On some numerical convergence studies of mixed finite element methods for flow in porous media

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Abstract

We consider an expanded mixed finite element method for solving second-order elliptic partial differential equations. We study the effects of nonmatching grids, discontinuous coefficients, and high variation in the coefficients on the accuracy of the numerical solution. The error in the case of nonmatching grids and smooth solutions occurs mainly along the interfaces and high accuracy is preserved in the interior. Discontinuous coefficients may lead to singular solutions and the polution from the singularity affects the accuracy in the whole domain. Our last set of examples shows that the dependence of the convergence rates and constants in front of the error terms on high variation in the coefficients is very weak.

1 Introduction

In this work we consider mixed finite element method for subdomain discretizations. Mixed methods owe their popularity to their local (element-wise) mass conservation property and the simultaneous and accurate approximation of two variables of physical interest, e.g., pressure and velocity in fluid flow. In many applications the complexity of the geometry or the behavior of the solution prompts the use of multiblock domain structure where the simulation domain is decomposed into a series of nonoverlapping subdomains (blocks). Each block is independently covered by a local grid. A non-overlpping domain decomposition algorithm was developed for matching grids by Glowinski and Wheeler [5, 3] ans was later extended to non-matching grids. Mortar finite elements are used to impose physically meaningfull matching conditions on the interfaces while mixed finite elements are applied locally on the subdomains (see [6, 1] for details).

In this work we consider a second-order elliptic equation which in porous medium applications models single phase Darsy flow. We solve for the pressure p and the velocity field $\mathbf{u}$ satisfying

$$\mathbf{u} = -K\,\nabla p \qquad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} + \alpha p = f \qquad \text{in } \Omega, \tag{2}$$

$$p = g^D \qquad \text{on } \Gamma^D, \tag{3}$$

$$\mathbf{u} \cdot \nu = g^N \qquad \text{on } \Gamma^N, \tag{4}$$

where $\alpha \geq 0$ represents the rock compressibility; $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3 is a multiblock domain; K is symmetric, uniformly positive definite tensor with smooth or perhaps piecewise smooth components representing the permeability divided by the viscosity; ν is outward unit normal vector on $\partial\Omega$; and $\partial\Omega$ is decomposed into Γ^D and Γ^N.

The problem was solved using the parallel domain decomposition code *Parcel* [4] with some modifications made by the author. The code implements an expanded mixed finite element method developed by Arbogast, Wheeler and Yotov [2] where mixed method with tensor coefficient is writen as a cell-centered finite difference method by incorporating certain quadrature rules.

In the case of nonmatching grids we study the convergence of interior velocity (far from subdomain interfaces). The results show that the interior velocity error is superconvergent of $O(h^2)$, which means that majority of the error occurs near the interfaces. Therefore we need to apply some local postprocessing to obtain better convergence rate for the velocity error.

Second group of tests was run in the case of discontinuous tensor for both mathing and nonmatching grids. As the results show, because of the strong singularity at the cross-point $(1/2, 1/2)$, there is no superconvergence even in the interior. The maximum rate of convergence for the interior velocity error is of $O(h)$. Therefore to control the error we need some local refinement near this cross-point.

Analyzing all test results in group 1 and group 2 we may conclude that interior velocity error depends on the smoothness of the solution in the whole domain Ω, but in a more weak sense, and that interior velocity error is better than the velocity error calculated over the whole domain.

The last group of tests studies the influence of the the low order term α in (2) on the constant C in the error estimate

$$||p - p_h|| \leq Ch^2.$$

We compared the results when $\alpha = 0$ (no low order term) and $\alpha = 1$. The results show that this method works very well for both cases even if there are big variations of K and that the constant increases very slowly when the ratio goes up.

The rest of the paper is organized as follows. Interior error estimates in the case of nonmatching grids are presented in Section 2. Error estimates in the case of discontinuous tensor are presented in Section 3. In section 4 the influence of the low order term on the constant in the error estimates is studied.

2 Interior error estimates in the case of nonmatching grids

To study the interior velocity error we used six tests with known analytic solutions. All examples are on the unit cube. The domain is divided into four subdomains with interfaces along the $x = 1/2$ and $y = 1/2$ lines. The boundary conditions are Dirichlet on the left and right face and Neumann on the rest of the boundary. In test#58 we have

$$p(x, y, z) = x^3y^2 + \sin(xy) \quad \text{and} \quad K = \begin{pmatrix} 10 + 5\cos(xy) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In test#59

$$p(x, y, z) = \cos(\frac{\pi x}{2})\cos(\frac{\pi y}{2}) \quad \text{and} \quad K = I.$$

In test#64 we have a problem with discontinuous coefficient

$$K = \begin{cases} I & , 0 \leq x < 1/2 \\ 10 * I & , 1/2 < x \leq 1 \end{cases}.$$

The solution

$$p(x, y, z) = \begin{cases} x^2y^3 + \cos(xy) & , 0 \leq x < 1/2 \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos(\frac{2x+9}{20}y) & , 1/2 < x \leq 1 \end{cases}$$

is chosen to be continuous and to have continuous normal flux at $x = 1/2$.

In the next three tests K is a full tensor.

In test#104

$$p(x, y, z) = x + y + z - 1.5 \quad \text{and} \quad K = \begin{pmatrix} x^2 + y^2 + 1 & 0 & 0 \\ 0 & z^2 + 1 & \sin(xy) \\ 0 & \sin(xy) & x^2y^2 + 1 \end{pmatrix}.$$

In test#107

$$p(x, y, z) = x^2(x-1)^2y^2(y-1)^2z^2(z-1)^2 \quad \text{and} \quad K = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Finally in test#110

$$p(x,y,z) = \begin{cases} xy & , 0 \le x \le 1/2 \\ xy + (x-1/2)(y+1/2) & , 1/2 \le x \le 1 \end{cases}$$

and

$$K = \begin{cases} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} & , 0 \le x < 1/2 \\ I & , 1/2 < x \le 1 \end{cases} .$$

For the $2d$-problems (## 58, 59, 64, 110) the initial nonmatching grids are given in Figure 1 and the initial mortar grids on all interfaces are given in Table 1. For $3d$-problems (#104, #107) we consider similar (but $3d-$) grids. We use $\alpha = 0.1$.

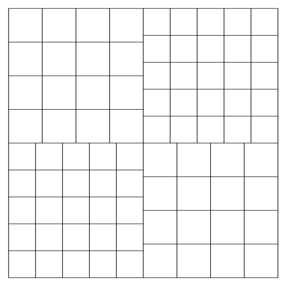


Figure 1: Initial non-matching grids for Cases 1–3

mortar	1	2	3	4
elements	3	3	1	5

Table 1: Initial number of elements in mortar grids for Cases 1–3

2.1 Case 1

The code was first modified to calculate the error over the interiors of all subdomains that have a one-element border around it. Tests were only run using mortar 4 (piecewise constant) because it was obvious that even there exist some improvement of the the rate of convergence of the interior error, it is not essential. The results for this case are in Table 2. The rates were established by running all tests for 5 levels of grid refinement (we halve the element diameters for each refinement) and computing a least squares fit to the error.

2.2 Case 2

The second modification of the code produced a **scaled** interior error. The calculation $||\mathbf{u} - \mathbf{u_h}||/||\mathbf{u}||$ over the interior subdomains was used in an attempt to eliminate any possible influence in the size of the interior subdomains would have on the error calculations. Again the improvement wasn't essential. The results for this case are in Table 3.

test	velocity L^2 error		vel. L^2 err. Int.		velocity L^∞error		vel. L^∞err. Int.	
	C_u	α_u	C_u	α_u	C_u	α_u	C_u	α_u
58	0.54367	0.78242	0.12750	0.55476	0.44620	0.26164	0.21674	0.18728
59	0.04109	0.52812	0.01805	0.62540	0.03793	0.04879	0.01935	0.18330
64	0.23754	1.15747	0.28418	1.40638	0.05094	0.19525	0.10549	0.61198
104	0.05610	0.47142	0.02044	0.51880	0.03826	-0.06980	0.02191	0.03434
107	0.01157	1.78657	0.00073	1.13243	0.01815	1.52227	0.00126	0.81571
110	0.06674	0.57901	0.01885	0.57524	0.05485	0.06251	0.08477	0.43350

