A NON-ITERATIVE, DOMAIN DECOMPOSITION METHOD WITH DIFFERENT TIME STEP SIZES FOR THE EVOLUTIONARY STOKES-DARCY MODEL

LI SHAN *, HAIBIAO ZHENG †, AND WILLIAM J. LAYTON ‡

Abstract. This report analyzes a partitioned time stepping algorithm, meaning a non-iterative, domain decomposition method, which *allows different time steps in the fluid region and the porous region for the fully evolutionary* Stokes-Darcy problem. The method presented requires only one, uncoupled Stokes and Darcy sub-physics and sub-domain solve per time step. Under a time step restriction of the form $\triangle t \leq C$*(physical parameters)* we prove stability and convergence of the method. Numerical tests are given confirming the convergence theory and demonstrating the computational efficiency of the partitioned method. They also show that in (the expected case) of greater fluid velocities in the free-flow region than in the porous media region, allowing smaller timesteps in the subregion with the faster velocities increases both accuracy and efficiency.

Keywords: Stokes and Darcy system, partitioned time stepping method, domain decomposition, asynchronous time stepping.

1. Introduction. The transport of substances between surface water and groundwater is an important problem of great current interest. The essential features of estimating penetration of a plume of pollution from surface water to ground water and remediation thereafter are that (i) the coupled problems in the fluid and porous media sub-regions are both inherently time dependent, (ii) the flows in the two regions act with different characteristic speeds, and (iii) the physical processes are sufficiently different that codes optimized for each individual sub-process ultimately will need to be used to solve the coupled problem. Thus, there are many open questions (beyond the results we present herein) connected to limiting cases of the various physical parameters. With these issues in mind, we analyze herein an asynchronous, uncoupled, partitioned method for the *fully evolutionary* Stokes-Darcy problem. The method allows different time steps in the two subregions (such methods are often called "asynchronous coupling" in geophysics) and requires only one, uncoupled Stokes solve and one Darcy solve per time step (with no iteration or construction of a fully coupled problem). The partitioning is based on simply lagging the interfacial coupling terms following a method analyzed by Mu and Zhu [17], see also [1] for its use in other applications. Connecting the different time steps at the interface adapts of Connors and Howell [7] for atmosphere-ocean coupling. The essential difficulty of both lagging terms and interpolation between meshes and time steps is doing so without creation of non-physical system energy. Partitioned methods have obvious and large advantages in efficiency over monolithic (fully coupled) discretizations followed by domain decomposition iteration at each timestep. However, partitioned methods for the Stokes-Darcy problem are in their infancy. We believe that partitioned methods will continue to evolve and improve.

The algorithms we present are an extension of the partitioned method in Mu and Zhu [17]. We shall thus follow the notations in [17] in specifying the problem (next). The mathematical model consists of the evolutionary Stokes equations in the fluid region coupled with the evolutionary Darcy equations in the porous medium, [9, 13, 15, 18, 19]. The key part is the interface coupling conditions of conservation of mass across the interface, balance of forces and the (tangential) Beavers-Joseph-Saffman conditions [2]. Consider thus a Stokes flow in Ω_f coupled with a porous media flow in Ω_p, where $\Omega_f, \Omega_p \subset R^d (d = 2 \ or \ 3)$ are bounded domains, $\Omega_f \cap \Omega_p = \emptyset$, and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. Denote by $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\mathbf{n}_f$ and $\mathbf{n}_p$ the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively, and $\tau_i, i = 1, \cdots, d-1$, the unit tangential vectors on the interface Γ. Note that $\mathbf{n}_p = -\mathbf{n}_f$ *on* Γ, see Figure 1.1 below. Let $T \geq 0$ be a finite time, the fluid flow is governed by

*College of Science, Xi'an Jiaotong University, Xi'an,710049,P.R.China. `li.shan13@gmail.com`. Partially supported by NSF of China (grant 10871156) and XJTU(grant 2009xjtujc30).

