\cdot\|_{G_N}$ defined by $\|v\|_{G_N} = (v, G_N v)$ is a norm on Ω equivalent to the $L^2(\Omega)$ norm, and $(\cdot,\, \cdot)_{G_N}$ defined by $(v,\, w)_{G_N} = (v, G_N w)$ is an inner product on Ω.*

Proof. See [BIL05]. $\square$

2.1 The models considered

We have now provided enough preliminaries to define the four LES models considered as well as the respective models' energies, helicities and enstrophies.

Definition 2.5. *The Stolz-Adams ADM:*
The ADM is given by

$$v_t + \overline{G_N v \cdot \nabla G_N v} + \nabla q - \nu\Delta v = 0, \quad \nabla \cdot v = 0. \tag{5}$$

If v is a solution to (5), then the energy, helicity, and enstrophy for the ADM are defined to be

$$\begin{aligned} E_{ADM} &= \|v\|_{G_N}^2 + \delta^2\|\nabla v\|_{G_N}, \\ H_{ADM} &= (v,\, \nabla \times v)_{G_N} + \frac{\delta^2}{2}\left(\nabla \times v,\, (\nabla \times)^2 v\right)_{G_N},\ \textit{and} \\ Ens_{ADM} &= \frac{1}{2}\|\nabla \times v\|_{G_N}^2 + \frac{\delta^2}{2}\|\Delta v\|_{G_N}. \end{aligned}$$

Definition 2.6. *The Leray/Leray-α model:*

The Leray model is given by

$$v_t + \overline{v} \cdot \nabla v + \nabla q - \nu \Delta v = 0, \quad \nabla \cdot v = 0. \tag{6}$$

If v is a solution to (6), then the energy, helicity, and enstrophy for the Leray model are defined to be

$$\begin{aligned} E_{Leray} &= \frac{1}{2}\|v\|^2, \\ H_{Leray} &= (v, \nabla \times v), \ \textit{and} \\ Ens_{Leray} &= \frac{1}{2}\|\nabla \times \overline{v}\|^2 + \frac{\delta^2}{2}\|\Delta \overline{v}\|^2. \end{aligned}$$

Definition 2.7. *The Bardina scale similarity model*

The Bardina scale similarity model is given by

$$v_t + v \cdot \nabla v + \nabla q - \nu \Delta v + \nabla \cdot (\overline{\overline{v}\overline{v}} - \bar{v}\bar{v}) = 0, \quad \nabla \cdot v = 0. \tag{7}$$

If v is a solution to (7), then the energy, helicity, and enstrophy for the Bardina model are defined to be

$$\begin{aligned} E_{Bardina} &= \frac{1}{2}\|v\|^2 + \frac{\delta^2}{2}\|\nabla \times v\|^2, \\ H_{Bardina} &= (v, \nabla \times v) + \delta^2(\nabla \times v, (\nabla \times)^2 v), \ \textit{and} \\ Ens_{Bardina} &= \frac{1}{2}\|\nabla \times v\|^2 + \frac{\delta^2}{2}\|\Delta v\|^2. \end{aligned}$$

Definition 2.8. *(Leray-deconvolution Model)*

The Leray devonvolution model is defined to be

$$v_t + G_N \overline{v} \cdot \nabla v + \nabla q - \nu \Delta v = 0, \quad \nabla \cdot v = 0 \tag{8}$$

and if v is a solution to (8), then the energy, helicity, and enstrophy for the Leray deconvolution model are defined to be

$$\begin{aligned} E_{LD} &= \frac{1}{2}\|v\|^2, \\ H_{LD} &= (v, \nabla \times v), \ \textit{and} \\ Ens_{LD} &= \frac{1}{2}\|\nabla \times v\|^2. \end{aligned}$$

The next lemma gives four useful vector identities.