Table 2: Velocity errors for Case 1 mortar 4

mortar	test	velocity L^2 error		vel. L^2 err. Int.		velocity L^∞error		vel. L^∞err. Int.	
		C_u	α_u	C_u	α_u	C_u	α_u	C_u	α_u
	58	2.66274	1.77322	0.10485	1.71342	2.00771	1.24683	0.04331	1.34017
	59	0.22929	1.69787	0.28510	1.92374	0.09980	0.94798	0.07066	1.31158
1	64	1.33161	1.78550	1.26313	1.96919	0.46868	0.86836	0.15226	1.34315
	104	0.14553	1.55720	0.03572	1.65734	0.08844	0.88043	0.00888	0.83621
	107	0.01688	1.95120	4.43731	1.62768	0.02860	1.72008	1.98867	1.17520
	110	0.25459	1.46322	0.12262	1.81978	0.17738	0.96047	0.02718	1.08433
	58	2.32761	1.71424	0.09601	1.66598	2.02304	1.24654	0.03760	1.29471
	59	0.26758	1.81129	0.28552	1.93455	0.11920	1.20358	0.09820	1.44864
2	64	1.95992	1.96678	1.33080	1.99125	0.51878	1.25044	0.58127	1.86492
	104	0.18661	1.61436	0.05468	1.76282	0.11574	0.96044	0.03387	1.17876
	107	0.02004	2.00490	5.77645	1.70399	0.02962	1.72968	2.37342	1.22172
	110	0.24784	1.45577	0.11780	1.80708	0.18424	0.96912	0.02429	1.06248
	58	0.20265	0.70098	0.01260	0.75098	0.21370	0.21300	0.00286	0.17287
	59	0.03542	0.75957	0.05237	1.12019	0.03058	0.24059	0.01960	0.51651
3	64	0.78962	1.65509	1.02173	1.90811	0.13897	0.74854	0.15094	1.39216
	104	0.21352	1.00307	0.09213	1.19911	0.18577	0.38702	0.01875	0.30153
	107	0.01509	1.91113	3.55414	1.55203	0.02489	1.66242	1.61587	1.12148
	110	0.04343	0.75239	0.01813	0.94637	0.03563	0.25080	0.00494	0.21490
	58	0.54367	0.78242	0.02276	0.75181	0.44620	0.26164	0.00897	0.31533
	59	0.04109	0.52812	0.04319	0.82989	0.03793	0.04879	0.01400	0.21155
4	64	0.23754	1.15747	0.47748	1.61093	0.05094	0.19525	0.03592	0.76608
	104	0.10682	0.48427	0.03329	0.61608	0.14464	-0.0051	0.00842	-0.2289
	107	0.01157	1.78657	2.88170	1.46652	0.01815	1.52227	0.92326	0.86969
	110	0.06674	0.57901	0.02669	0.77457	0.05485	0.06251	0.00924	0.13240

Table 3: Velocity errors for Case 2

2.3 Case 3

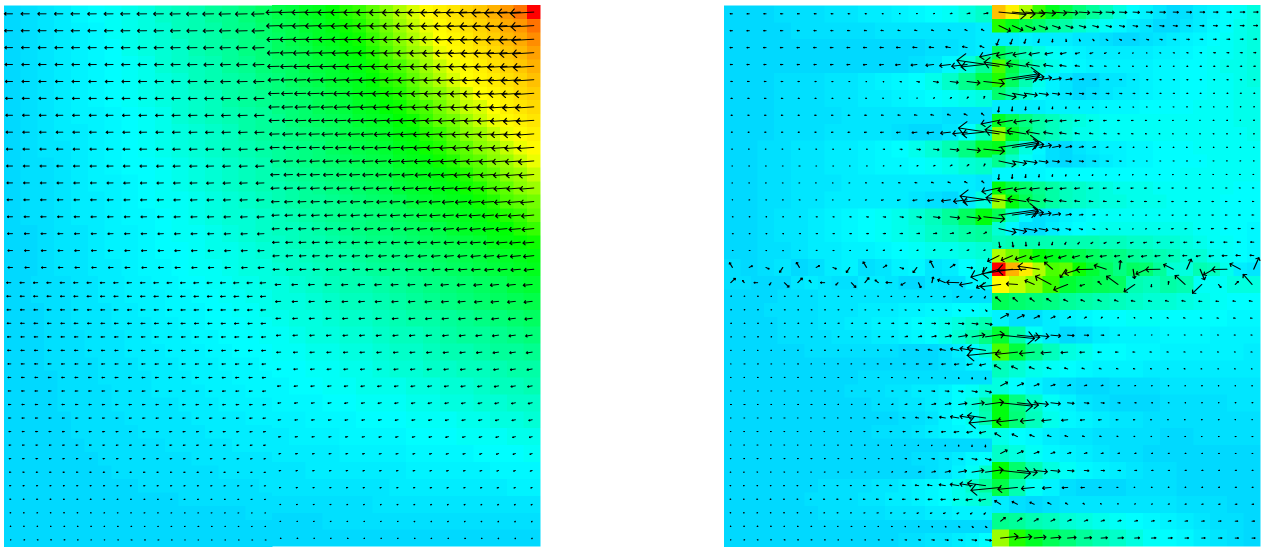
Thirdly, the code was modified to calculate the errors over **fixed** interior domains for each level of refinement. In this case it seems that the interior velocity error is superconvergent of $O(h^2)$, which means that majority of the error occurs near the interfaces. Therefore we need to apply some local postprocessing to obtain better convergence rate for the velocity error. The results for this case are in Table 4 and Table 5. Plots of the computed solution and the numerical error for the case of mortar 4 are shown in Figure 2 through Figure 7.

mor		*flux error*		*pressure L^2error*		*λ error*	
tar	*test*	C_f	α_f	C_p	α_p	C_λ	α_λ
1	58	0.86618	1.14348	0.13977	2.00716	0.25063	1.92904
	59	0.13611	1.02397	0.15428	1.99830	0.18123	1.99922
	64	0.42816	0.99543	0.03591	2.00812	0.07926	1.92863
	104	0.20796	1.34665	0.20782	2.02484	0.10682	1.93479
	107	0.00634	1.51331	0.00132	2.01518	0.00314	2.03247
	110	0.27076	1.30103	0.13936	2.00022	0.12387	1.92116
2	58	0.75732	1.06461	0.14012	2.00717	0.24363	1.91795
	59	0.06147	0.96515	0.15483	1.99924	0.16323	1.99847
	64	0.07839	1.00963	0.03109	1.95955	0.08556	1.96633
	104	0.32708	1.41856	0.20910	2.02645	0.15509	2.02785
	107	0.02873	2.03787	0.00152	2.05609	0.00422	2.12012
	110	0.19697	1.19006	0.13997	2.00161	0.14532	1.95482
3	58	0.11513	0.12640	0.08315	1.79005	0.73177	1.06345
	59	0.02008	0.08901	0.13470	1.96377	0.06971	0.99552
	64	0.00923	0.22353	0.03283	1.98314	0.02382	0.97858
	104	0.06591	0.19293	0.30498	1.93876	0.06700	0.74369
	107	0.00238	1.24266	0.00127	2.00185	0.00108	1.417078
	110	0.02073	0.05298	0.11937	1.94448	0.08108	0.96348
4	58	0.21224	0.11464	0.13560	1.77063	0.12086	1.23240
	59	0.03114	-0.0819	0.09478	1.76410	0.07444	1.10489
	64	0.02227	0.10901	0.02811	1.78406	0.01445	1.07005
	104	0.06653	-0.1480	0.14887	1.53792	0.13933	1.01947
	107	0.00107	0.81488	0.00127	1.98400	0.00231	1.87368
	110	0.05203	-0.0242	0.11900	1.78721	0.08153	1.10277