†College of Science, Xi'an Jiaotong University, Xi'an,710049,P.R.China. `hbzheng13@gmail.com`. Partially supported by NSF of China (grant 10871156) and XJTU(grant 2009xjtujc30).

‡Department of Mathematics, University of Pittsburgh, Pittsburgh, PA,15260, USA. `wjl@pitt.edu`. Partially supported by NSF grant 0810385.

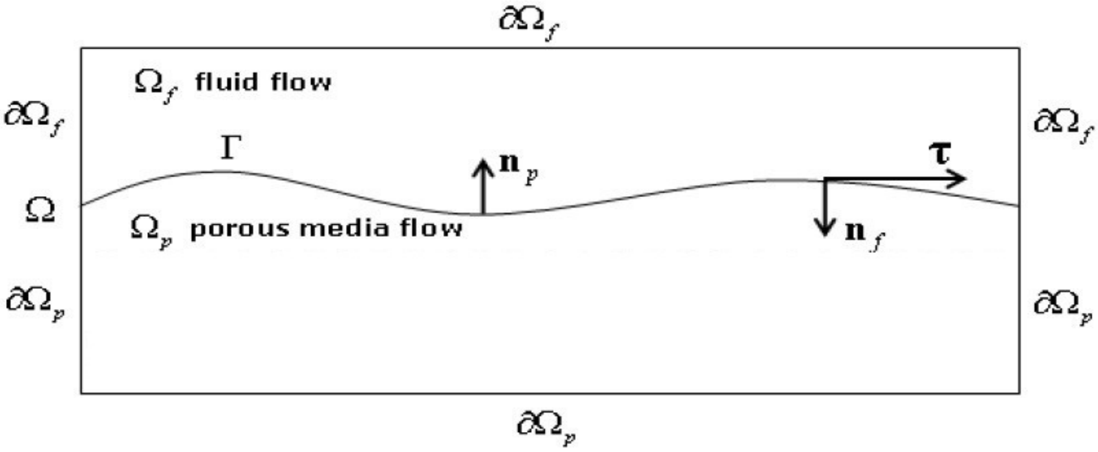

FIG. 1.1. *The global domain* Ω *consisting of the fluid region* Ω_f *and the porous media region* Ω_p *separated by the interface* Γ.

the Stokes equations for the fluid velocity and pressure in Ω_f, $u(x,t)$ and $p(x,t)$:

$$u_t - \nu\Delta u + \nabla p = f_1 \quad \texttt{in } \Omega_f \times (0,T], \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \texttt{in } \Omega_f \times (0,T], \tag{1.2}$$

$$u(x,0) = u_0 \quad \texttt{in } \Omega_f, \tag{1.3}$$

$$u = 0 \quad \texttt{on } \partial\Omega_f \setminus \Gamma. \tag{1.4}$$

Here $f_1(x,t)$ is the external force, and ν is the kinematic viscosity.

The porous media flow is governed by the following equations on Ω_p for the piezometric head $\phi(x,t)$:

$$S_0\phi_t + \nabla \cdot q = f_2 \quad \texttt{in } \Omega_p \times (0,T], \tag{1.5}$$

$$q = -\mathbf{K}\nabla\phi \quad \texttt{in } \Omega_p \times (0,T], \tag{1.5}$$

$$u_p = \frac{q}{n} \quad \texttt{in } \Omega_p \times (0,T], \tag{1.6}$$

$$\phi(x,0) = \phi_0 \quad \texttt{in } \Omega_p, \tag{1.7}$$

$$\phi = 0 \quad \texttt{on } \partial\Omega_p \setminus \Gamma. \tag{1.8}$$

Here q is the specific discharge defined as the volume of the fluid flowing per unit time through a unit cross-sectional area normal to the direction of the flow, ξ is the fluid velocity in Ω_p, S_0 is the specific mass storativity coefficient, $\mathbf{K}$ represents the hydraulic conductivity tensor, n is the volumetric porosity, and f_2 is the source term. Note that $\phi = z + \frac{P_p}{\rho g}$, the sum of elevation from a reference level plus pressure head, where P_p is the pressure of the fluid in Ω_p, ρ is the density of the fluid, g is the gravitational acceleration. (The usage of g as gravitational vector or source term will be clear from the context in which it occurs).