Lemma 2.9. *For sufficiently smooth u,*

$$u \cdot \nabla u = \frac{1}{2}\nabla u^2 - u \times (\nabla \times u)$$

For sufficiently smooth, periodic u, v,

$$(u, \nabla \times v) = (\nabla \times u, v)$$

For sufficiently smooth, periodic u, v with v divergence free,

$$(u, \Delta v) = -(\nabla \times u, \nabla \times v)$$

For suffiently smooth, periodic, two dimensional u

$$(u \cdot \nabla u, \Delta u) = 0$$

For proofs, see, for example, [Fr95],[GS98].

3 Conservation Laws

We develop conservation laws for the models considered together for momentum, mass, energy, helicity and enstrophy. The conservation laws are presented for inviscid flow (i.e. $\nu = 0$ or the Euler equations) and without external force ($f = 0$). However, we leave ν arbitrarily non-negative until the final step of each proof, as the case when dissipation is present is also be of interest because it gives a clue about the decay of these quantities in the presence of dissipation.

3.1 Momentum and Mass

Solutions to each of the models conserve momentum and mass. The conservation of a model mass comes directly from $\nabla \cdot v = 0$. Conservation of momentum follows for each model because each term in all the models, except for the time derivative term, is a spatial derivative (the nonlinear terms all can be expressed as spatial derivatives because of the commutation of differential operators under periodic boundary conditions coupled with the constraint that v be divergence free). Hence, integrating the first equation of each model over Ω vanishes all terms except the time derivative. Hence if v is a solution to any of the models, we have the relation

$$\frac{d}{dt}\int_\Omega v = 0$$

for that model. Thus integrating this equation from 0 to T yields

$$\int_\Omega v(T, x) = \int_\Omega v(0, x),$$

which establishes conservation of model momentum.

3.2 Energy

The ADM, Leray and Leray deconvolution models *exactly* conserve a model energy, whereas the Bardina model conserves a model energy only approximately (asymptotically as $\delta \to 0$). For smooth flows and as $\delta \to 0$, the energy estimate for the Bardina model of three dimensional flow is $O(\delta^2)$, and for two dimensional flow is $O(\delta^4)$. However, for flows with chaotic behavior or when large δ is required, a blow up to infinity of $E_{Bardina}$ cannot be ruled out.

Theorem 3.1. *The following energy conservation laws hold,* $\forall T > 0$.

$$\begin{aligned} E_{ADM}(T) &= E_{ADM}(0) \\ E_{Leray}(T) &= E_{Leray}(0) \\ E_{LD}(T) &= E_{LD}(0) \end{aligned}$$

The Bardina model satisfies

$$\begin{aligned} E^{3d}_{Bardina}(T) &= E^{3d}_{Bardina}(0) - \delta^2 \int_0^T \{(v \cdot \nabla v - \overline{v} \cdot \overline{v}, \Delta v) - (\overline{v} \cdot \nabla \overline{v}, \Delta \overline{v})\} dt \\ E^{2d}_{Bardina}(T) &= E^{2d}_{Bardina}(0) - \delta^4 \int_0^T \{(\overline{v} \cdot \overline{v}, \Delta^2 \overline{v})\} dt \end{aligned}$$

Proof. For the ADM, multiplying (5) by $AG_N v$ and integrating over Ω, we obtain

$$(v_t, AG_N v) + (\overline{G_N v \cdot \nabla G_N v}, AG_N v) + (\nabla q, AG_N v) - \nu(\Delta v, AG_N v) = 0. \quad (9)$$

As the operator A is self adjoint, the nonlinear term in (9) vanishes.

$$(\overline{G_N v \cdot \nabla G_N v}, AG_N v) = (G_N v \cdot \nabla G_N v, G_N v) = 0$$

The pressure term also vanishes, which can be seen by applying Green's Theorem, and using commutativity of the differential operators under periodic boundary conditions.

$$(\nabla q, AG_N v) = -(q, \nabla \cdot AG_N v) = (q, AG_N(\nabla \cdot v)) = 0$$

