TRUNCATION OF SCALES BY TIME RELAXATION *

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Abstract. We study a time relaxation regularization of flow problems proposed and tested extensively by Stolz and Adams. The aim of the relaxation term is to drive the unresolved fluctuations in a computational simulation to zero exponentially fast by an appropriate and often problem dependent choice of its coefficient; this relaxation term is thus intermediate between a tunable numerical stabilization and a continuum modeling term. Our aim herein is to understand how this term, by itself, acts to truncate solution scales and to use this understanding to give insight into parameter selection.

Key words. energy cascade, similarity theory, turbulence, deconvolution, time relaxation

1. Introduction.

Direct numerical simulation of a $3d$ turbulent flow typically requires $N_{dof}^{NSE} \simeq O(Re^{+9/4})$ mesh points in space per time step, and thus is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport. Thus, various numerical regularizations for truncating the small structures and turbulence models of the large structures are used for simulations seeking to predict flow statistics or averages. The resulting simulations are typically complex with many uncertainties and fitting/tuning parameters whose effects upon the computed solutions are often poorly understood. Thus, it is important to understand how these regularizations and models (and their parameters) act to truncate the scales in a simulated flow to be representable on a computationally feasible grid.

In this report we study one such model/regularization: a time relaxation operator introduced as a numerical regularization by Stolz and Adams, e.g., Stolz, Adams and Kleiser [39], [40], based on theoretical work on regularizations of Chapman-Enskog expansions in Rosenau [32], Schochet and E. Tadmor [35]. This operator aims precisely to truncate the small scales in a solution without altering appreciably the solution's large scales. This regularization operator has many attractive features. It is a lower order perturbation and thus (since the equation does not change order or type) questions of well-posedness and boundary conditions are transparent; it ensures sufficient numerical entropy dissipation for numerical solution of conservation laws, Adams and Stolz [2], p.393; in combination with a large eddy simulation model, it has produced positive results for the Navier-Stokes equations at high Reynolds numbers. It can also be used quite independently of any turbulence model (and has been so used in compressible flow calculations). As a stand alone regularization, it has been successful for the Euler equations for shock-entropy wave interaction and other tests, [2],[39] , [40], [41], including aerodynamic noise prediction and control, Geunanff [18]. Because this term has proven to be widely useful, we isolate its effects by studying the sizes of the persistent scales in the Navier-Stokes equations+Relaxation term. We focus on the expected case when *the Reynolds number is high enough that all dissipation and scale truncation is created by precisely this relaxation term* (up to negligible effects).

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In Section 5.1 we shall examine this assumption and see that it is satisfied provided (essentially) the filter length-scale δ is larger than the Kolmogorov micro-scale and the relaxation parameter $\chi > O(1)$.

To introduce the time relaxation term which, when added to the Navier-Stokes equations, we consider as a continuum model, let $\Omega = (0, L)^3$ and suppose periodic with zero mean boundary conditions are imposed on $\partial\Omega$:

$$\phi(x + Le_j, t) = \phi(x, t) \text{ and } \int_\Omega \phi(x, t)dx = 0 \text{ for } \phi = u, p, f, u_0.$$

A local spacial averaging operator associated with a length-scale δ must be selected and many are possible, e.g., [5], John [21], Sagaut [34]. For specificity, we choose a simple differential filter, Germano [16] (related to a Gaussian, e.g.,[15]): given an L-periodic $\phi(x)$, its average $\overline{\phi}$ is the unique L-periodic solution of

$$-\delta^2 \triangle \overline{\phi} + \overline{\phi} = \phi, \text{ in } \Omega.$$

The correct extension to no-slip boundary conditions is developed in [28].This filtering operation is often denoted $\overline{\phi} = G\phi$ and it will be convenient to let $A := -\delta^2 \triangle + I$. The Nth van Cittert approximate deconvolution operator G_N is defined compactly by

$$G_N \overline{\phi} := \sum_{n=0}^{N} (I - G)\overline{\phi}, N = 0, 1, 2, \cdots.$$

See Section 2.1 for more detail. The (bounded) operator G_N is a approximation to the (unbounded) inverse of the filter G in the sense that for very smooth functions and as $\delta \to 0$

$$\phi = G_N \overline{\phi} + O(\delta^{2N+2}),$$

e.g., Adams and Stolz [1], [38], Dunca and Epshteyn [9], and [5].

The model we consider is to find a L-periodic (with zero mean) velocity and pressure satisfying

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p + \nu \triangle u + \chi(u - G_N \overline{u}) &= f, \text{in } \Omega \times (0, T) \\ u(x, 0) &= u_0(x), \text{ in } \Omega \text{ and} \\ \nabla \cdot u &= 0, \text{in } \Omega \times (0, T) \ . \end{aligned} \tag{1.1}$$

The relaxation coefficient χ must be specified and has units $\frac{1}{time}$. The term $u - G_N \overline{u}$ is a generalized fluctuation included to drive fluctuations below $O(\delta)$ to zero rapidly as $t \to \infty$ without affecting the order of accuracy of the model's solution u as an approximation to the resolved ($\geq O(\delta)$)scales.

REMARK 1.1. *Another option is to use a related formulation of the relaxation term given by $\chi(I - G_N G)^2 u$. There is little difference in the above (linear) case between these two possibilities. In the nonlinear case (Section 5) the difference might be more significant.*

The simplest interesting case is $N = 0$. Here $G_0 \overline{u} = \overline{u}$ represents the part of the velocity that can be represented on an $O(\delta)$ mesh, while $u' := u - \overline{u}$ represents the part of the velocity varying over scales $l \leq O(\delta)$. When $N = 0$ the above model reduces to

$$u_t + u \cdot \nabla u + \nu \triangle u + \nabla p + \chi u' = f, \text{in } \Omega \times (0, T) \ . \tag{1.2}$$

When $N = 0$, $u' = u - \overline{u} = -\delta^2 \triangle \overline{u}$ so the term $\chi u'$ represents a smoothed viscous term and some sort of scale truncation is plausible.

To use time relaxation, the relaxation parameter χ must be chosen. Analytical guidance concerning its appropriate scaling with respect to other problem parameters is essential. In Schochet and Tadmor [35] asymptotic analysis suggested the scaling $\chi \sim C_0 + C_1/\delta$ but this value was found too large in tests reported in Adams and Stolz [2] p. 403. Herein we consider parameter selection for the Navier-Stokes equations as a part of broader issues for the relaxation model, including: What is the length scale of the smallest persistent eddy in the above model's solution? (This length scale for (1.1) corresponds to the Kolmogorov dissipation length scale for a turbulent flow of an incompressible, viscous, Newtonian fluid.) Do solutions of the Navier-Stokes equations + relaxation term exhibit an energy cascade and, if so, what are the details of their energy cascade? And, How does the relaxation term act to truncate the small eddies? Our work herein has been inspired by Muschinsky's study of the Smagorinsky model [29] and Foias, Holm and Titi [11] study of the Camassa-Holm / Navier-Stokes-alpha model. The answers to these questions will come from two simple but powerful tools: a precise energy balance for (1.1) in Section 3 together with Kolmogorov's similarity theory, e.g., [5], [12], [30], [25], [34], suitably adapted. Interestingly, similarity theory in section 5 yields $\chi \sim C\delta^{-\frac{2}{3}}$ which is smaller than the value obtained by Schochet and Tadmor [35] but consistent with it within the accuracy of an asymptotic expansion.