The presentation of the coupled problem with separate discretizations and differing time steps involves substantial notation. We therefore make some simplifying assumptions to reduce the notational complexity. In particular, we assume $z = 0$ and that $\mathbf{K} = diag(K, \cdots, K)$ with $K \in L^\infty(\Omega_p), K > 0$, which implies that the porous media is homogeneous. By using Darcy's law, (1.5) can be rewritten in the parabolic form

$$S_0\phi_t - \nabla \cdot (\mathbf{K}\nabla\phi) = f_2 \quad \texttt{in } \Omega_p \times (0,T], \tag{1.10}$$

$$\phi(x,0) = \phi_0 \quad \texttt{in } \Omega_p. \tag{1.11}$$

For the Stokes-Darcy model, the interface conditions of conservation of mass (1.9), balance of forces

(1.10) and the Beavers-Joseph-Saffman condition are imposed herein:

$$u \cdot \mathbf{n}_f + u_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma \times (0,T], \tag{1.9}$$

$$p - \nu \mathbf{n}_f \frac{\partial u}{\partial \mathbf{n}_f} = \rho g \phi \quad \text{on } \Gamma \times (0,T], \tag{1.10}$$

$$-\nu \tau_i \frac{\partial u}{\partial \mathbf{n}_f} = \frac{\alpha}{\sqrt{\tau_i \cdot \mathbf{K} \tau_i}} u \cdot \tau_i, \ i = 1, \cdots, d-1 \quad \text{on } \Gamma \times (0,T]. \tag{1.11}$$

In (1.11) α is a positive parameter depending on the properties of the porous medium and must be experimentally determined. The condition (1.9) can be rewritten as

$$u \cdot \mathbf{n}_f = \frac{\mathbf{K}}{n} \frac{\partial \phi}{\partial \mathbf{n}_p} \quad on \ \ \Gamma \times (0,T]. \tag{1.12}$$

In the last ten years there has been an explosion of work on numerical analysis of coupling surface water to ground water. For a comprehensive overview of other work on this important problem, see [10] and the 125 references therein. Much of the work has studied the equilibrium problem, e.g., [9, 10, 15]. Discacciati [8] presents results for a monolithic method for the evolutionary problem which is uncoupled at each timestep by domain decomposition iteration. Various quasi-static models (not considered herein) have also been proposed with time dependence in one region and in the other at equilibrium. To our knowledge, justification of the quasi-static assumption based on the rates of return to equilibrium in either sub problem in the context of the fully evolutionary setting is still open. Among the many fewer papers (so far) on the numerical analysis of the *fully evolutionary* Stokes-Darcy problem (considered herein), beyond [8], Mu and Zhu [17] study a partitioned method which we build upon herein. Cao, Gunzburger, Hu, Hua, Wang and Zhao [4, 3] study a fully, monolithically coupled implicit method for the much harder and physically more accurate case of Beavers-Joseph coupling conditions (without Saffman's simplification) as well as an interesting approach to partitioning in [5].

1.1. Variational formulation of the continuous problem. Denote $W = H_f \times H_p$ and $Q = L^2(\Omega_f)$, where

$$H_f = \{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \setminus \Gamma\}, H_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \setminus \Gamma\}.$$

The space $L^2(D)$, where $D = \Omega_f$ or Ω_p, is equipped with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $||\cdot||_{L^2}(D)$. The spaces H_f and H_p are equipped with the following norms:

$$||u||_{H_f} = ||\nabla u||_{L^2(\Omega_f)} = \sqrt{(\nabla u, \nabla u)_{\Omega_f}} \qquad \forall\ u \in H_f, \tag{1.13}$$

$$||\phi||_{H_p} = ||\nabla \phi||_{L^2(\Omega_p)} = \sqrt{(\nabla \phi, \nabla \phi)_{\Omega_p}} \qquad \forall\ \phi \in H_p. \tag{1.14}$$