The time derivative and dissipation terms do not vanish, and so we rewrite (9) and simplify by decomposing A.

$$-\delta^2(v_t, \Delta G_N v) + (v_t, G_N v) + \delta^2\nu(\Delta v, \Delta G_N v) - \nu(\Delta v, G_N v) = 0$$

Green's Theorem and the fact that Δ and $G_N v$ commute under periodic boundary conditions allows this to be written as

$$\frac{1}{2}\frac{d}{dt}\|v\|_{G_N}^2 + \frac{\delta^2}{2}\frac{d}{dt}\|\nabla v\|_{G_N}^2 = -\nu\|\nabla v\|_{G_N}^2 - \delta^2\nu\|\Delta v\|_{G_N}^2. \tag{10}$$

Setting $\nu = 0$ and integrating over time in (10) gives the stated result.

For Leray and Leray deconvolution energy, the stated laws follow immediately by simply multiplying each model by its respective solution, integrating over the domain, setting $\nu = 0$, and integrating over time.

The equality for the Bardina model requires a little more effort. We begin by multiplying (7) by Av, where v is a solution of the Bardina model.

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{\delta^2}{2}\frac{d}{dt}\|\nabla v\|^2 + (v \cdot \nabla v, Av) + (\nabla \cdot (\overline{vv} - \bar{v}\bar{v}), Av) = 0$$

The first nonlinear term, after decomposing A, may be written as

$$(v \cdot \nabla v, Av) = -\delta^2(v \cdot \nabla v, \Delta v) + (v \cdot \nabla v, v) = -\delta^2(v \cdot \nabla v, \Delta v).$$

For the second nonlinear term, using the fact that A is self adjoint, differential operators commute under periodic boundary conditions and $A\overline{\phi} = \phi$, we have

$$(\nabla \cdot \overline{v}\overline{v}, Av) = (A\overline{\nabla \cdot vv}, v) = (\nabla \cdot vv, v) = 0$$

The third nonlinear term reduces by decomposing A, using the identity $v = A\overline{v}$, and decomposing A again.

$$\begin{aligned}
(\overline{v} \cdot \nabla\overline{v}), Av) &= -\delta^2(\overline{v} \cdot \nabla\overline{v}, \Delta v) + (\overline{v} \cdot \nabla\overline{v}, v) \\
&= -\delta^2(\overline{v} \cdot \nabla\overline{v}, \Delta v) + (\overline{v} \cdot \nabla\overline{v}, A\overline{v}) \\
&= -\delta^2(\overline{v} \cdot \nabla\overline{v}, \Delta v) - \delta^2(\overline{v} \cdot \nabla\overline{v}, \Delta\overline{v})
\end{aligned}$$

Recombining the terms, setting $\nu = 0$ and integrating over time, yields the three dimensional Bardina energy result.

In two dimensions, this result will reduce further. By Lemma 2.9,

$$(v \cdot \nabla v, \Delta v) = (\bar{v} \cdot \nabla\bar{v}, \Delta\bar{v}) = 0.$$

The remaining extra term can be decomposed as

$$(\bar{v} \cdot \nabla\bar{v}, \Delta v) = (\bar{v} \cdot \nabla\bar{v}, \Delta A\bar{v}) = -\delta^2(\bar{v} \cdot \nabla\bar{v}, \Delta^2\bar{v}).$$

Inserting these reductions into the three dimensional Bardina energy result yields the two dimensional Bardina energy result.

$\square$

3.3 Helicity

We now present the helicity conservation of the models. Only the ADM was found to exactly conserve an model helicity. The other three models were found to only approximately (asymptotically as $\delta \to 0$) conserve a model helicity. For each of these other three models, a blow up of helicity cannot be ruled out in this analysis.