1.1. Summary of results. The results are presented in the following sections with full details. We give here an overview of the main results of this report keeping notation as transparent as possible and describing only the most interesting cases.

Section 3 reviews the analytic framework of the space-periodic problem. Using simple energy estimates (which are similar in spirit and detail to the numerical analysis of penalty methods) we show that the component of the solution of (1.1) fluctuating below $O(\delta)$ must $\to 0$ in $L^2(\Omega \times (0,T))$ as $\chi \to \infty$.This result follows directly from the continuum equations (1.1) and validates the relaxation term as a general computational strategy but it sheds no light into parameter selection or the details of how scales are truncated by the relaxation term. K41 phenomenology is briefly reviewed in the appendix to establish that it is indeed applicable to the case herein of NSE + Time relaxation. In Section 4 we delineate those details by developing a similarity theory for (1.1) following the K-41 Theory of the Navier-Stokes equations. There are several interesting cases, but the most important consequence for practical computing is the following predicted optimal scaling of the relaxation parameter which forces the model's micro-scale $\eta_{model} = O(\delta)$:

$$\chi \simeq \frac{U}{L^{\frac{1}{3}}} 2^{\frac{N+1}{3}} \delta^{-\frac{2}{3}}. \tag{1.3}$$

Note that $\chi = O(\delta^{-\frac{2}{3}}) \to \infty$ as $\delta \to 0$ as required in the analytic estimates of Section 3. For this value of the relaxation term the consistency error of the relaxation term is

$$|\chi(u - G_N \overline{u})| = O(\chi \delta^{2N+2}) = O(\delta^{2N+\frac{4}{3}}).$$

Section 5 considers extension to a nonlinear relaxation term. For the most physically appealing choice of the nonlinear term a heuristic analysis of Lilly [27], famous in the large eddy simulation community, is adapted to give a prediction of an optimal

χ based upon a different but equally valid physical principle. Interestingly, with the proper form of relaxation parameter, this analysis also yields the scaling: $\chi = O(\delta^{-\frac{2}{3}})$ as $\delta \to 0$. Section 6 collects conclusions and open problems.

2. Preliminaries. The de-convolution problem is central in both image processing, [4] and turbulence modeling in large eddy simulation, [5], [17], [23], [26], [24]. The basic problem in approximate de-convolution is: given $\overline{u}$ find *useful* approximations of u. In other words, solve the following equation for an approximation which is appropriate for the application at hand

$$G u = \overline{u}, \text{ solve for } u.$$

For most averaging operators, G is symmetric and positive semi-definite. Typically, G is not invertible or at least not stably invertible due to small divisor problems. Thus, this de-convolution problem is ill-posed.

2.1. The van Cittert Algorithm. The de-convolution algorithm we consider was studied by van Cittert in 1931. For each $N = 0, 1, ...$ it computes an approximate solution u_N to the above de-convolution equation by N steps of a fixed point iteration, [4]. Rewrite the above de-convolution equation as the fixed point problem:

$$\text{given } \overline{u} \text{ solve } u = u + \{\overline{u} - G u\} \text{ for } u.$$

The de-convolution approximation is then computed as follows.

ALGORITHM 2.1 (van Cittert approximate de-convolution algorithm). $u_0 = \overline{u}$,
for n=1,2,...,N-1, perform
$u_{n+1} = u_n + \{\overline{u} - G u_n\}$

Clearly, this is nothing but the first order Richardson iteration for the operator equation $G u = \overline{u}$ involving a possibly non-invertible operator G. Since the de-convolution problem is ill posed, convergence as $N \to \infty$ is not expected.

DEFINITION 2.2. *The Nth van Cittert approximate deconvolution operator* $G_N : L^2(\Omega) \to L^2(\Omega)$ *is the map* $G_N : \overline{u} \to u_N$ *, or* $G_N(\overline{u}) = u_N$ *.*

H_N *denotes the map* $H_N : L^2(\Omega) \to L^2(\Omega)$ *by* $H_N(\phi) := G_N G \phi = G_N \overline{\phi}$.

By eliminating the intermediate steps, it is easy to find an explicit formula for the N^{th} de-convolution operator G_N :

$$G_N \phi := \sum_{n=0}^{N} (I - G)^n \phi. \tag{2.1}$$

For example, the approximate de-convolution operator corresponding to $N = 0, 1, 2$ are $G_0 \overline{u} = \overline{u}$, and $G_1 \overline{u} = 2\overline{u} - \overline{\overline{u}}$, and $G_2 \overline{u} = 3\overline{u} - 3\overline{\overline{u}} + \overline{\overline{\overline{u}}}$. The consistency error of G_N as an approximate inverse of G is known to be $O(\delta^{2N+2})$.

LEMMA 2.3 (Error in approximate de-convolution). *For any* $\phi \in L^2(\Omega)$,

$$\begin{aligned} \phi - G_N \overline{\phi} &= (I - A^{-1})^{N+1} \phi \\ &= (-1)^{N+1} \delta^{2N+2} \triangle^{N+1} A^{-(N+1)} \phi \\ &= O(\delta^{2N+2}) \textit{ as } \delta \to 0 \textit{ for smooth } \phi. \end{aligned} \tag{2.2}$$

Proof. See [9] , [5] . □

3. Energy estimates. Recall that we impose the zero mean condition $\int_\Omega \phi dx = 0$ on $\phi = u, p, f$, and u_0. We can thus expand the fluid velocity in a Fourier series

$$u(\mathbf{x}, t) = \sum_{\mathbf{k}} \widehat{u}(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}, \text{ where } \mathbf{k} = \frac{2\pi\mathbf{n}}{L} \text{ is the wave number and } \mathbf{n} \in \mathbb{Z}^3.$$

The Fourier coefficients are given by

$$\widehat{u}(\mathbf{k}, t) = \frac{1}{L^3} \int_\Omega u(x, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}.$$

Magnitudes of $\mathbf{k}, \mathbf{n}$ are defined by

$$|\mathbf{n}| = \{|n_1|^2 + |n_2|^2 + |n_3|\}^{\frac{1}{2}}, |\mathbf{k}| = \frac{2\pi|\mathbf{n}|}{L},$$
$$|\mathbf{n}|_\infty = \max\{|n_1|, |n_2|, |n_3|\}, |\mathbf{k}|_\infty = \frac{2\pi|\mathbf{n}|_\infty}{L}.$$

The length-scale of the wave number $\mathbf{k}$ is defined by $l = \frac{2\pi}{|\mathbf{k}|_\infty}$. Parseval's equality implies that the energy in the flow can be decomposed by wave number as follows. For $u \in L^2(\Omega)$,

$$\frac{1}{L^3} \int_\Omega \frac{1}{2} |u(x, t)|^2 dx = \sum_{\mathbf{k}} \frac{1}{2} |\widehat{u}(\mathbf{k}, t)|^2 =$$
$$= \sum_k \left(\sum_{|\mathbf{k}|=\mathbf{k}} \frac{1}{2} |\widehat{u}(\mathbf{k}, t)|^2 \right), \text{where } \mathbf{k} = \frac{2\pi\mathbf{n}}{L} \text{ is the wave number and } \mathbf{n} \in \mathbb{Z}^3.$$