We equip the space W with the following norms: $\forall\ \mathbf{u} = (u, \phi) \in W$,

$$||\mathbf{u}||_0 = \sqrt{n(u,u)_{\Omega_f} + \rho g S_0 (\phi,\phi)_{\Omega_p}}, \tag{1.15}$$

$$||\mathbf{u}||_W = \sqrt{n\nu(\nabla u, \nabla u)_{\Omega_f} + \rho g (\mathbf{K}\nabla\phi, \nabla\phi)_{\Omega_p}}, \tag{1.16}$$

where $(\cdot,\cdot)_D$ refers to the scalar product $(\cdot,\cdot)$ in the corresponding domain D for $D = \Omega_f$ or Ω_p. For simplicity, we assume that n, ρ, g, S_0 and ν are constants.

We also recall Poincaré and trace inequalities which are useful in the analysis. There exist constants P_1 and C_0 which only depend on Ω_f such that

$$||v||_{L^2(\Omega_f)} \le P_1 ||v||_{H_f}, \ ||v||_{L^2(\Gamma)} \le C_0 ||v||^{1/2}_{L^2(\Omega_f)} ||v||^{1/2}_{H_f}, \ \ \forall\ v \in H_f. \tag{1.17}$$

There exist constants P_2 and $\tilde{C}_0$ that only depend on Ω_p such that

$$||\psi||_{L^2(\Omega_p)} \le P_2 ||\psi||_{H_p}, \ ||\psi||_{L^2(\Gamma)} \le \tilde{C}_0 ||\psi||^{1/2}_{L^2(\Omega_p)} ||\psi||^{1/2}_{H_p}, \ \forall\ \psi \in H_p. \tag{1.18}$$

The weak formulation of the time-dependent Stokes-Darcy model reads as follows: find $\mathbf{u} = (u, \phi) \in W$ and $p \in Q$, such that , $\forall\ t \in (0, T]$,

$$
\begin{aligned}
(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad & \texttt{in}\ \Omega, \\
b(\mathbf{u}, q) = 0 \qquad & \texttt{in}\ \Omega, \\
\mathbf{u}(0) = \mathbf{u}_0 \qquad & \texttt{in}\ \Omega,
\end{aligned}
\tag{1.19}
$$

where

$$
\begin{aligned}
(\mathbf{u}_t, \mathbf{v}) &= n(u_t, v) + \rho g S_0(\phi_t, \psi), \\
a(\mathbf{u}, \mathbf{v}) &= a_f(u, v) + a_p(\phi, \psi) + a_\Gamma(\mathbf{u}, \mathbf{v}), \\
a_f(u, v) &= n\nu(\nabla u, \nabla v)_{\Omega_f} + \sum_{i=1}^{d-1} \int_\Gamma \frac{n\alpha}{\sqrt{\tau_i \cdot \mathbf{K}\tau_i}} (u \cdot \tau_i)(v \cdot \tau_i), \\
a_p(\phi, \psi) &= \rho g(\mathbf{K}\nabla\phi, \nabla\psi)_{\Omega_p}, \\
a_\Gamma(\mathbf{u}, \mathbf{v}) &= n\rho g \int_\Gamma (\phi v \cdot \mathbf{n}_f - \psi u \cdot \mathbf{n}_f), \\
b(\mathbf{v}, p) &= -n(p, \nabla \cdot v)_{\Omega_f}, \\
(\mathbf{f}, \mathbf{v}) &= n(f_1, v)_{\Omega_f} + \rho g(f_2, \psi)_{\Omega_p}.
\end{aligned}
$$