Theorem 3.2. *The ADM conserves a model helicity:* $\forall T > 0$.

$$H_{ADM}(T) \quad = \quad H_{ADM}(0)$$

The remaining models satisfy, $\forall T > 0$,

$$\begin{aligned}
H_{Leray}(T) &= H_{Leray}(0) + 2\delta^2 \int_0^T ((\overline{v} \cdot \nabla v, \nabla \Delta \overline{v}) + (\overline{v} \cdot \nabla(\Delta \overline{v}), \nabla \times \overline{v}))\ dt \\
H_{Bardina}(T) &= H_{Bardina}(0) + 2\delta^2 \int_0^T (\ (v \cdot \nabla v, \nabla \times \Delta v) \\
&\quad -(\overline{v} \cdot \nabla \overline{v}, \nabla \times \Delta v) - (\overline{v} \cdot \nabla \overline{v}, \nabla \times \Delta \overline{v})\ )\ dt \\
H_{LD}(T) &= H_{LD}(0) + (-2)^N \delta^{2N+2} \int_0^T (\Delta^{N+1} A^{-(N+1)} v \cdot \nabla v, \nabla \times v)\ dt
\end{aligned}$$

Proof. The proof for ADM helicity is similar to that of ADM energy. Multiply (5) by $(\nabla \times AG_N v)$, where v solves (5), and integrate over Ω.

$$\begin{aligned}
&(v_t, \nabla \times AG_N v) + (\overline{G_N v \cdot \nabla G_N v}, \nabla \times AG_N v) + \\
&\qquad (\nabla q, \nabla \times AG_N v) - \nu(\Delta v, \nabla \times AG_N v) = 0 \quad (11)
\end{aligned}$$

As in the energy proof, the nonlinear term vanishes. To show this, we use the commutativity of differential operators under periodic boundary conditions, the fact that A is self adjoint, and apply Lemma 2.9.

$$
\begin{aligned}
&(\overline{G_N v \cdot \nabla G_N v}, \nabla \times A G_N v) \\
&= (G_N v \cdot \nabla G_N v, \nabla \times G_N v) \\
&= (\frac{1}{2}\nabla((G_N v)^2), \nabla \times G_N v) - (G_N v \times (\nabla \times G_N v), \nabla \times G_N v) \\
&= \frac{1}{2}(\nabla \times \nabla((G_N v)^2), G_N v) - 0 \\
&= 0
\end{aligned}
$$

The pressure term also vanishes.

$$(\nabla q, \nabla \times A G_N v) = (\nabla \times (\nabla q), A G_N v) = 0$$

The time derivative term is simplified using commutativity of the differential operators after decomposing A and applying Lemma 2.9.

$$
\begin{aligned}
(v_t, \nabla \times (-\delta^2 \Delta + I) G_N v) &= -\delta^2 (v_t, \nabla \times \Delta(G_N v)) + (v_t, \nabla \times G_N v) \\
&= \delta^2((\nabla \times v)_t, \nabla \times G_N(\nabla \times v)) + (v_t, \nabla \times G_N v) \\
&= \frac{\delta^2}{2}\frac{d}{dt}\left(\nabla \times v,\, \nabla \times)^2 v\right)_{G_N} + \frac{1}{2}\frac{d}{dt}\left(v,\, \nabla \times v\right)_{G_N}
\end{aligned}
$$

The dissipation term simplifies by decomposing A and applying Lemma 2.9.

$$
\begin{aligned}
-\nu(\Delta v, \nabla \times A G_N v) &= \delta^2 \nu (\Delta v, \nabla \times (\Delta G_n v)) - \nu(\Delta v, \nabla \times G_N v) \\
&= \delta^2 \nu \left((\nabla \times)^2 v,\, (\nabla \times)^3 v\right)_{G_N} + \nu\left(\nabla v,\, (\nabla \times)^2 v\right)_{G_N} \qquad (12)
\end{aligned}
$$

Recombining all the terms and setting $\nu = 0$ gives

$$\frac{\delta^2}{2}\frac{d}{dt}\left(\nabla \times v,\, \nabla \times)^2 v\right)_{G_N} + \frac{1}{2}\frac{d}{dt}\left(v,\, \nabla \times v\right)_{G_N} = 0 \qquad (13)$$

Integrating over time give the stated conservation law.