Let $< \cdot >$ denote long time averaging (e.g., Reynolds, [31])

$$< \phi > (\mathbf{x}) := \lim_{T\to\infty} \sup \frac{1}{T} \int_0^T \phi(\mathbf{x}, t) dt. \tag{3.1}$$

DEFINITION 3.1. *The kinetic energy distribution functions are defined by*

$$E(k, t) = \frac{L}{2\pi} \sum_{|\mathbf{k}|=k} \frac{1}{2} |\widehat{u}(\mathbf{k}, t)|^2, \textit{ and}$$
$$E(k) \ : \ =< E(k, t) >,$$

Parseval's equality thus can be rewritten as

$$\frac{1}{L^3} \int_\Omega \frac{1}{2} |u(x, t)|^2 dx = \frac{2\pi}{L} \sum_k E(k, t), \text{ and}$$
$$< \frac{1}{L^3} \int_\Omega \frac{1}{2} |u(x, t)|^2 dx >= \frac{2\pi}{L} \sum_k E(k).$$

The analysis of time relaxation involves dimensional analysis coupled with precise mathematical knowledge of (1.1) 's kinetic energy balance. This requires information on the action of the operator H_N.

LEMMA 3.2. *Define the bounded linear operator $H_N : L^2(\Omega) \to L^2(\Omega)$ by $H_N\phi = G_N G\phi$. Then, H_N and $I - H_N$ are both symmetric, positive semi-definite operators on $L_0^2(\Omega)$. For $u \in L_0^2(\Omega)$*

$$\int_\Omega (u - H_N u) \cdot u dx \geq 0, \int_\Omega (H_N u) \cdot u dx \geq 0.$$

Proof. Both H_N and G_N are functions of the SPD operator G so symmetry is immediate and positivity is easily established in the periodic case by a direct calculation using Fourier series. To begin, expand $u(\mathbf{x}, t) = \sum_{\mathbf{k}} \widehat{u}(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}$, where $\mathbf{k} = \frac{2\pi\mathbf{n}}{L}$ is the wave number and $\mathbf{n} \in \mathbb{Z}^3$. Then, by direct calculation using Parseval's equality

$$\frac{1}{2L^3} \int_\Omega (H_N u) \cdot u dx = \frac{2\pi}{L} \sum_k \widehat{H}_N(k) E(k, t) \text{ , where}$$

$$\widehat{H}_N(k) = \frac{1}{1 + z^2} \sum_{n=0}^{N} (1 - \frac{1}{1 + z^2})^n \text{ , where } z = \delta k.$$

The expression for $\widehat{H}_N(k)$ can be simplified by summing the geometric series. This gives

$$\widehat{H}_N(k) = 1 - (\frac{z^2}{1 + z^2})^{N+1} \text{ , where } z = \delta k \text{ .}$$

Since z is real, $0 \leq \frac{z^2}{1+z^2} \leq 1$, and $0 \leq 1 - (\frac{z^2}{1+z^2})^{N+1} \leq 1$. Thus we have shown

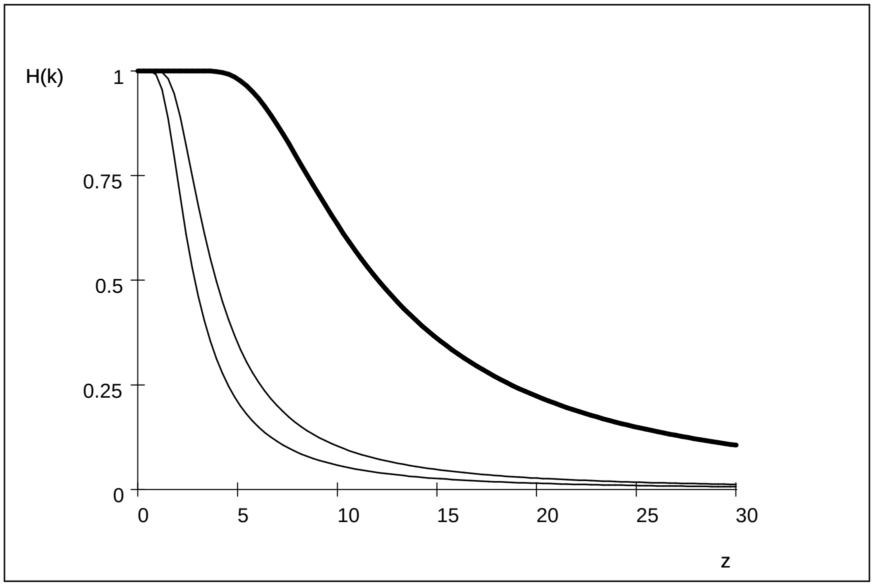
$$0 \leq \int_\Omega (H_N u) \cdot u dx \leq \int_\Omega |u|^2 dx.$$

Similarly, we show $0 \leq 1 - \widehat{H}_N(k) \leq 1$ and

$$0 \leq \int_\Omega (u - H_N u) \cdot u dx \leq \int_\Omega |u|^2 dx,$$

which completes the proof. □

It is insightful to plot the transfer function $\widehat{H}_N(k) = 1 - (\frac{z^2}{1+z^2})^{N+1}$ for a few values of N. We do so below for $N = 5, 10, 100$.



Transfer function $\widehat{H}_N(\cdot)$, $N = 5, 10, \mathbf{100}$.

Examining these graphs, we observe that $H_N(u)$ is very close to u for the low frequencies/largest solution scales and that $H_N(u)$ attenuates small scales/high frequencies. The breakpoint between the low frequencies and high frequencies is somewhat arbitrary. The following is convenient for our purposes and fits our intuition of an approximate spectral cutoff operator[1].

Definition 3.3 (Cutoff-Frequency). *The cutoff frequency of H_N is*

$$k_c := \textit{greatest integer}(\widehat{H}_N^{-1}(\frac{1}{2})).$$

In other words, the frequency for which $\widehat{H}_N$ most closely attains the value $\frac{1}{2}$.

From the above explicit formulas, it is easy to verify that the cutoff frequency grows to infinity slowly as $N \to \infty$ for fixed δ and as $\delta \to 0$ for fixed N. Other properties of the operator $H_N(\cdot)$ follow similarly easily from its transfer function.

Proposition 3.4. *H_N is a compact operator. Let Π_N denote the orthogonal L^2 projection into span$\{e^{i\mathbf{k}\cdot\mathbf{x}} : |\mathbf{k}| \leq k_c\}$.For all $u \in L^2(\Omega)$:*

$$(H_N u, u)_{L^2(\Omega)} \geq C||\Pi_N u||^2, \qquad (3.2)$$
$$(u - H_N u, u)_{L^2(\Omega)} \geq C||(I - \Pi_N)u||^2.$$

Proof. Compactness follows since $\widehat{H}_N(k) \to 0$ as $k \to \infty$. The second and third claims follow from the definition of the cutoff frequency, the explicit formula for the transfer function and a calculation. □

The theory of (1.1) begins, like the Leray theory of the Navier-Stokes equations, with a clear global energy balance;

[1] For periodic problems, spectral cutoff is, of course, best done by spectral cutoff. The van Cittert deconvolution operators give an approximate spectral cutoff operator that is computationally cheap and extends to the non-periodic case.