Table 4: Velocity errors for Case 3 Part I

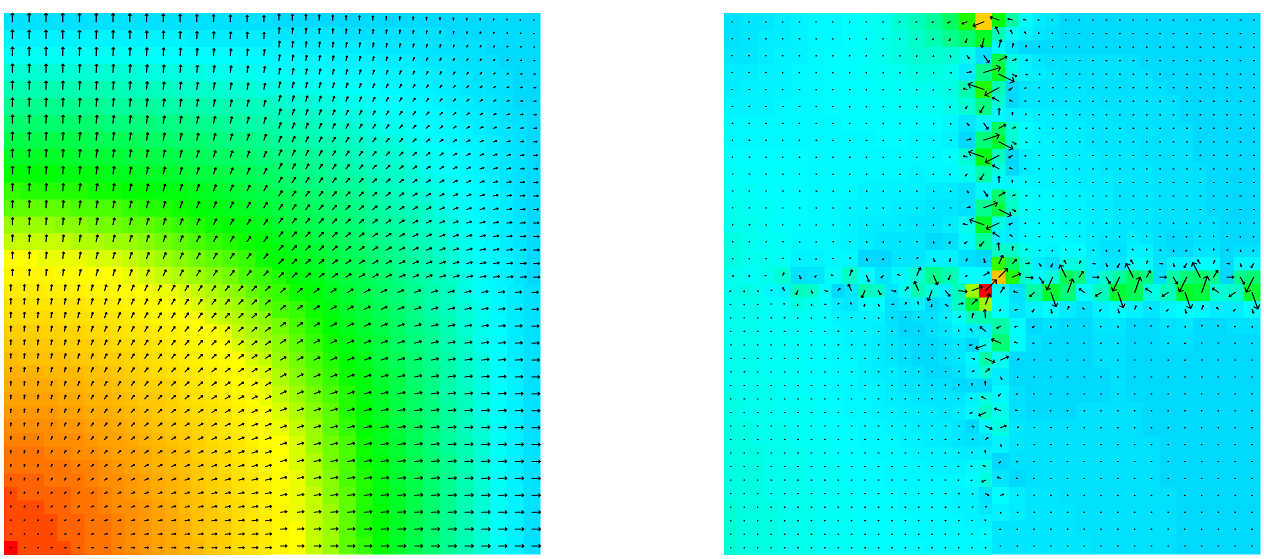
mor		*velocity L^2 error*		*vel. L^2 err. Int.*		*velocity L^∞error*		*vel. L^∞err. Int.*	
tar	*test*	C_u	α_u	C_u	α_u	C_u	α_u	C_u	α_u
	58	2.66274	1.77322	1.83728	2.07849	2.00771	1.24683	5.25444	1.90940
	59	0.22929	1.69787	0.20752	2.01600	0.09980	0.94798	0.59001	1.97308
1	64	1.33161	1.78550	1.10879	1.98649	0.46867	0.86836	3.50649	1.97183
	104	0.14553	1.55720	0.08091	1.96243	0.08844	0.88043	0.20929	1.77685
	107	0.01688	1.95120	0.00615	2.06281	0.02860	1.72008	0.01078	1.74192
	110	0.25459	1.46322	0.17284	1.97959	0.17738	0.96047	0.42815	1.89051
	58	2.32761	1.71424	2.02425	2.08969	2.02304	1.24654	6.64471	1.94481
	59	0.26758	1.81129	0.20011	2.00933	0.11920	1.20358	0.57503	1.96715
2	64	1.95992	1.96678	1.13106	1.99507	0.51878	1.25044	3.45547	1.96728
	104	0.18661	1.61436	0.13642	2.10844	0.11574	0.96044	0.82962	2.14136
	107	0.02004	2.00490	0.00796	2.13705	0.02962	1.72968	0.01424	1.81625
	110	0.24784	1.45577	0.17167	1.97908	0.18424	0.96912	0.39475	1.87154
	58	0.20265	0.70098	3.75547	2.08824	0.21370	0.21300	5.03022	1.66747
	59	0.03542	0.75957	0.32442	2.03956	0.03058	0.24059	1.10888	1.98844
3	64	0.78962	1.65509	1.03225	1.97148	0.13897	0.74854	3.40828	1.96204
	104	0.21352	1.00307	0.78365	2.04442	0.18577	0.38702	0.75907	1.47399
	107	0.01509	1.91113	0.00503	1.99090	0.02489	1.66242	0.00767	1.61258
	110	0.04343	0.75239	0.25975	2.03322	0.03563	0.25080	0.78292	1.97619
	58	0.54367	0.78242	8.40882	2.17624	0.44620	0.26164	18.1699	1.86383
	59	0.04109	0.52811	0.42804	1.99718	0.03793	0.04879	1.14213	1.87719
4	64	0.23754	1.15745	1.14140	1.98235	0.05094	0.19525	3.52201	1.97291
	104	0.10682	0.48427	0.42934	1.63100	0.14464	-0.0051	0.36614	0.97479
	107	0.01157	1.78657	0.00446	1.94802	0.01815	1.52227	0.00687	1.59490
	110	0.06674	0.57901	0.54528	2.04151	0.05485	0.06251	1.06262	1.82550

Table 5: Velocity errors for Case 3 Part II



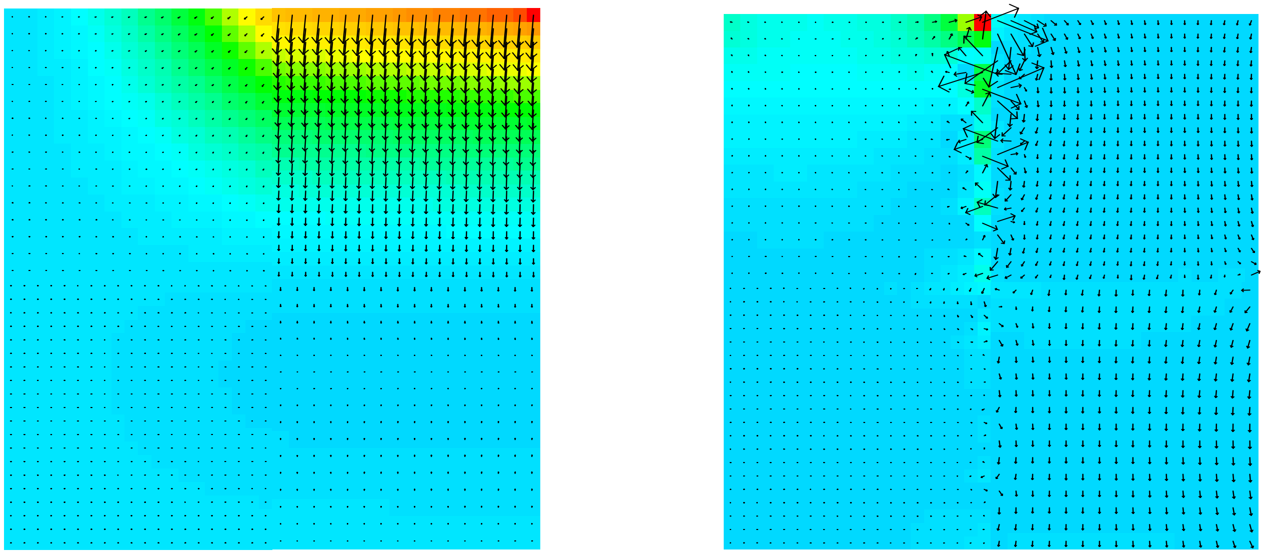
A. *Computed pressure and velocity* B. *Pressure and velocity error*

Figure 2: Solution and error (magnifi ed) for test#58 mortar4



A. *Computed pressure and velocity* B. *Pressure and velocity error*

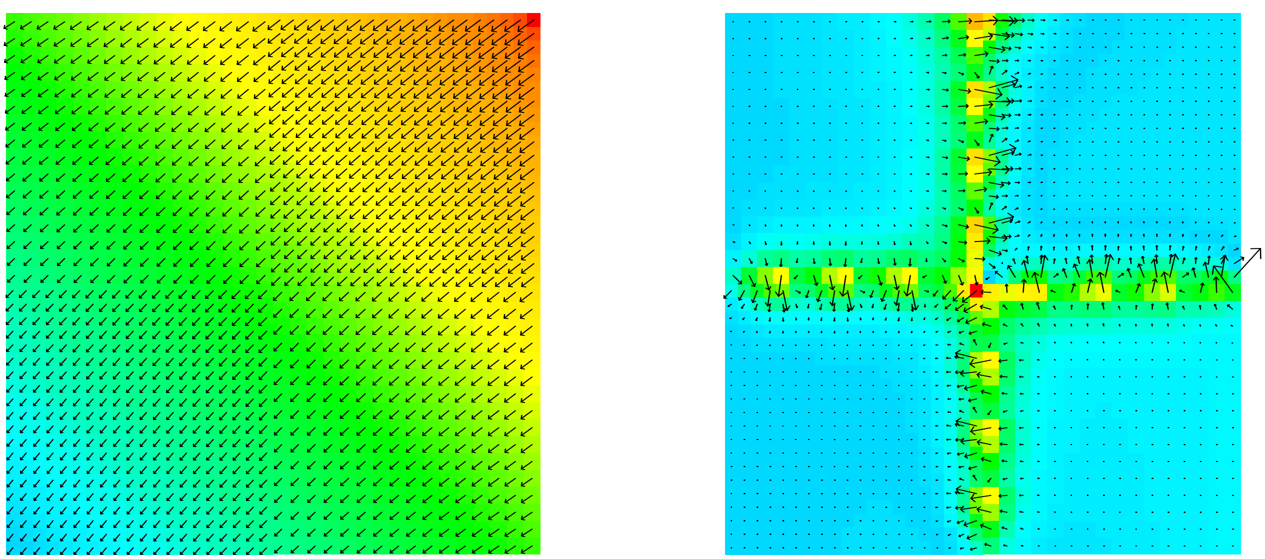
Figure 3: Solution and error (magnifi ed) for test#59 mortar4