Lemma 1.1. *Assume that*

$$
f_1 \in L^2(0, T; L^2(\Omega_f)^2), f_2 \in L^2(0, T; L^2(\Omega_p)), \mathbf{K} \in L^\infty(\Omega_p)^{2\times 2}, \tag{1.20}
$$

and $\mathbf{K}$ *is uniformly bounded and positive definite in* Ω_p*: there exist* $k_{min}, k_{max} > 0$ *such that*

$$
k_{min}|x|^2 \leq \mathbf{K}x \cdot x \leq k_{max}|x|^2 \ a.e.\ x \in \Omega_p. \tag{1.21}
$$

In addition, let $u_0 \in L^2(\Omega_f)^2, \phi_0 \in L^2(\Omega_p)$*, then any solution* $(u, p, \phi) \in (L^2(0, T; H_f) \cap H^1(0, T; L^2(\Omega_f)^2)) \times L^2(0, T; Q) \times L^2(0, T; H_p)$ *of (1.1)-(1.11) is also a solution to (1.19). Conversely any solution to (1.19) satisfies (1.1)-(1.11).*

Proof. The well-posedness of the Stokes-Darcy model(1.19) can be found in [8, 9, 15] for the stationary case and is assumed to hold similarly for the non-stationary case. □

From the assumption (1.21), we have

$$
\frac{1}{\sqrt{k_{max}}} ||\mathbf{K}^{1/2}\nabla\psi||_{L^2(\Omega_p)} \leq ||\psi||_{H_p} \leq \frac{1}{\sqrt{k_{min}}} ||\mathbf{K}^{1/2}\nabla\psi||_{L^2(\Omega_p)}. \tag{1.22}
$$

Furthermore, $a_\Gamma(\cdot, \cdot)$ satisfy the following properties:

$$
a_\Gamma(\mathbf{u}, \mathbf{v}) = -a_\Gamma(\mathbf{v}, \mathbf{u}), \qquad a_\Gamma(\mathbf{u}, \mathbf{u}) = 0, \qquad \forall\ \mathbf{u}, \mathbf{v} \in W. \tag{1.23}
$$

The first partitioned method of Mu and Zhu [17] uncouples the Stokes-Darcy problem by the implicit method in time and the explicit method for the coupling terms. Herein we extend the partitioned method to allow for different size time steps for the decoupled subproblems, say $\triangle t$ on Ω_f and $\triangle s$ on Ω_p, with any integer ratio $r = \triangle s/\triangle t$ between them. The reason for using different time step size is that physical processes happen at different rates, e.g., [11] whose analysis is consistent with the intuition that fluid flow is faster than that in the porous medium. The methods extend immediately to the case where the regions of small and large time steps are reversed. The natural CFL condition demands $\frac{\mathbf{v}\triangle t}{h} \leq 1$ where $\mathbf{v}$ denotes the velocity in the sub-domain. Since different domain have different flow velocities, practical computing often will require different time steps and even possibly adapting $\triangle t$ separately in each sub-region.

Remark. Coupled fluid flow with flow in porous media occurs in such a wide range of applications that many parameters regimes are important. While the focus of this paper is asynchronous time stepping (treating the differing flow rates in free flow and in filtration flow), we also try to estimate the dependencies of the worst case constants upon k_{min} (which is often small) and S_0 (also often small). The constants in the analysis also depend upon the final time T, indicated as $C(T)$. We have not attempted to optimize any estimate with respect to any of these important parameters. We anticipate that different methods will be the preferable for different parameter values.

The rest of the paper is organized as follows. Both coupled and decoupled algorithms are presented in Section 2. The stability of the decoupled algorithm is given in Section 3. In Section 4, we analyze its error. Numerical tests are reported in Section 5, followed by conclusions in Section 6.