Proposition 3.5. *Let* $u_0 \in L^2_0(\Omega), f \in L^2(\Omega \times (0,T))$, *and* $\int_\Omega f(x,t)dx = 0$. *For* $\delta > 0$, *let the averaging be* $(-\delta^2\triangle + 1)^{-1}$. *There exists a weak solution to (1.1) which is unique if it is additionally a strong solution. If* u *is a strong solution of (1.1) ,* u *satisfies*

$$\frac{1}{L^3}\frac{1}{2}\left\|u(t)\right\|^2 + \int_0^t \frac{1}{L^3}\int_\Omega \nu|\nabla u|^2 + \chi(u - H_N u)\cdot u dx dt' = \frac{1}{L^3}\frac{1}{2}\left\|u_0\right\|^2 + \int_0^t \frac{1}{L^3}\int_\Omega f \cdot u dx dt'.$$

The above energy bound with equality replaced by "$\leq$*" is also satisfied by weak solutions.*

Proof. The model (1.1) is a lower order, linear perturbation of the Navier Stokes equations so this follows the Navier-Stokes case very closely, e.g., Galdi [13], [14] for a clear and beautiful presentation. For example, for the energy equality, multiply (1.1) by u, integrate over the domain Ω, then integrate from 0 to t. □

Remark 3.6. *By the above lemma and energy estimate, the model's relaxation term thus extracts energy from resolved scales. Thus, we can define an energy dissipation rate induced by time relaxation for (1.1) as*

$$\varepsilon_{\ model}(u)(t) := \frac{1}{L^3}\int_\Omega \chi(u - H_N u)\cdot u dx \tag{3.3}$$

The models kinetic energy is the same as for the Euler equations

$$E_{\ model}(u)(t) := \frac{1}{L^3}\frac{1}{2}\left\|u(t)\right\|^2 \tag{3.4}$$

The following analytic estimate of the effect of the relaxation term follows easily from the above energy estimate.

Theorem 3.7. *Let* u *be a weak solution of (1.1). If* $u_0 \in L^2(\Omega)$ *and* $f \in L^2(0,T;L^2(\Omega))$ *then there is a* $C = C(u_0, f, T)$ *such that*

$$\int_{\Omega\times(0,T)} |(I - \Pi_N)u|^2 dx dt \leq \frac{C}{\chi}, \tag{3.5}$$

and thus $(I - \Pi_N)u \to 0$ *in* $L^2(\Omega \times (0,T))$ *as* $\chi \to \infty$.

Proof. With the stated regularity of the body force, we may use the Cauchy-Schwarz inequality in the RHS of the energy inequality and apply Gronwall's inequality. After this, drop every term on the LHS except the time relaxation term giving

$$\int_0^T \int_\Omega \chi(u - H_N u)\cdot u dx dt' \leq C(u_0, f, T).$$

The result follows from this and the previous proposition. □

4. A similarity theory of time relaxation. We consider now the Navier-Stokes equations with time relaxation at a high enough Reynolds number and large enough relaxation coefficient that viscous dissipation is negligible. The first question is: *Does the time relaxation term induce a truncation of persistent solution scales?*

This question is linked to another: *Does the NSE + time relaxation share the common features of the Navier-Stokes equations which make existence of an energy cascade likely?* Since (1.1) has the same nonlinearity as the Navier-Stokes equations, the conditions remaining are that (i) the solution satisfies an energy equality in which its kinetic energy and energy dissipation are readily discernible, and (ii) in the absence of relaxation (for $\chi = 0$) the model's kinetic energy is conserved through a large ranger of scales/wave-numbers. Since both conditions are satisfied we are proceed to develop a quantitative similarity theory of (1.1) , along the lines of the K-41 theory of turbulence.

Since the time relaxation term is not scale invariant, it is critical to formulate the problem in a way that is as simple, clear and physically correct as possible. The first step is to find the model's equivalent of the large scales' Reynolds number of the Navier-Stokes equations. Recall the Reynolds number for the Navier Stokes equations is, in simplest terms, the ratio of nonlinearity to viscous terms action on the largest scales:

$$\text{for the NSE: } Re \simeq \frac{|u \cdot \nabla u|}{|\nu \triangle u|} \simeq \frac{U\frac{1}{L}U}{\nu\frac{1}{L^2}U} = \frac{UL}{\nu}.$$

The NSE's Reynolds numbers with respect to the smallest scales is obtained by replacing the large scales velocity and length by their small scales equivalent as in $Re_{small} = \frac{u_{small}\eta}{\nu}$. To proceed we must find the physically appropriate and mathematically analogous quantity for the NSE equations + time relaxation. Again, this derivation is under the assumption that viscous dissipation is negligible compared to dissipation due to time relaxation.

Proceeding analogously, it is clear that the ratio of nonlinearity to dissipative effects should be the analogous quantity, and it should correspond to

$$R_N \simeq \frac{|u \cdot \nabla u|}{|\chi(u - H_N u)|}.$$

For example, if $N = 0$, and keeping in mind that for the large scales $(\frac{\delta}{L})^2 << 1$, then we have

$$\begin{aligned} R_0 &\simeq \frac{|u \cdot \nabla u|}{|\chi(u - \overline{u})|} = \frac{|u \cdot \nabla u|}{|\chi\delta^2\triangle\overline{u}|} = \\ &= \frac{|u \cdot \nabla u|}{|\chi\delta^2\triangle(-\delta^2\triangle + 1)^{-1}u|} \simeq \frac{U\frac{1}{L}U}{\chi\delta^2\frac{1}{L^2}(\frac{\delta^2}{L^2} + 1)^{-1}U} \\ &= \frac{LU}{\chi\delta^2}(\frac{\delta^2}{L^2} + 1) \simeq \frac{LU}{\chi\delta^2} \end{aligned}$$

In the general case, and using Lemma 2.3, we have

$$\begin{aligned} R_N &\simeq \frac{|u \cdot \nabla u|}{|\chi(u - H_N u)|} \simeq \frac{U^2\frac{1}{L}}{\chi\delta^{2N+2}(\frac{1}{L^2})^{N+1}(\frac{\delta^2}{L^2} + 1)^{-(N+1)}U} \\ &= \frac{L^{2N+1}U}{\chi\delta^{2N+2}}(\frac{\delta^2}{L^2} + 1)^{N+1} \simeq \frac{L^{2N+1}U}{\chi\delta^{2N+2}}. \end{aligned}$$

This parameter definition can also be obtained by non-dimensionalization. For example, for $N = 0$, denoting the non-dimensionalized quantities with an over-$\hat{}$, we non-dimensionalize in the usual manner and obtain the following system. The term R_0 multiplies is $O(1)$ for the large scales-as it should be after non-dimensionalization.