A. *Computed pressure and velocity*

B. *Pressure and velocity error*

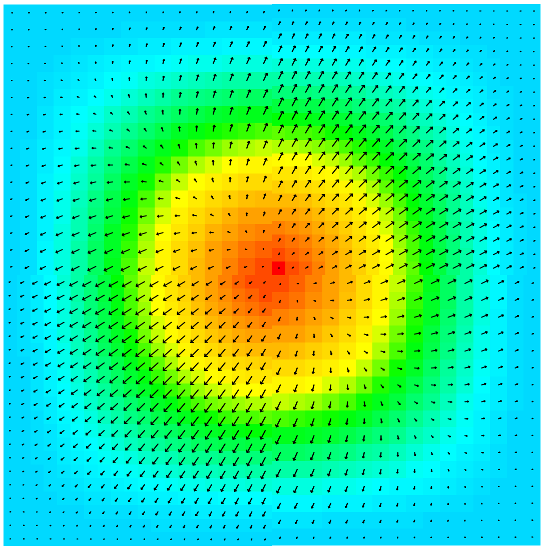
Figure 4: Solution and error (magnifi ed) for test#64 mortar4



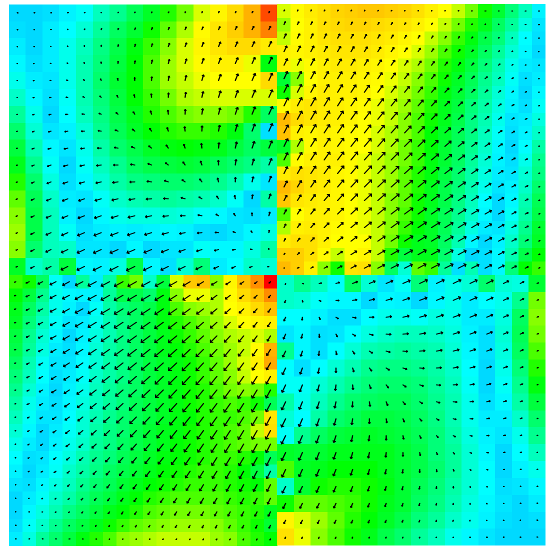
A. *Computed pressure and velocity*

B. *Pressure and velocity error*

Figure 5: Solution and error (magnifi ed) for test#104 mortar4

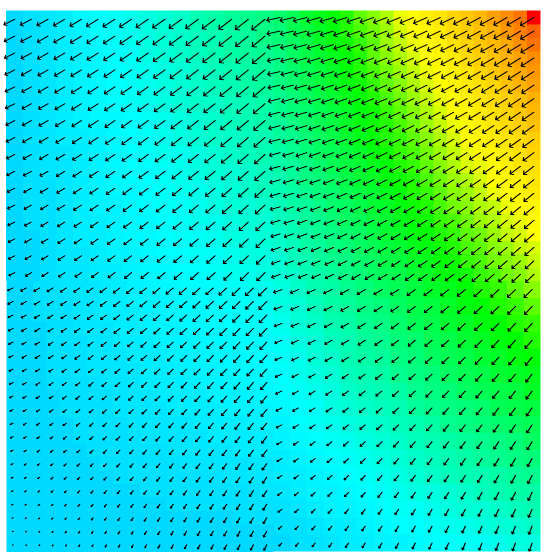


A. *Computed pressure and velocity*

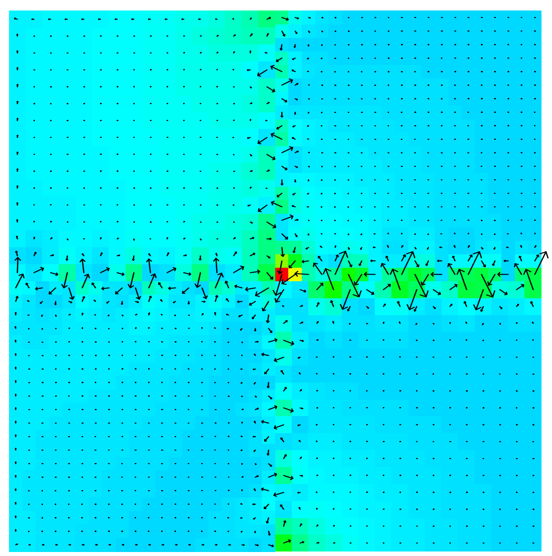


B. *Pressure and velocity error*

Figure 6: Solution and error (magnifi ed) for test#107 mortar4



A. *Computed pressure and velocity*



B. *Pressure and velocity error*

Figure 7: Solution and error (magnifi ed) for test#110 mortar4

3 Error estimates in the case of discontinuous tensor

Because it is hard to find problems with discontinuous tensor and known true solution for which the right-hand side f is a smooth function, we needed to make a bigger modification in the code. Thus, first the code was run for the finest grid and the solution was saved in files. Then the stored solution from this initial run was used to calculate the errors for all coarser grids.

Again all examples are on the unit cube; the domain was divided into four equal subdomains. The initial grid in the case of matching grids was chosen to be 128×128. The initial nonmatching and mortar grids are shown in Table 6.

64×64	80×80
80×80	64×64

Non-matching grids

mortar	1	2	3	4
elements	48	48	16	80

Mortar grids

Table 6: Initial grids for Case 4

In this case different test problems were tested. In problems 70 through 75 the permeability tensors were diagonal with piecewise constant diagonal elements. The prototype for the permeability tensor is

$$K = \begin{pmatrix} a(x,y) & 0 & 0 \\ 0 & a(x,y) & 0 \\ 0 & 0 & a(x,y) \end{pmatrix}$$

where

$$a(x,y) = \begin{cases} 10^n & \text{, if } x < 1/2, y < 1/2 \\ 10^n & \text{, if } x > 1/2, y > 1/2 \\ 1 & \text{, otherwise} \end{cases}$$

For test problem#70,$n = 1$ and then n increments by 1 with each test problem through test#73. For test problem#74

$$a(x,y) = \begin{cases} 10^2 & , x < 1/2, y < 1/2 \\ 10 & , x > 1/2, y > 1/2 \\ 1 & \text{, otherwise} \end{cases}$$

For test problem#75

$$a(x,y) = \begin{cases} 10 & , x < 1/2, y < 1/2 \\ 10^2 & , x > 1/2, y > 1/2 \\ 1 & \text{, otherwise} \end{cases}$$

Test problem#170 is with full tensor

$$K = \begin{pmatrix} a(x,y) & .1a(x,y) & 0 \\ .1a(x,y) & a(x,y) & 0 \\ 0 & 0 & a(x,y) \end{pmatrix}$$

where $a(x,y)$ is as in test problem#70.

The boundary conditions are Dirichlet on the left and right face and Neumann (**no flow**) on the rest of the boundary. For test problems#70,#71,#72, #73,#170

$$p|_{x=0} = 1$$

while for test problems#74 and #75

$$p|_{x=0} = 10$$

For all tests

$$p|_{x=1} = 0$$

The results for this case are in Table 7 and Table 8. Plots of the computed solution and the numerical error for test problems#71 and #170 are shown in Figure 8 and Figure 9. As the results show, because of the strong singularity at the cross-point $(1/2, 1/2)$, there is no superconvergence even in the interior. The maximum rate of convergence for the interior velocity error is of $O(h)$. Therefore to control the error we need some local refinement near this cross-point.

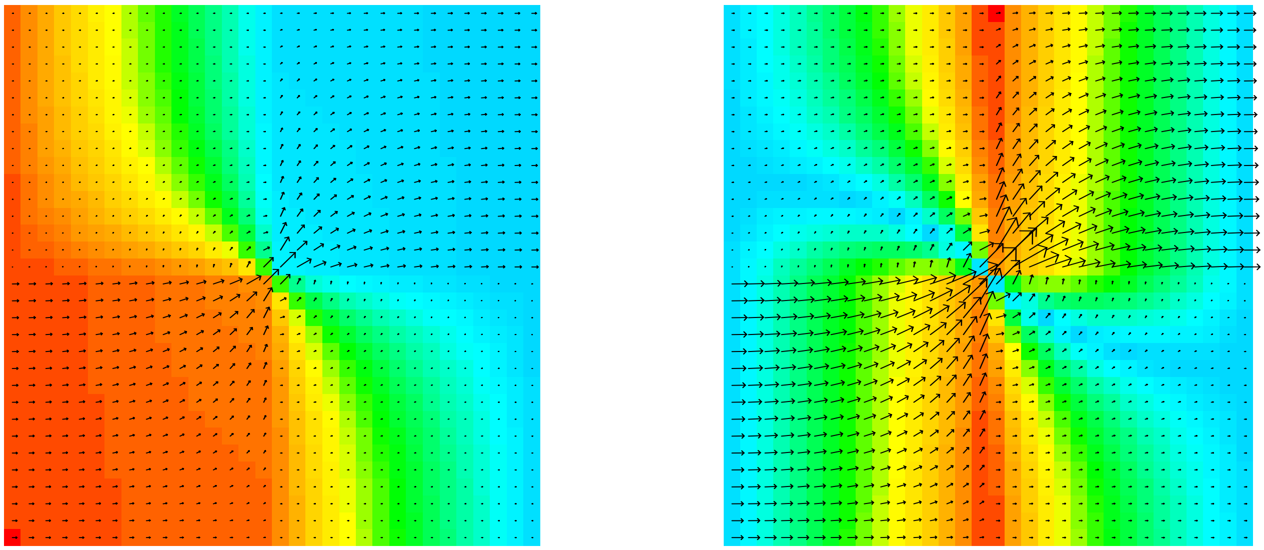
Conclusion: Analyzing all test results in Section 2 and Section 3 we may conclude that interior velocity error depends on the smoothness of the solution in the whole domain Ω, but in more weak sense, and that interior velocity error is better than the velocity error calculated over the whole domain.