For the Leray helicity relation, we multiply (6) by the curl of its solution, $(\nabla \times v)$, and integrate over the domain. After simplifying, this yields

$$\frac{1}{2}\frac{d}{dt}(v, \nabla \times v) = -\nu(\nabla \times v, (\nabla \times)^2 v) - (\overline{v} \cdot \nabla v, \nabla \times v) \tag{14}$$

Expand the nonlinear term by using the identity $v = A\overline{v}$ and simplifying.

$$\begin{aligned}
(\overline{v} \cdot \nabla v, \nabla \times v) &= (\overline{v} \cdot \nabla v, \nabla \times A\overline{v}) \\
&= -\delta^2(\overline{v} \cdot \nabla v, \nabla \times \Delta\overline{v}) + (\overline{v} \cdot \nabla v, \nabla \times \overline{v}) \\
&= -\delta^2(\overline{v} \cdot \nabla v, \nabla \times \Delta\overline{v}) - \delta^2(\overline{v} \cdot \nabla(\Delta\overline{v}), \nabla \times \overline{v})
\end{aligned}$$

Recombing terms and setting $\nu = 0$ gives

$$\frac{1}{2}\frac{d}{dt} = \delta^2(\overline{v} \cdot \nabla v, \nabla \times \Delta\overline{v}) + \delta^2(\overline{v} \cdot \nabla(\Delta\overline{v}), \nabla \times \overline{v}) \tag{15}$$

Integrating over time will now give the stated Leray helicity conservation.

For Bardina, multiply (7) by $(\nabla \times Av)$, where v solves (7). Performing analysis very similar to that in the ADM proof for the time derivative, dissipation, and pressure terms reduces this to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(v, \nabla \times v) + \frac{\delta^2}{2}\frac{d}{dt}(\nabla \times v, (\nabla \times)^2 v) &= -\nu(\nabla \times v, (\nabla \times)^2 v) \\
-(v \cdot \nabla v, \nabla \times Av) - (\nabla \cdot \overline{vv}, \nabla \times Av) &+ (\nabla \cdot \overline{vv}, \nabla \times Av)
\end{aligned} \tag{16}$$

The first nonlinear term is expanded by decomposing A and simplifying.

$$\begin{aligned}
(v \cdot \nabla v, \nabla \times Av) &= -\delta^2(v \cdot \nabla v, \nabla \times \Delta v) + (v \cdot \nabla v, \nabla \times v) \\
&= -\delta^2(v \cdot \nabla v, \nabla \times \Delta v)
\end{aligned}$$

The second nonlinear term vanishes by using the fact the A is self adjoint and that differential operators commute under periodic boundary conditions.

$$(\nabla \cdot \overline{vv}, \nabla \times Av) = (A\overline{\nabla \cdot vv}, \nabla \times v) = (\nabla \cdot vv, \nabla \times v) = 0 \qquad (17)$$

For the third nonlinear term, we decompose A and use the identity $v = A\overline{v}$.

$$\begin{aligned}
(\nabla \times \overline{vv}, \nabla \times Av) &= -\delta^2(\overline{v} \cdot \nabla\overline{v}, \nabla \times \Delta v) + (\overline{v} \cdot \nabla\overline{v}, \nabla \times v) \\
&= -\delta^2(\overline{v} \cdot \nabla\overline{v}, \nabla \times \Delta v) + (\overline{v} \cdot \nabla\overline{v}, \nabla \times A\overline{v}) \\
&= -\delta^2(\overline{v} \cdot \nabla\overline{v}, \nabla \times \Delta v) - \delta^2(\overline{v} \cdot \nabla\overline{v}, \nabla \times \Delta\overline{v})
\end{aligned}$$

Recombining terms, setting $\nu = 0$, and integrating over time gives the Bardina helicity conservation.