$$\widehat{u}_t + \widehat{u} \cdot \widehat{\nabla}\widehat{u} + \widehat{\nabla}\widehat{p} + \widehat{\nu}\widehat{\triangle}\widehat{u} + R_0^{-1}(\frac{\widehat{u} - \widehat{\overline{u}}}{(\frac{\delta}{L})^2}) = \widehat{f}, \text{in } \Omega \times (0,T) \ .$$

DEFINITION 4.1. *The non-dimensionalized time relaxation parameter for the NSE equations + time relaxation is*

$$R_N = \frac{L^{2N+1}U}{\chi\delta^{2N+2}} \ , \textit{ for } N = 0, 1, 2, \cdots. \tag{4.1}$$

Next we must form the small scales parameters which measure the ratio of non-linearity to dissipation at the smallest persistent scales. Let u_{small} denote a characteristic velocity of the smallest persistent eddies and let η_{model} denote the length scale associated with them. Then, exactly as above we calculate

$$\begin{aligned} R_{N-small} &\simeq \frac{|u_{small} \cdot \nabla u_{small}|}{|\chi(u_{small} - H_N u_{small})|} \\ &\simeq \frac{u_{small}^2 \frac{1}{\eta_{model}}}{\chi\delta^{2N+2}(\frac{1}{\eta_{model}^2})^{N+1}(\frac{\delta^2}{\eta_{model}^2} + 1)^{-(N+1)} u_{small}} \\ &= \frac{\eta_{model}^{2N+1} u_{small}}{\chi\delta^{2N+2}}(\frac{\delta^2}{\eta_{model}^2} + 1)^{N+1}. \end{aligned}$$

For the small scales it is no longer reasonable to suppose δ is small with respect to η_{model}.

DEFINITION 4.2. *Let η_{model}, u_{small} denote, respectively, a characteristic length and velocity of the smallest persistent structures in the flow. The non-dimensionalized parameter associated with the smallest persistent scales of the NSE equations + time relaxation is*

$$R_{N-small} = \frac{\eta_{model}^{2N+1} u_{small}}{\chi\delta^{2N+2}}(\frac{\delta^2}{\eta_{model}^2} + 1)^{N+1} \tag{4.2}$$

The estimate of the smallest resolved scales is based upon two principles:

$$R_{N-small} = O(1) \ \text{ at length-scale } \eta_{model}$$

and statistical equilibrium in the form *energy input at large scales = dissipation at small scales.* As in the Navier-Stokes equations, the NSE equations + relaxation term's energy cascade is halted by dissipation caused by the time relaxation effects grinding down eddies exponentially fast when $R_{N-small} = O(1)$ at length-scale η_{model}. The largest eddies have energy which scales like $O(U^2)$ and associated time scale $\tau = O(\frac{L}{U})$. The rate of energy transfer/energy input is thus $O(\frac{U^2}{\tau}) = O(\frac{U^3}{L})$ exactly as in the Navier-Stokes case. The dissipation at the smallest resolved scales, estimated carefully, is

$$\begin{aligned} \text{dissipation at small scales } &\simeq \chi(u - H_N u)u \simeq \text{ (by Lemma 2.3)} \\ &\simeq \chi\delta^{2N+2}(\triangle^{N+1}A^{-(N+1)}u)u \simeq \text{ (at the smallest scales)} \\ &\simeq \chi\delta^{2N+2}(\frac{1}{\eta_{model}^2})^{N+1}(1 + \frac{\delta^2}{\eta_{model}^2})^{-(N+1)} u_{small}^2. \end{aligned}$$

These two conditions thus give the pair of equations

$$\frac{\eta_{model}^{2N+1} u_{small}}{\chi \delta^{2N+2}} (\frac{\delta^2}{\eta_{model}^2} + 1)^{N+1} \simeq 1 \text{ , and} \tag{4.3}$$

$$\frac{U^3}{L} \simeq \chi \delta^{2N+2} (\frac{1}{\eta_{model}^2})^{N+1} (1 + \frac{\delta^2}{\eta_{model}^2})^{-(N+1)} u_{small}^2 \ .$$

The first equation gives an estimate of the characteristic velocity of the smallest eddy in terms of the other parameters; solving for u_{small} gives

$$u_{small} \simeq \frac{\chi \delta^{2N+2}}{\eta_{model}^{2N+1} (1 + \frac{\delta^2}{\eta_{model}^2})^{N+1}}.$$

Inserting this value into the second equation gives the following equation determining the model's micro-scale

$$\frac{U^3}{L} \simeq \chi \delta^{2N+2} (\frac{1}{\eta_{model}^2})^{N+1} (1 + \frac{\delta^2}{\eta_{model}^2})^{-(N+1)} [\frac{\chi \delta^{2N+2}}{\eta_{model}^{2N+1} (1 + \frac{\delta^2}{\eta_{model}^2})^{N+1}}]^2. \tag{4.4}$$

This is the fundamental equation determining the model's micro-scale. There are three cases: $\delta < \eta_{model}$, $\delta > \eta_{model}$ and $\delta = \eta_{model}$. This third and last case is the important one.

Case 1: Fully resolved. In this case $\delta < \eta_{model}$ so that $1 + \frac{\delta^2}{\eta_{model}^2} \simeq 1$.
In this case the equation for the micro-scale reduces to

$$\frac{U^3}{L} \simeq \chi \delta^{2N+2} (\frac{1}{\eta_{model}^2})^{N+1} [\frac{\chi \delta^{2N+2}}{\eta_{model}^{2N+1}}]^2,$$

which implies

$$\eta_{model} \simeq (\frac{\chi^3 L}{U^3})^{\frac{1}{6N+4}} \delta^{1+\frac{1}{3N+2}}. \tag{4.5}$$

Case 2: Under resolved. In this case $\delta > \eta_{model}$ so that $1 + \frac{\delta^2}{\eta_{model}^2} \simeq \frac{\delta^2}{\eta_{model}^2}$.
In this case we have

$$\frac{U^3}{L} \simeq \chi \delta^{2N+2} (\frac{1}{\eta_{model}^2})^{N+1} (\frac{\delta^2}{\eta_{model}^2})^{-(N+1)} [\frac{\chi \delta^{2N+2}}{\eta_{model}^{2N+1} (\frac{\delta^2}{\eta_{model}^2})^{N+1}}]^2, \tag{4.6}$$

which gives, after simplification,

$$\eta_{model} \simeq (\frac{U^3}{\chi^3 L})^{\frac{1}{2}}. \tag{4.7}$$

At this point, we do not know how to interpret this estimate because it predicts that in this case increasing χ *decreases* the model's micro-scale. However, this case is not the expected one in practical computations so perhaps the simple interpretation is that solution scales should be resolved and odd results can occur otherwise.

Case 3: Perfect resolution. In this case $\delta = \eta_{model}$ so that $1 + \frac{\delta^2}{\eta_{model}^2} \simeq 2$.