		flux error		*pressure L^2error*		*λ error*	
	test	C_f	α_f	C_p	α_p	C_λ	α_λ
matching grids	70	2.63071	0.14077	0.15364	1.05311	0.77287	1.03731
	71	5.74425	0.09065	0.04845	0.67407	0.25210	0.70538
	72	6.48755	0.07803	0.05149	1.03670	0.03005	0.64297
	73	6.56845	0.07630	0.06634	1.11448	0.00306	0.63601
	74	11.6789	0.12253	2.12416	0.91930	8.95074	0.90491
	75	11.6788	0.12253	2.12416	0.91930	8.95073	0.90491
	170	3.46109	0.28146	0.15389	1.05316	0.77345	1.03728
mortar 1	70	2.87058	0.14170	0.20148	1.06326	1.00424	1.04344
	71	6.10402	0.09222	0.05755	0.67933	0.30058	0.70927
	72	6.86057	0.07994	0.06501	1.03706	0.03521	0.64615
	73	6.94565	0.07842	0.08507	1.11449	0.00358	0.63884
	74	12.5847	0.12298	2.66909	0.92547	11.2241	0.90994
	75	12.5850	0.12298	2.66909	0.92547	11.2240	0.90994
	170	4.14805	0.28343	0.20179	1.06329	1.00500	1.04342
mortar 2	70	2.65103	0.11212	0.20343	1.18225	0.99408	1.03561
	71	5.76999	0.06879	0.05757	0.67919	0.27619	0.64336
	72	6.74202	0.07343	0.06499	1.03704	0.03521	0.64616
	73	6.88443	0.07526	0.08507	1.11449	0.00358	0.63921
	74	11.7169	0.09095	2.68899	0.92641	11.1267	0.90294
	75	11.6919	0.09022	2.68410	0.92583	11.1120	0.90253
	170	3.53285	0.20636	0.20374	1.06415	0.99484	1.03559
mortar 3	70	2.64355	0.10357	0.20035	1.05759	0.30514	0.50878
	71	6.01293	0.08727	0.05943	0.70188	0.10796	-0.0291
	72	6.57508	0.07120	0.05660	1.11356	0.09621	-0.1143
	73	5.47429	-0.0072	0.01310	0.54227	0.41838	0.07096
	74	11.7193	0.09411	2.64884	0.92685	3.72005	0.42120
	75	11.7236	0.09421	2.64983	0.92693	3.71763	0.42069
	170	3.52772	0.20854	0.20067	1.05763	0.30547	0.50888
mortar 4	70	2.81382	0.13225	0.20157	1.07050	1.03948	1.05073
	71	6.06643	0.08742	0.05798	0.68121	0.30610	0.71235
	72	6.83717	0.07650	0.06511	1.03735	0.03578	0.64901
	73	6.92509	0.07519	0.08508	1.11451	0.00364	0.64196
	74	12.4547	0.11595	2.70630	0.93120	11.5505	0.91556
	75	12.4545	0.11595	2.70630	0.93120	11.5504	0.91556
	170	3.98991	0.26285	0.20186	1.07051	1.04027	1.05071

Table 7: Errors for Case 4 Part I

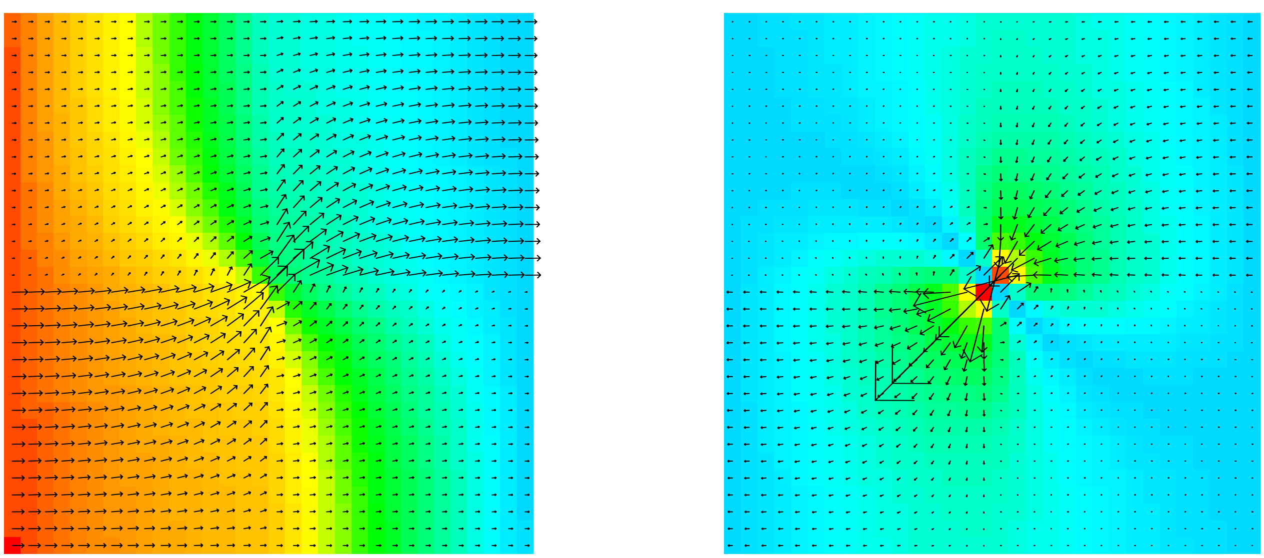
	test	velocity L^2 error		vel. L^2 err. Int.		velocity L^∞ error		vel. L^∞ err. Int.	
		C_u	α_u	C_u	α_u	C_u	α_u	C_u	α_u
matching grids	70	5.02826	0.80760	3.16430	1.13060	2.24790	-0.1318	7.24114	0.98681
	71	19.5352	0.60120	9.33316	0.73624	9.49735	-0.2476	22.2446	0.62329
	72	23.7806	0.55261	11.0525	0.66849	11.9794	-0.2722	26.6700	0.56086
	73	24.2655	0.54673	11.2627	0.66136	12.2589	-0.2759	27.2236	0.83473
	74	92.7638	0.73567	50.8235	0.96406	43.5174	-0.1777	123.086	0.83473
	75	92.7634	0.73567	50.8245	0.96407	43.5213	-0.1777	123.085	0.83473
	170	5.03173	0.80762	3.16635	1.13047	2.24876	-0.1318	7.24688	0.98681
mortar 1	70	6.35111	0.80766	20.5492	1.64331	2.86067	-0.1191	22.0366	1.23625
	71	23.4454	0.60740	13.5818	0.73726	11.5272	-0.2309	28.8878	0.60806
	72	28.1822	0.55933	15.8126	0.66910	14.4447	-0.2539	33.5560	0.54250
	73	28.7154	0.55350	16.0688	0.66135	14.7786	-0.2572	34.1529	0.53564
	74	115.035	0.73837	77.4494	0.96422	54.3464	-0.1640	172.361	0.82124
	75	115.030	0.73836	77.4548	0.96425	54.3452	-0.1640	172.411	0.82136
	170	6.35561	0.80768	4.98072	1.13034	2.86191	-0.1191	10.6355	0.97250
mortar 2	70	6.23825	0.79586	4.92724	1.12636	2.69003	-0.1528	10.0805	0.95716
	71	23.3687	0.60462	13.5857	0.73701	10.9453	-0.2546	28.4547	0.60378
	72	28.0979	0.55838	15.8298	0.66939	14.0227	-0.2649	33.4987	0.54209
	73	28.6897	0.55336	16.0938	0.66192	14.4348	-0.2653	34.0587	0.53482
	74	113.537	0.73029	77.0655	0.96194	51.1419	-0.1942	166.450	0.81096
	75	113.523	0.73029	77.0163	0.96173	51.1524	-0.1941	166.580	.81125
	170	6.24275	0.79589	4.93047	1.12626	2.69127	-0.1528	10.0881	0.95719
mortar 3	70	6.07333	0.79065	20.2001	1.63663	2.89122	-0.1171	21.2636	1.22538
	71	24.7832	0.63538	14.4814	0.76852	12.7548	-0.1867	30.5703	0.63750
	72	27.4119	0.56672	15.1191	0.66777	14.8135	-0.2258	32.1567	0.54070
	73	24.6847	0.49919	13.7162	0.58979	12.0942	-0.3260	31.2366	0.47901
	74	111.059	0.73119	76.1713	0.96367	55.0603	-0.1539	167.261	0.81792
	75	111.091	0.73135	76.3615	0.96482	55.1061	-0.1537	167.482	0.81866
	170	6.07766	0.79067	4.89698	1.12373	2.89246	-0.1171	10.2909	0.96263
mortar 4	70	6.34711	0.80107	5.03046	1.13980	2.83992	-0.1323	10.8196	0.98179
	71	23.6418	0.60633	13.7003	0.73942	11.6003	-0.2349	29.2555	0.61080
	72	28.4218	0.55910	15.9559	0.67129	14.5522	-0.2560	34.0070	0.54555
	73	28.9680	0.55346	16.2248	0.66390	14.8990	-0.2586	34.6187	0.53904
	74	115.837	0.73468	78.5609	0.97004	54.5911	-0.1724	176.555	0.82858
	75	111.091	0.73135	76.3615	0.96482	54.5922	-0.1724	176.507	0.82848
	170	6.35174	0.80110	5.03374	1.13970	2.84132	-0.1323	10.8275	0.98181