For the Leray deconvolution model, the analysis is exactly the same as the Leray model except for the nonlinear term, after multiplying (8) by the curl of its solution. The nonlinear term can be written as

$$(G_N\overline{v} \cdot \nabla v, \nabla \times v) = ((v - (-1)^{N+1}\delta^{2N+2}\Delta^{N+1}A^{-(N+1)}v) \cdot \nabla v, \nabla \times v). \quad (18)$$

Thus we have the stated result, since $(v \cdot \nabla v, \nabla \times v) = 0$. $\square$

3.4 Enstrophy (2d)

The ADM and Leray model exactly conserve 2d model enstrophy. As in the helicity case, the other models have approximate laws which may not be useful without restrictive assumptions on the size of higher derivatives and the size of δ.

Theorem 3.3. *The ADM and Leray model conserve enstrophy in 2d:* $\forall T > 0$.

$$\begin{aligned} Ens_{ADM}(T) &= Ens_{ADM}(0) \\ Ens_{Leray}(T) &= Ens_{Leray}(0) \end{aligned}$$

The remaining models, in 2d, satisfy

$$\begin{aligned} Ens_{Bard}(T) &= Ens_{Bard}(0) + \delta^2 \int_0^T \left(\overline{v} \cdot \nabla \overline{v}, \Delta^2 (v + \overline{v}) \right) - (v \cdot \nabla v, \Delta^2 v) \Big) \ dt \\ Ens_{LD}(T) &= Ens_{LD}(0) + (-1)^N \delta^{2N+2} \int_0^T (\Delta^{N+1} A^{-(N+1)} v \cdot \nabla v, , \Delta v) \ dt \end{aligned}$$

Proof. To prove the (2d) ADM enstrophy relation, we multiply (5) by $\Delta A G_N v$ where v solves (5) and integrate over Ω.

$$(v_t, \Delta A G_N v) + (\overline{G_N v \cdot \nabla G_N v}, \Delta A G_N v) + (\nabla q, \Delta A G_N v) \\ - \nu(\Delta v, \Delta A G_N v) = 0 \quad (19)$$

The nonlinear term is handled differently than in any of the previous proofs, and it is this term which makes the stated enstrophy relation hold only in two dimensions (it does not necessarily vanish in 3d). We use that A is self adjoint, A and Δ commute, and that $G_N v$ is two dimensional.

$$\begin{aligned} (\overline{G_N v \cdot \nabla G_N v}, \Delta A G_N v) &= (G_N v \cdot \nabla G_N v, \Delta G_N v) \\ &= 0 \end{aligned}$$

The pressure vanishes.

$$(\nabla q, \Delta A G_N v) = -(\nabla \times \nabla q, \nabla \times A G_N v) = 0$$

For the time derivative, decompose A, apply Lemma 2.9, and simplify.

$$\begin{aligned}(v_t, \Delta A G_N v) &= -\delta^2(v_t, \Delta\Delta G_N v) + (v_t, \Delta G_N v)\\ &= -\delta^2((\Delta v)_t, \Delta v) - ((\nabla \times v)_t, \nabla \times v)\\ &= -\frac{\delta^2}{2}\frac{d}{dt}\|\Delta v\|_{G_N}^2 - \frac{1}{2}\frac{d}{dt}\|\nabla \times v\|_{G_N}^2\end{aligned}$$

The dissipation term also requires decomposition of A and Lemma 2.9.

$$\begin{aligned}-\nu(\Delta v, \Delta A G_N v) &= -\delta^2\nu(\nabla \times \Delta v, \nabla \times \Delta G_N v) + \nu(\nabla \times \Delta G_N v, \nabla \times G_N v)\\ &= -\delta^2\nu\|\nabla \times \Delta v\|_{G_N}^2 - \nu\|\Delta G_N v\|_{G_N} \quad (20)\end{aligned}$$

Recombining the terms and setting $\nu = 0$ gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla \times v\|_{G_N}^2 + \frac{\delta^2}{2}\frac{d}{dt}\|\Delta v\|_{G_N}^2 = 0. \quad (21)$$

Integrating over time now gives the stated ADM 2d enstrophy conservation law.