In this case the interesting question is to determine the choice of relaxation parameter that enforces $\delta = \eta_{model}$. Setting $\delta = \eta_{model}$ and solving for χ gives

$$\chi \simeq \frac{U}{L^{\frac{1}{3}}} 2^{N+1} \delta^{-\frac{2}{3}}. \tag{4.8}$$

When perfectly resolved, the consistency error of the relaxation term (evaluated for smooth flow fields) is, for this scaling of relaxation parameter,

$$|\chi(u - G_N \overline{u})| = O(\chi \delta^{2N+2}) = O(\delta^{2N+\frac{4}{3}}).$$

4.1. Interpreting the assumption that viscous dissipation is negligible. Our assumption that viscous dissipation is negligible compared to dissipation caused by time relaxation holds provided the Kolmogorov micro-scale for the Navier-Stokes equations is smaller than the model's micro-scale induced by the relaxation term. This is because the K41 theory is asymptotic at infinite Reynolds number meaning that viscous dissipation is considered negligible at scales above the micro-scale. Thus, one tenant of K41 is that above the Kolmogorov micro-scale the NSE acts like the Euler equations. At high enough Reynolds number and large enough relaxation parameter, it is certainly plausible that relaxation dominates viscosity and that the latter is negligible. The estimates derived in this section give some insight into how large "large enough" is.

The first interpretation of "large enough" is that $\eta_{model} >> \eta_{Kolmogorov}$. If $\eta_{model} >> \eta_{Kolmogorov}$ then practical considerations suggest that we are most commonly in the fully-resolved case or the perfectly resolved case. In the latter, $\eta_{model} = \delta$ and the condition is that $\delta >> \eta_{Kolmogorov}$, i.e., computational resources are insufficient for a DNS. In the fully-resolved case $\eta_{model} > \eta_{Kolmogorov}$ is equivalent to

$$(\frac{\chi^3 L}{U^3})^{\frac{1}{6N+4}} \delta^{1+\frac{1}{3N+2}} > \eta_{Kolmogorov} = \mathrm{Re}^{-\frac{3}{4}} L, \text{ which implies}$$

$$\chi > (\mathrm{Re}^{-\frac{3}{4}} L)^{2N+\frac{4}{3}} \frac{U}{L^{\frac{1}{3}}} \delta^{-2(N+1)} \tag{4.9}$$

In the typical case of $\delta >> \eta_{Kolmogorov}$ and χ large this places almost no constraint upon the relaxation parameter.

The second interpretation is that at $\eta = \eta_{Kolmogorov}$, $\mathrm{Re}_{small} >> R_{N-small}$; this also gives the following mild condition, satisfied by any reasonable scaling of χ, including those derived herein,

$$\chi > \nu(\frac{\delta}{\eta})^{-2N} \delta^{-2} (1 + (\frac{\delta}{\eta})^2)^{N+1}.$$

5. Nonlinear time relaxation. Nonlinear time relaxation mechanisms endeavors to focus the dissipative effects further on smaller scales by localization in physical as well as wave number space. Nonlinear relaxation, especially quadratic relaxation, is also a more physical realization due to the connection to friction (which is quadratic being proportional to the square of the speed and acting to oppose the direction of motion). For this reason we focus on the quadratic case; the extension to a more general nonlinearity is immediate. In the quadratic case, the following is the correct frictional relaxation model: find a L-periodic (with zero mean) velocity and pressure satisfying

$$u(x,0) = u_0(x), \text{ in } \Omega \ ,$$
$$\nabla \cdot u = 0, \text{in } \Omega \times (0,T) \ ,$$
$$u_t + u \cdot \nabla u + \nabla p + \nu \triangle u + \chi^{\frac{3}{2}}(I - H_N)\{|u - H_N u|(u - H_N u)\} = f, \text{in } \Omega \times (0,T) \ .$$

The dissipation in the above is given by

$$\begin{aligned} \varepsilon_{model}(u)(t) &= \frac{1}{L^3}\int_\Omega \chi^{\frac{3}{2}}(I - H_N)\{|u - H_N u|(u - H_N u)\} \cdot u dx \\ &= \frac{1}{L^3}\int_\Omega \chi^{\frac{3}{2}}|u - H_N u|(u - H_N u) \cdot (u - H_N u) dx \\ &= \frac{1}{L^3}\int_\Omega (\chi^{\frac{1}{2}}|u - H_N u|)^3 dx. \end{aligned}$$

Note that $\varepsilon_{model} \geq 0$ precisely because of the form chosen for the nonlinear term[2] .

5.1. Parameter determination via $<\varepsilon>=<\varepsilon_{model}>$. For the special value $\mu = 3$ the derivation of Lilly [27] for the Smagorinsky model can be adapted to nonlinear time relaxation. This derivation is heuristic but gives another useful indication of the scaling of the relaxation parameter with respect to the other model parameters. Since this analysis is very well known in large eddy simulation, e.g., [30], [34], [5], we give an abbreviated summary here. The idea of Lilly is to equate $<\varepsilon>=<\varepsilon_{model}>$ and evaluate the RHS by assuming (among other things) that the velocity field arises from homogeneous isotropic turbulence. To use energy spectrum information a further assumption is needed that the following two are of comparable orders of magnitude:

$$<||u - H_N u||^3_{L^3(\Omega)}> \simeq <||u - H_N u||^2_{L^2(\Omega)}>^{\frac{3}{2}} .$$

Under these assumptions we calculate

$$\begin{aligned} <\varepsilon_{model}> &= [\int_{k_{\min}}^{k_{\max}} (1 - \widehat{H_N(k)})^2 E(k) dk]^{\frac{3}{2}}, \text{ where} \\ <\varepsilon_{model}> &= (4\pi\chi)^{\frac{3}{2}}[\sum_k (1 - \widehat{H_N(k)})^2 E(k)]^{\frac{3}{2}}. \end{aligned}$$

Using the formula for $\widehat{H_N(k)}$ and $E(k) \simeq \alpha\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$, $\alpha =$ Kolmogorov constant, we get

$$<\varepsilon_{model}> = (4\pi\chi\alpha)^{\frac{3}{2}} <\varepsilon> [\sum_k (\frac{(\delta k)^2}{1 + (\delta k)^2})^{2N+2} k^{-\frac{5}{3}}]^{\frac{3}{2}}$$

The above infinite series is convergent and its value depends upon both δ and N. We are interested in its asymptotics as $\delta \to 0$ for N fixed. Its sum can be majorized by a few initial terms plus a convergent improper integral. With this majorization, the infinite series is bounded as follows

$$\begin{aligned} \sum_k (\frac{(\delta k)^2}{1 + (\delta k)^2})^{2N+2} k^{-\frac{5}{3}} &\leq \beta_N \delta^{\frac{2}{3}} + \text{higher order terms in } \ \delta \\ &\simeq \beta_N \delta^{\frac{2}{3}}. \end{aligned}$$

[2] The choice of relaxation parameter ($\chi^{\frac{3}{2}}$ instead of χ) is motivated by the resemblance of this last expression with the one arising in the linear case.

The value of β_N can be estimated by the value of the integral

$$\beta_N \simeq \int_0^\infty (\frac{z^2}{1+z^2})^{2N+2} z^{-\frac{5}{3}} dz$$

It can be shown that $\beta_N = O(1)$ for the first few values of N and is decreasing as N increases[3]. Some estimates of values of β_N , obtained by numerical integration, are given below.