Table 8: Errors for Case 4 Part II



A. *Computed pressure and velocity* B. *Pressure and velocity error*

Figure 8: Solution and error (magnifi ed) for test#71 matching grids



A. *Computed pressure and velocity* B. *Pressure and velocity error*

Figure 9: Solution and error (magnifi ed) for test#170 matching grids

4 Influence of the low order term on the constant in the error estimates

Theory indicates that the constant C in the error estimate

$$||p - p_h|| \leq Ch^2$$

depends on K_{max}/K_{min}. We study the dependence of the constant on the low order term α in (2). We tested three groups of problems. For all of them the permeability tensor K was chosen to be diagonal matrix with diagonal elements

$$a(x) = e^{-\beta(x-\frac{1}{2})^2}$$

where β is a real, nonnegative parameter. The values of β and corresponding values (approximately) of the ratio K_{max}/K_{min} are given in Table 9.

β	0	9	18.5	28	37	46
ratio	10^0	10^1	10^2	10^3	10^4	10^5

Table 9: Values of $\frac{K_{max}}{K_{min}}$

Test problems#81 and #82 used true analytic solutions. For test#81 $p = 1 - x$ and for test#82

$$p = x^3y^4 + x^2 + \sin(xy)\cos(y)$$

For test#83 we used again files to save the solution for the finest grid. For this test

$$f \equiv 0, \quad p|_{x=0} = 1, \quad p|_{x=1} = 0 \text{ and } \mathbf{u} \cdot \nu = 0 \text{ on } \Gamma^N.$$

For all test problems we used matching grids and the boundary conditions were Dirichlet on the left and right face and Neumann on the rest of the boundary.

We compared the results when $\alpha = 0$ (no low order term) and $\alpha = 1$. The results are in Table 10 through Table 15.. Plots of the computed solution and the numerical error for $\alpha = 0$ and $\alpha = 1$ are shown in Figure 10 through Figure 15. They show that this method works very well for both cases even if there are big variations of K and that the constant increases very slowly when the ratio goes up.

ratio	α	*flux error*		*pressure L^2error*		*λ error*	
		C_f	α_f	C_p	α_p	C_λ	α_λ
10^0	0	1.520E-05	-1.0184	5.223E-09	-0.4163	2.623E-08	-0.4938
	1	2.520E-05	-0.8979	9.024E-09	-0.3304	4.684E-08	-0.3745
10^1	0	1.18544	1.00582	1.11186	1.99857	1.57440	1.99900
	1	0.88688	1.01210	1.05436	1.99651	1.49051	1.99636
10^2	0	1.72450	1.00050	2.55276	1.98550	3.61007	1.98549
	1	1.13756	1.02740	2.16116	1.97790	3.05399	1.97764
10^3	0	2.53682	1.00673	4.54492	1.96592	6.42815	1.96596
	1	1.92024	1.04562	2.87016	1.94721	4.05745	1.94707
10^4	0	3.10393	1.01237	6.74732	1.94627	9.53957	1.94618
	1	2.49763	1.04793	3.31429	1.95384	4.68600	1.95375
10^5	0	6.35081	2.03410	9.10429	1.92649	12.8703	5.29718
	1	4.45480	2.09359	3.74429	1.95838	1.92635	1.95851

Table 10: Errors for Test#81 Part I

ratio	α	*velocity L^2 error* C_u	α_u	*vel. L^2 err. Int.* C_u	α_u	*velocity L^∞ error* C_u	α_u	*vel. L^∞ err. Int.* C_u	α_u
10^0	0	1.959E-08	-0.8815	2.822E-08	-0.1845	1.595E-08	-1.3747	4.406E-08	-0.4704
	1	1.117E-08	-0.9970	3.896E-08	-0.1997	2.907E-08	-1.2267	9.665E-08	-0.3208
10^1	0	0.91658	2.00282	0.24832	1.97709	1.44057	1.94041	0.58024	1.85035
	1	1.04796	2.00076	0.29905	1.97719	1.71072	1.95797	0.72373	1.86202
10^2	0	1.60093	2.00913	0.26881	1.93611	3.00145	1.96145	0.53670	1.81068
	1	1.87700	2.00235	0.25119	1.91043	3.60231	1.96361	0.39762	1.65979
10^3	0	2.06907	2.00956	0.66296	2.00152	3.53206	1.93215	1.55599	1.95890
	1	2.37246	1.99846	0.32107	1.99301	4.50453	1.93396	0.42730	1.73236
10^4	0	2.51460	2.00756	1.04688	2.03072	3.83614	1.89745	2.75755	2.00510
	1	2.79471	1.99700	0.67433	2.11495	5.05376	1.90684	1.86876	2.06975
10^5	0	2.83901	1.99846	1.30681	2.03978	4.02057	1.86281	3.41880	2.00177
	1	3.11272	1.99131	0.95971	2.12676	5.36164	1.87915	2.31484	2.04601

Table 11: Errors for Test#81 Part II

ratio	α	*flux error* C_f	α_f	*pressure L^2 error* C_p	α_p	*λ error* C_λ	α_λ
10^0	0	0.48058	0.92198	0.29886	1.99667	0.39792	2.00045
	1	0.39681	0.87120	0.27677	1.99533	0.37104	1.99594
10^1	0	1.58922	1.00151	2.06764	1.99694	2.85774	2.00007
	1	1.31857	0.99968	1.94081	1.99468	2.64292	1.99813
10^2	0	2.16148	0.99954	4.54800	1.98593	6.26899	1.98799
	1	1.55603	1.00823	3.86108	1.97652	5.28056	1.97920
10^3	0	3.12151	1.00590	8.23068	1.96967	12.5596	1.97820
	1	2.36137	1.03718	4.90257	1.94078	6.93021	1.94673
10^4	0	3.81211	1.01168	11.8719	1.94853	17.8129	1.97492
	1	3.05579	1.04467	5.39610	1.94485	7.68368	1.94838
10^5	0	9.61315	2.03300	16.4131	1.76500	23.0464	1.71409
	1	6.68535	2.09079	6.10213	1.95496	8.61460	1.95559

Table 12: Errors for Test#82 Part I

ratio	α	*velocity L^2 error* C_u	α_u	*vel. L^2 err. Int.* C_u	α_u	*velocity L^∞ error* C_u	α_u	*vel. L^∞ err. Int.* C_u	α_u
10^0	0	0.51967	1.98672	0.18760	1.99842	1.32760	1.79569	0.90474	1.91797
	1	0.52388	1.97723	0.19538	1.99463	1.27471	1.77775	0.91529	1.91845
10^1	0	1.70894	1.99882	0.68123	1.99665	4.27849	1.91655	2.55625	1.89986
	1	1.87290	1.99771	0.71191	1.99283	4.47469	1.90753	2.84745	1.90317
10^2	0	2.74123	2.00611	0.69310	1.97321	9.01643	1.95499	1.59156	1.72722
	1	3.16910	2.00064	0.71311	1.96552	10.2015	1.95629	2.62397	1.78597
10^3	0	3.46317	2.00788	1.12040	1.99457	11.7216	1.95136	2.96550	1.92528
	1	3.93653	1.99727	0.69361	1.98080	13.7282	1.95030	1.57091	1.68975
10^4	0	4.16972	2.00827	1.64383	2.02131	13.4855	1.93658	5.16595	1.99412
	1	4.60159	1.99738	1.05977	2.06600	15.9724	1.93683	2.13064	1.81785
10^5	0	4.73005	2.00294	2.02860	2.03314	14.9040	1.91828	6.82549	2.00437
	1	5.14499	1.99477	1.45510	2.09869	17.6947	1.92183	4.56553	2.01597