For the Leray enstrophy result, multiply (6) by $\Delta\overline{v}$, where v solves (6), integrate over the domain, and write $v = A\overline{v}$.

$$((A\overline{v})_t, \Delta\overline{v}) + (\overline{v} \cdot \nabla(A\overline{v}), \Delta\overline{v}) + (q, \Delta\overline{v}) - \nu(\Delta(A\overline{v}, \Delta\overline{v}) = 0$$

Next decompose each A, and simplify. The pressure term vanishes by applying Lemma 2.9.

$$\begin{aligned}(\overline{v}_t, \Delta\overline{v}) - \delta^2(\Delta\overline{v}_t, \Delta\overline{v}) - \delta^2(\overline{v} \cdot \nabla(\Delta\overline{v}), \Delta\overline{v}) + (\overline{v} \cdot \nabla\overline{v}, \Delta\overline{v})&\\ - \nu\delta^2\|\nabla \times (\Delta\overline{v})\|^2 - \nu\|\Delta\overline{v}\|^2 = 0&\end{aligned}$$

Since both nonlinear terms vanish, this expression can be simplified and rewritten as

$$\frac{1}{2}\frac{d}{dt}\|\nabla \times \overline{v}\|^2 + \delta^2\|\Delta\overline{v}\|^2 = -\nu\|\Delta\overline{v}\|^2 - \delta^2\|\nabla \times \Delta\overline{v}\|^2$$

Setting $\nu = 0$ and integrating over time gives the result.

For the Leray-deconvolution enstrophy, multiply (8) by Δv, where v is a solution to (8), integrate over the domain, and simplify. This gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 = -\nu\|\Delta v\|^2 + (G_N\overline{v} \cdot \nabla v, \Delta v) \tag{22}$$

For the nonlinear term, we reduce by expanding the $G_N\overline{v}$ term.

$$(G_N\overline{v} \cdot \nabla v, \Delta v) = (v - (-1)^{N+1}\delta^{2N+2}\Delta^{N+1}A^{-(N+1)}v \cdot \nabla v, \Delta v) \tag{23}$$

Applying Lemma 2.9, setting $\nu = 0$ and integrating over time will then give the desired result.

For the Bardina model, we multiply (7) by ΔAv, where v solves (7) and integrate over the domain. We do the analysis term by term, except for the pressure term, which will vanish in the same manner as in all other cases. Rewrite the time derivative by decomposing A and simplifying.

$$(v_t, \Delta Av) = -\delta^2(v_t, \Delta\Delta v) + (v_t, \Delta v) = -\frac{1}{2}\frac{d}{dt}\|\nabla \times v\|^2 - \frac{\delta^2}{2}\frac{d}{dt}\|\Delta v\|^2 \tag{24}$$

For the first nonlinear term, we decompose A.

$$(v \cdot \nabla v, \Delta Av) = -\delta^2(v \cdot \nabla v, \Delta\Delta v) + (v \cdot \nabla v, \Delta v) = -\delta^2(v \cdot \nabla v, \Delta\Delta v) \tag{25}$$

The dissipation term also gets expanded by decomposing A.

$$-\nu(\Delta v, \Delta Av) = \delta^2\nu(\Delta v, \Delta\Delta v) - \nu(\Delta v, \Delta v) = -\nu\|\Delta v\|^2 - \delta^2\nu\|\nabla \times \Delta v\|^2 \tag{26}$$

The second nonlinear term vanishes, using the fact the A is self adjoint, the restriction to 2d, and Lemma 2.9.

$$(\nabla \cdot \overline{v}\overline{v}, \Delta A v) = (\nabla \cdot vv, \Delta v) = 0 \tag{27}$$