N =0	1	2	3	4	5	6	7	8
β_N $\simeq$1.21	0.895	0.766	0.689	0.635	0.596	0.564	0.538	0.517

Setting $< \varepsilon > = < \varepsilon_{model} >$ thus gives the following value of the relaxation parameter (after simplification)

$$\chi = [4\pi\alpha\delta^{\frac{2}{3}}\beta_N]^{-1} .$$

These calculations reiterate the scaling of the linear case, obtained by a different physical principle, in the nonlinear case

$$\chi \sim O(\delta^{-\frac{2}{3}}). \tag{5.1}$$

6. Conclusions and open problems. The Navier-Stokes equations + time relaxation possesses an energy cascade that truncates the energy spectrum at a point that depends upon the relaxation parameter, the global velocity and length scale and the averaging radius δ . This time relaxation term does not dissipate appreciable energy for the resolved scales of the flow for N large enough. The action of this time relaxation term is to induce a micro-scale, analogous to the Kolmogorov micro-scale in turbulence, and to trigger decay of eddies at the model's micro-scale. The extra dissipation at the cutoff length scale induced by time relaxation must reduce the number of degrees of freedom needed (per time step) for a 3d turbulent flow simulation. With proper scaling of χ this extra dissipation will also balance the transfer of energy to those scales from the flow's power input and thus prevent a non-physical accumulation of energy around the cutoff length scale as well as force the model's micro-scale to coincide with the averaging radius δ.

With the formula derived herein, $\chi \simeq \frac{U}{L^{\frac{1}{3}}} 2^{N+1}\delta^{-\frac{2}{3}}$, the model's micro-scale is δ and the number of degrees of freedom (per time step) needed for a 3d turbulent flow simulation with the model (1.1) is

$$N_{dof} \simeq (\frac{L}{\delta})^3, \text{ independent of } Re \text{ !}$$

This leads to a *huge* computational speedup using (1.1) over a DNS of

$$(\frac{N_{dof}^{NSE}}{N_{dof}})^{\frac{4}{3}} \simeq (\frac{\mathrm{Re}^{\frac{9}{4}}}{L^3\delta^{-3}})^{\frac{4}{3}} = (\frac{\delta}{L})^4 \,\mathrm{Re}^3 .$$

[3] The proof that β_N is decreasing can be done by differentiation with respect to "N". It is also our instinct as mathematicians to ask the limit of β_N as $N \to \infty$. We have shown in fact that $\beta_N \to 0$ as $N \to \infty$. However interesting this case $N \to \infty$, the important case is N fixed.

Finally, the time relaxation studied herein, since it is a lower order term, is ideal to be used with many other models (e.g., the NSE-alpha model) to reduce further the computational complexity of simulations with them by accelerating the truncation of scales without altering a model's accuracy on the resolved scales. The above value of χ is derived for fully developed, turbulent flow. While it is smaller than other theoretical values, it is also possible that other flow settings, such as transition, would require other, still smaller, values- an important open problem.

There are many other open problems connected to finding rigorous proofs of this description of the effects of time relaxation directly from the Navier-Stokes equations and without assumptions of homogeneity or isotropy. There are also other possible scale-dependent relaxation strategies which should be developed and compared to find the best tool for a given flow problem. It is also important to study the time relaxation operator used in a synthesis with other good models of turbulence. There does not seem to be a clear strategy of developing a general theory of such mixed models so the effect of such combinations must be investigated on a case by case basis. Lastly, we have studied, as a first step, time relaxation as the continuum model (1.1). Simulations are of course performed using a chosen discretization of (1.1). Thus understanding the effects of these terms, when discretized, and performing a rigorous numerical analysis of the combination is a very important next step.

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Appendix. A synopsis of K41 phenomenology.

Turbulent flows consist of three dimensional eddies of various sizes. In 1941, I. Kolmogorov gave a remarkable, universal description of the eddies in turbulent flow by combining a judicious mix of physical insight, conjecture, mathematical analysis and dimensional analysis, e.g., Frisch [12], Pope [30]. In his description, the largest eddies are deterministic in nature. Those below a critical size are dominated by viscous forces, and die very quickly due to these forces. This critical length scale (the Kolmogorov micro-scale) is $\eta = O(Re^{-3/4})$[4] in $3d$. From this estimate, it follows that

[4] The length scale of the smallest persistent eddy is traditionally denoted by η rather than l .

direct numerical simulation of a $3d$ flow thus requires $\Delta x = \Delta y = \Delta z = O(Re^{-3/4})$ giving $O(Re^{+9/4})$ mesh points in space per time step, and thus is often not computationally economical or even feasible. This estimate is based upon existence of an energy cascade in turbulent flow problems and Kolmogorov's above estimate of the micro-scale at the bottom of the energy cascade. Since this energy cascade theory is extended herein (and in other papers as well) beyond the Navier-Stokes equations, the answers to important questions about it must be reviewed.

Why do solutions of the Navier-Stokes equations exhibit an energy cascade? And, should it be expected that solutions of (1.1) have their own energy cascade? The answer to the first question has been understood since the work of L. F. Richardson and I. Kolmogorov. We shall briefly review the answer (which is given also in Chapter 1 of most books on turbulence) because its answer also contains the answer to the second question (which we have developed in this report). The Navier-Stokes equations and their solutions have the following well-known features;

- If $\nu = 0$ the total kinetic energy of the flow is exactly conserved[5]:

$$E(u)(t) = E(u)(0) + \int_0^t \frac{1}{L^3}\int_\Omega f \cdot u d\mathbf{x} dt.$$

- The nonlinearity conserves energy globally (since $\int_\Omega u \cdot \nabla u \cdot u dx = 0$) but acts to transfer energy to smaller scales by breaking down eddies into smaller eddies (for example, if $u \simeq (U \sin(\frac{\pi x_1}{l}), 0, 0)^{tr}$ has wave length l and frequency $\frac{\pi}{l}$ then $u \cdot \nabla u \simeq \frac{U^2 \pi}{2l}(\sin(\frac{\pi x_1}{l/2}), 0, 0)^{tr}$ has shorter wave length $\frac{l}{2}$).
- If $\nu > 0$, them the viscous terms dissipate energy from the flow globally:

$$E(u)(t) + \int_0^t \varepsilon(u)(t')dt' = E(u)(0) + \int_0^t \frac{1}{L^3}\int_\Omega f \cdot u d\mathbf{x} dt, \text{ where } \varepsilon(u)(t') \geq 0.$$

- For Re large the energy dissipation due to the viscous terms is negligible except on very small scales of motion. For example, if $u \simeq (U \sin(\frac{\pi x_1}{l}), 0, 0)^{tr}$ then

$$\text{viscous term on this scale} = -\nu \triangle u \simeq \pi^2 \frac{\nu U}{l^2}(\sin(\frac{\pi x_1}{l}), 0, 0)^{tr}, \text{from which:}$$

$$\text{energy dissipation on this scale } = \varepsilon(u) \simeq \frac{C}{L^3}\frac{\nu U^2}{l^2}.$$

  Thus the nonlinear term dominates and the viscous term is negligible if

$$\frac{U^2}{l} >> \frac{\nu U}{l^2} \text{ , i.e., } \frac{lU}{\nu} >> 1.$$

- The forces driving the flow input energy persistently into the largest scales of motion.