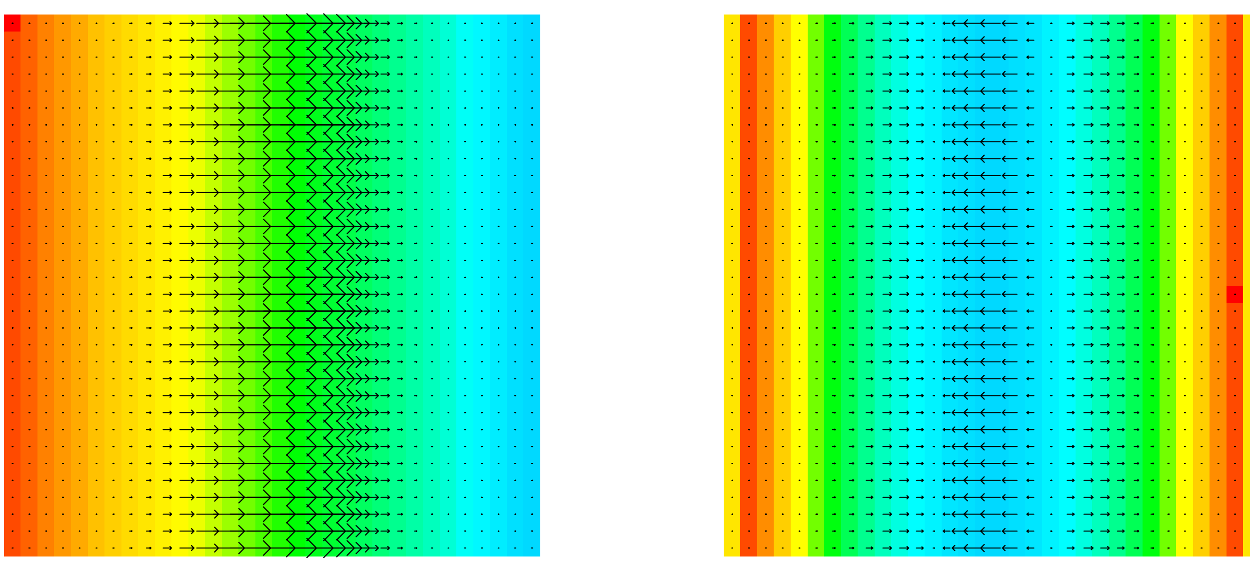
Table 13: Errors for Test#82 Part II

ratio	α	*flux error* C_f	α_f	*pressure* L^2*error* C_p	α_p	λ *error* C_λ	α_λ
	0	4.342E-06	0.38460	6.309E-08	0.33978	3.564E-07	0.19340
10^0	1	3.803E-06	0.38418	6.318E-08	0.35544	3.321E-07	0.20265
	0	1.69532	2.08386	1.02830	2.06002	1.45382	2.05990
10^1	1	1.28574	2.08814	0.98923	2.06111	1.39853	2.06099
	0	1.66590	2.08796	2.56514	2.00134	3.62735	2.00131
10^2	1	0.56148	2.22665	2.41972	2.01461	3.42218	2.01463
	0	0.66009	2.10511	3.59111	1.92632	5.07830	1.92629
10^3	1	0.75200	1.96891	3.24561	1.97698	4.58996	1.97698
	0	0.19089	2.13192	4.01591	1.85285	5.67940	1.85285
10^4	1	0.88100	1.93894	2.32435	1.92743	3.28695	1.92741
	0	0.04673	2.16905	4.09979	1.78221	5.79796	1.78221
10^5	1	0.74871	1.88571	1.42642	1.80294	2.01720	1.80293

Table 14: Errors for Test#83 Part I

ratio	α	*velocity* L^2 *error* C_u	α_u	*vel.* L^2 *err. Int.* C_u	α_u	*velocity* L^∞*error* C_u	α_u	*vel.* L^∞*err. Int.* C_u	α_u
	0	6.315E-07	-0.0915	8.596E-08	0.39917	5.106E-07	-0.7172	2.225E-07	0.15937
10^0	1	6.366E-07	-0.0597	1.033E-07	0.39344	4.386E-07	-0.7236	3.156E-07	0.25133
	0	1.20316	2.08523	0.60161	2.08525	1.08091	2.04545	1.20239	2.08498
10^1	1	1.02448	2.09032	0.47528	2.09072	1.15857	2.02808	0.95384	2.06170
	0	1.17719	2.08771	0.58860	2.08771	1.13300	2.07356	1.17688	2.08757
10^2	1	0.59393	2.13819	0.19821	2.22732	0.71378	1.91463	0.35860	2.08571
	0	0.46627	2.10473	0.23314	2.10473	0.45246	2.09366	0.46596	2.10447
10^3	1	0.50664	1.96340	0.28805	1.97335	0.62015	1.97963	0.61683	1.97954
	0	0.13499	2.13195	0.06750	2.13195	0.13153	2.12240	0.13498	2.13192
10^4	1	0.57140	1.93648	0.31943	1.93825	0.71538	1.95218	0.70779	1.94979
	0	0.03302	2.16883	0.01651	2.16883	0.03205	2.15785	0.03302	2.16882
10^5	1	0.45583	1.87963	0.23836	1.86526	0.61667	1.90274	0.60471	1.89878

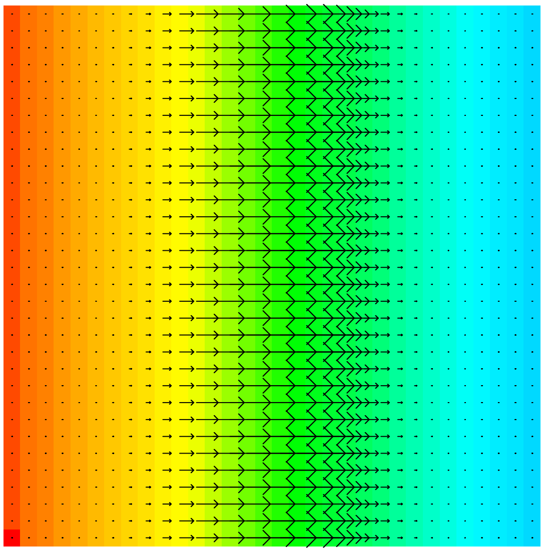
Table 15: Errors for Test#83 Part II



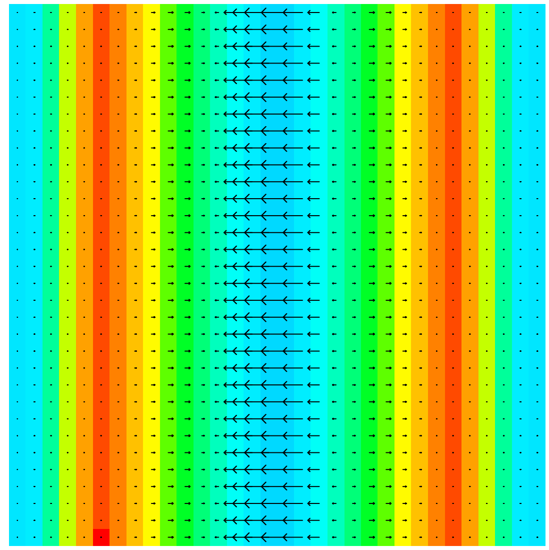
A. *Computed pressure and velocity*

B. *Pressure and velocity error*

Figure 10: Solution and error (magnified) for test#81 $\alpha = 0$ $\beta = 46$ matching grids