The third nonlinear term takes the most work. First we decompose A, then we use the identity $v = A\overline{v}$ and Lemma 2.9.

$$\begin{aligned}(\nabla \cdot \overline{v}\overline{v}, \Delta A v) &= -\delta^2(\overline{v} \cdot \nabla \overline{v}, \Delta\Delta v) + (\overline{v} \cdot \nabla \overline{v}, \Delta v) \\ &= -\delta^2(\overline{v} \cdot \nabla \overline{v}, \Delta\Delta v) - \delta^2(\overline{v} \cdot \nabla \overline{v}, \Delta\Delta \overline{v})\end{aligned} \tag{28}$$

Recombining all the terms, setting $\nu = 0$, and integrating over time gives the Bardina result. $\square$

4 Conclusions

This report studied conservation laws in the Bardina, ADM, Leray and Leray deconvolution models in an effort to establish which of these models had conservation laws analogous to those of the Navier Stokes equations. All of the models exactly conserved a model mass and model momentum. However, only the ADM was found to exactly conserve a model helicity, and only the ADM and Leray model exactly conserved a model enstrophy. The Bardina model was the only model found to not conserve a model energy. This is consistent with the stability problems reported in simulations of the Bardina model.

Nonconservation a rotational quantity such as helicity or enstrophy could significantly affect the dynamics of a model's prediction, causing serious inaccuracy. Hence our results suggest that if one is modelling a flow with rotation, then in three dimensions, the ADM appears to be the best of these four models, and in two dimensions, the ADM and Leray models apprear to be the best.

5 References

AS01 Adams, N.A. and S. Stolz. Deconvolution methods for subgrid-scale approximation in large eddy simulation, *Modern Simulation Strategies for Turbulent Flow*, R.T. Edwards, 2001.

BIL05 Berselli, L., Iliesu, T. and W. Layton. Mathematics of Large Eddy Simulation of Turbulent Flows. Springer, Berlin, 2005.

CCE03 Chen, Q., Chen, S. and G. Eyink. The joint cascade of energy and helicity in three dimensional turbulence. *Physics of Fluids*, 2003, Vol. 15, No. 2: 361-374.

CCEH03 Chen, Q., Chen, S., Eyink, G. and D. Holm. Intermittency in the joint cascade of energy and helicity. *Physics Review Letters*, 2003, 90: 214503.

DE05 Dunca, A. and Y. Epshteyn. On the Stolz-Adams deconvolution LES model. To appear in: *SIAM Journal of Math. Anal.*, 2005.

DP01 Ditlevson, P. and P. Guiliani. Cascades in helical turbulence. *Physical Review E*, Vol. 63, 2001.

FHT01 Foias, C., Holm, D. and E. Titi. The Navier-Stokes-alpha model of fluid turbulence. *Physica D*, May 2001: 505-519.

Fr95 Frisch, U. *Turbulence*. Cambridge University Press, 1995.

GS98 Gresho, P. and R. Sani. *Incompressible Flow and the Finite Element Method*, Vol. 2. Wiley, 1998.

LL05 Layton, W. and R. Lewandowski. On the Leray deconvolution model. Technical Report, University of Pittsburgh, 2005.

LL05 Layton, W. and R. Lewandowski. On a well-posed turbulence model. To appear in *Discrete and Continuous Dynamical Systems, B*, 2005.

LW04 Lui, J. and W. Wang. Energy and helicity preserving schemes for hydro and magnetohydro-dynamics flows with symmetry. *Journal of Computational Physics* 2004, 200: 8-33.

MT92 Moffatt, H., and A. Tsoniber. Helicity in laminar and turbulent flow. *Annual Review of Fluid Mechanics*, 1992, 24:281-312.

SA99 Stolz, S. and N.A. Adams. On the approximate deconvolution procedure for LES, *Physics of Fluids*, II(1999): 1699-1701.