The picture of the energy cascade that results from these effects is thus: *energy is input into the largest scales of the flow. There is an intermediate range in which nonlinearity drives this energy into smaller and smaller scales and conserves the global energy because dissipation is negligible. Eventually, at small enough scales dissipation is nonnegotiable and the energy in those smallest scales is driven to zero exponentially fast.* This is the physical reasoning behind Richardson's famous description:

[5] For the physical reasoning in this appendix and sections 4 it is perhaps appropriate to suppose that the energy equality holds and sidestep the deeper questions concerning weak vs. strong solutions and energy equality vs. energy inequality, e.g., [13], [14].

"Big whirls have little whirls
That feed on their velocity,
And little whirls have lesser whirls,
And so on to viscosity."

Inspired by this description, in 1941 I. Kolmogorov gave a quantitative and universal characterization of the energy cascade (often called the K-41 theory). The most important components of the K-41 theory are the time (or ensemble) averaged energy dissipation rate, ε, and the distribution of the flows averaged kinetic energy across wave numbers, $E(k)$. Given the velocity field of a particular flow, $u(\mathbf{x}, t)$, the (time averaged) energy dissipation rate of that flow is defined to be

$$< \varepsilon >:=< \frac{1}{L^3} \int_\Omega \nu |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} \ > . \tag{A.1}$$

To present the K-41 theory's conclusions, recall that the time-averaged kinetic energy distribution in wave number space, Section 2, is denoted by $E(k)$. The K-41 theory states that at high enough Reynolds numbers there is a range of wave numbers

$$0 < k_{\min} := U\nu^{-1} \leq k \leq < \varepsilon >^{\frac{1}{4}} \nu^{-\frac{3}{4}} =: k_{\max} < \infty, \tag{A.2}$$

known as the inertial range, beyond which the kinetic energy in a turbulent flow is negligible, and in this range

$$E(k) \doteq \alpha < \varepsilon >^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{A.3}$$

where α is the universal Kolmogorov constant whose value is generally believed to be between 1.4 and 1.7 (for example, Wyngaard and Pao [43] found a value of $\alpha = 1.62$ in studies of atmospheric turbulence), k is the wave number and ε is the particular flow's energy dissipation rate. In this formula, the energy dissipation rate $< \varepsilon >$ is the only parameter which differs from one flow to another. Indeed, in Pope [30], figure 6.14 page 235 in [30], the power spectrums of 17 different turbulent flows taken from Saddoughi and Veeravalli [33, SV94] (which also contains the references to the particular experiments) are plotted on log-log plots. The slope of the linear region in this plot has the universal value of $-\frac{5}{3}$ for all 17 turbulent flows, exactly corresponding to the $k^{-\frac{5}{3}}$ law.

We review this argument of Kolmogorov, which is adapted in the next section. It begins with a physical conjecture that:

Conjecture A.1. *The time averaged kinetic energy only depends on the time averaged energy dissipation rate ε and the wave number k .*

Beginning with this, postulate a simple power law dependency of the form

$$E(k) \simeq C < \varepsilon >^a k^b. \tag{A.4}$$

If this relation is to hold the units, denoted by $[\cdot]$ on the LHS must be the same as the units on the RHS, $[LHS] = [RHS]$. The three quantities in the above have the units

$$[k] = \frac{1}{length}, [< \varepsilon >] = \frac{length^2}{time^3}, [E(k)] = \frac{length^3}{time^2}.$$

Inserting these units into the above relation gives

$$\frac{length^3}{time^2} = \frac{length^{2a}}{time^{3a}} \frac{1}{length^b} = length^{2a-b} time^{-3a}, \text{ giving}$$

$$3a = 2, 2a - b = 3, \text{ or } a = \frac{2}{3}, b = -\frac{5}{3}.$$

Thus, Kolmogorov's law follows

$$E(k) = \alpha < \varepsilon >^{\frac{2}{3}} k^{-\frac{5}{3}}, \text{ over the inertial range } 0 < k \leq C(LRe^{-\frac{3}{4}})^{-1}.$$

The above estimate $\eta \sim LRe^{-\frac{3}{4}}$ for the Kolmogorov micro-scale is derived by similar physical reasoning. Let the reference large scale velocity and length (which are used in the definition of the Reynolds number) be denoted by U, L. At the scales of the smallest persistent eddies (the bottom of the inertial range) we shall denote the smallest scales of velocity and length by v_{small}, η. We form two Reynolds numbers:

$$Re = \frac{UL}{\nu}, Re_{small} = \frac{v_{small}\eta}{\nu}.$$

The global Reynolds number measures the relative size of viscosity on the large scales and when Re is large the effects of viscosity on the large scales are then negligible. The smallest scales Reynolds number similarly measures the relative size of viscosity on the smallest persistent scales. Since it is non-negligible we must have

$$Re_{small} \simeq 1, \text{ equivalently } \frac{v_{small}\eta}{\nu} \simeq 1.$$

Next comes an assumption of statistical equilibrium: Energy Input at large scales = Energy dissipation at smallest scales. The largest eddies have energy which scales like $O(U^2)$ and associated time scale $\tau = O(\frac{L}{U})$. The rate of energy transfer/energy input is thus $O(\frac{U^2}{\tau}) = O(\frac{U^3}{L})$[6]. The small scales energy dissipation from the viscous terms scales like

$$\varepsilon_{small} \simeq \nu |\nabla u_{small}|^2 \simeq \nu (\frac{v_{small}}{\eta})^2.$$

Thus we have the second ingredient:

$$\frac{U^3}{L} \simeq \nu (\frac{v_{small}}{\eta})^2.$$

Solving the first equation for v_{small} gives $v_{small} \simeq \frac{\nu}{\eta}$. Inserting this value for the small scales velocity into the second equation, solving for the length-scale η and rearranging the result in terms of the global Reynolds number gives the following estimate for η which determines the above estimate for the highest wave-number in the inertial range:

$$\eta = \eta_{Kolmogorov} \simeq Re^{-\frac{3}{4}} L.$$

[6] It is known for many turbulent flows that, as predicted by K-41, ε scales like $\frac{U^3}{L}$. This estimate expresses statistical equilibrium in K-41 formalism, [12], [25], [30], [36], [37] and has been proven as an upper bound directly from the Navier Stokes equations without any assumptions of homogeneity or isotropy for turbulent flows in bounded domains driven by persistent shearing of a moving boundary, Constantin and Doering [7] , and Wang [42], . The same estimate has been proven, Foias [10], Doering and Foias [8], Childress, Kerswell and Gilbert [6] (others have also contributed to this important theory as well), when the flow is driven by a persistent body force, the boundary conditions are periodic and the forcing acts on the largest modes/ largest scales.

This estimate for the size of the smallest persistent solution scales is the basis for the estimates of $O(\mathrm{Re}^{\frac{9}{4}})$ mesh-points in space leading to complexity estimates of $O(\mathrm{Re}^3)$ for DNS of turbulent flows.