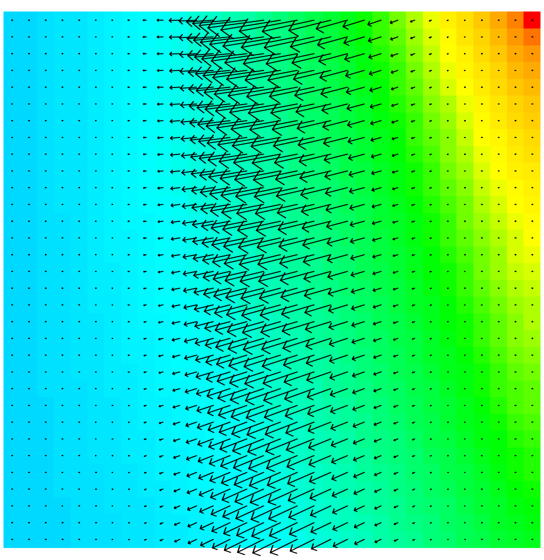


A. *Computed pressure and velocity*

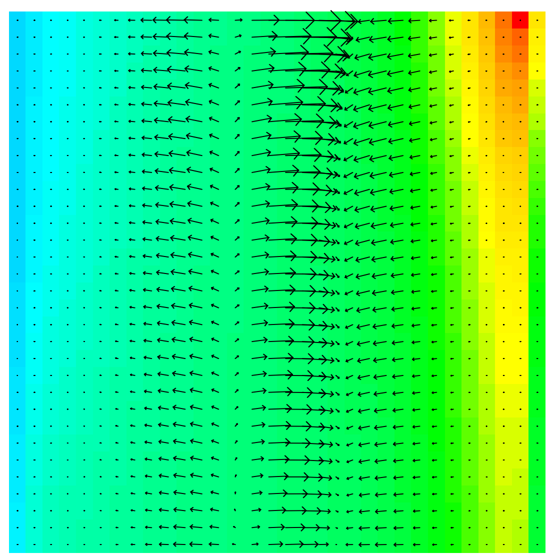


B. *Pressure and velocity error*

Figure 11: Solution and error (magnified) for test#81 $\alpha = 1$ $\beta = 46$ matching grids

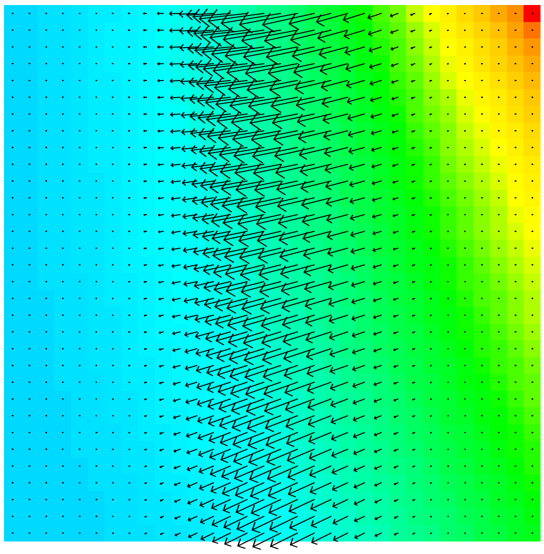


A. *Computed pressure and velocity*

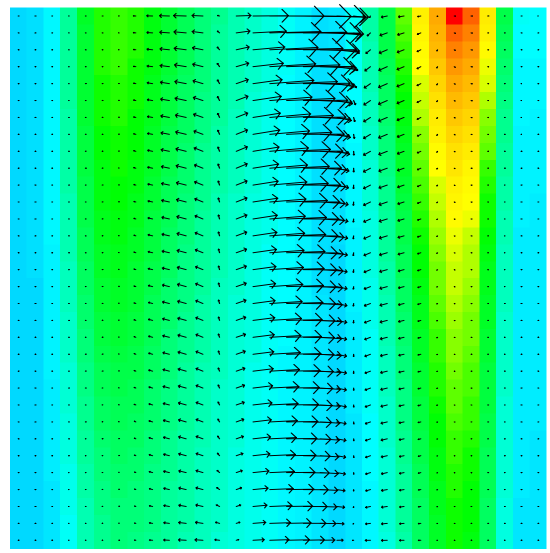


B. *Pressure and velocity error*

Figure 12: Solution and error (magnified) for test#82 $\alpha = 0$ $\beta = 46$ matching grids

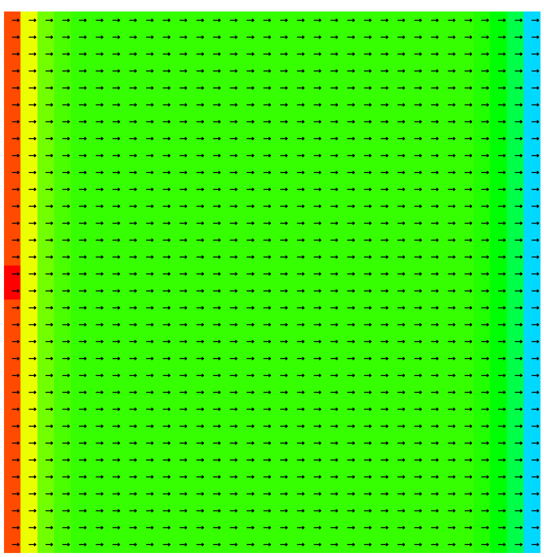


A. *Computed pressure and velocity*

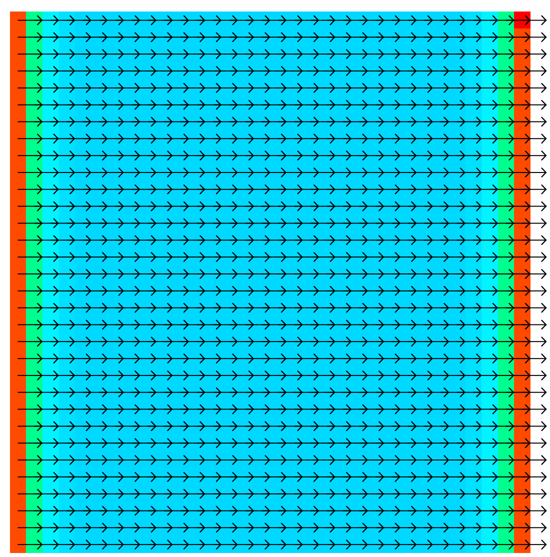


B. *Pressure and velocity error*

Figure 13: Solution and error (magnifi ed) for test#82 $\alpha = 1$ $\beta = 46$ matching grids

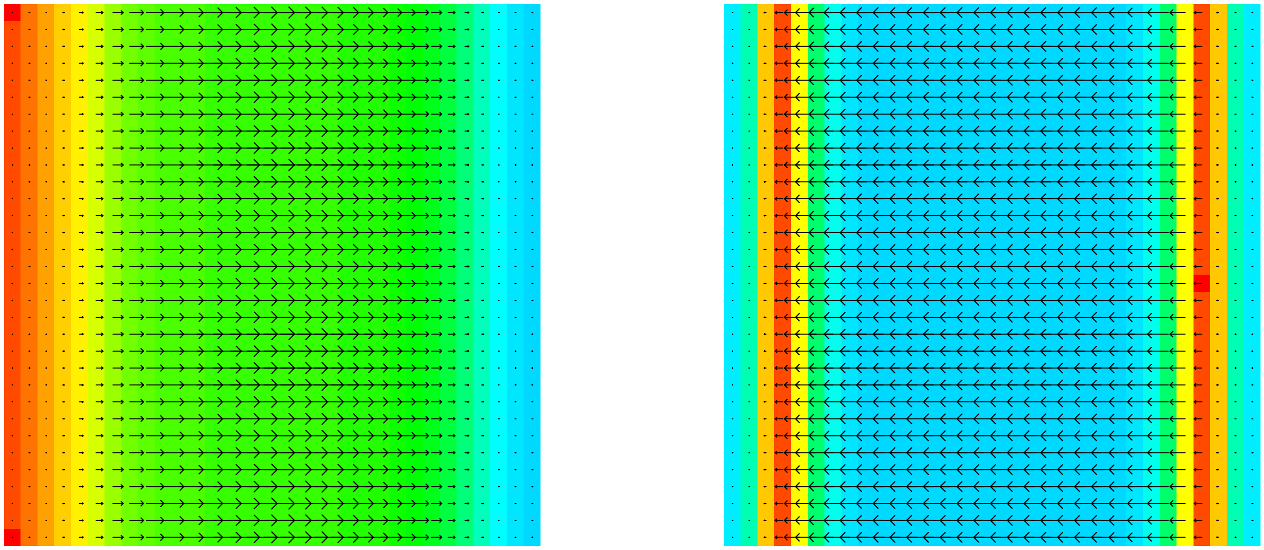


A. *Computed pressure and velocity*



B. *Pressure and velocity error*

Figure 14: Solution and error (magnifi ed) for test#83 $\alpha = 0$ $\beta = 46$ matching grids



A. *Computed pressure and velocity*

B. *Pressure and velocity error*

Figure 15: Solution and error (magnifi ed) for test#83 $\alpha = 1$ $\beta = 46$ matching grids

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