2. Numerical algorithms. We consider a triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra if $d = 3$. Let $W_h = H_{fh} \times H_{ph} \subset W$ and $Q_h \subset Q$ denote the finite element subspaces. The finite element spaces H_{fh} and Q_h approximating velocity and pressure in the fluid flow region are assumed to satisfy the well-known discrete inf-sup condition: there exists a positive constant β, independent of h, such that $\forall\ q_h \in Q_h, \exists\ v_h \in H_{fh}, v_h \neq 0$,

$$b(v_h, q_h) \geq \beta ||v_h||_{H_f} ||q_h||_{L^2(Q)}. \tag{2.1}$$

Moreover, we need the inverse inequalities in both H_{fh} and H_{ph}: there exist constants C_1 and $\tilde{C}_1$ which depend on the domain Ω_f and Ω_p, respectively, such that

$$||v_h||_{H_f} \leq C_1 h^{-1} ||v_h||_{L^2(\Omega_f)} \quad \forall\ v_h \in H_{fh}, \tag{2.2}$$

$$||\psi_h||_{H_p} \leq \tilde{C}_1 h^{-1} ||\psi_h||_{L^2(\Omega_p)}, \ \forall\ \psi_h \in H_{ph}. \tag{2.3}$$

The following estimates on the coupling term are useful in our analysis.

LEMMA 2.1. $\forall\ \mathbf{u}, \mathbf{v} \in \mathbf{W}$, *there exists* $C_2 \geq 0$, *such that* $\forall \varepsilon \geq 0$,

$$|a_\Gamma(\mathbf{u}, \mathbf{v})| \leq \varepsilon ||\mathbf{u}||_W^2 + \frac{n\rho g C_2}{4\varepsilon k_{min}} ||\mathbf{v}||_W^2. \tag{2.4}$$

Further, we have $\forall\ \mathbf{u}, \mathbf{v} \in \mathbf{W}$, *there exists* $C_3 \geq 0$ *such that .*

$$|a_\Gamma(\mathbf{u}, \mathbf{v})| \leq \frac{\varepsilon}{2}(||\mathbf{u}||_W^2 + ||\mathbf{v}||_W^2) + \frac{n\rho g C_3}{4\varepsilon\sqrt{\nu S_0 k_{min}}}(||\mathbf{u}||_0^2 + ||\mathbf{v}||_0^2). \tag{2.5}$$

In addition, if the finite element spaces satisfy the inverse inequality, then $\forall\ \mathbf{u}_h, \mathbf{v}_h \in \mathbf{W}_h$, *there exists* $C_4 \geq 0$ *such that .*

$$|a_\Gamma(\mathbf{u}_h, \mathbf{v}_h)| \leq \varepsilon ||\mathbf{u}_h||_W^2 + \frac{n\rho g C_4}{4\varepsilon h} ||\mathbf{v}_h||_0^2. \tag{2.6}$$

Proof. By using trace and Poincaré inequalities (1.17)-(1.18), we have

$$\begin{aligned}
&|a_\Gamma(\mathbf{u}, \mathbf{v})| = n\rho g \Big| \int_\Gamma (\phi v \cdot \mathbf{n}_f - \psi u \cdot \mathbf{n}_f) \Big| \\
&\leq n\rho g ||\phi||_{L^2(\Gamma)} ||v \cdot \mathbf{n}_f||_{L^2(\Gamma)} + n\rho g ||\psi||_{L^2(\Gamma)} ||u \cdot \mathbf{n}_f||_{L^2(\Gamma)} \\
&\leq n\rho g C_0 \tilde{C}_0 (||\phi||_{L^2(\Omega_p)}^{1/2} ||\phi||_{H_p}^{1/2} ||v||_{L^2(\Omega_f)}^{1/2} ||v||_{H_f}^{1/2} + ||\psi||_{L^2(\Omega_p)}^{1/2} ||\psi||_{H_p}^{1/2} ||u||_{L^2(\Omega_f)}^{1/2} ||u||_{H_f}^{1/2}) \\
&\leq n\rho g C_0 \tilde{C}_0 P_1^{1/2} P_2^{1/2} (||\phi||_{H_p} ||v||_{H_f} + ||\psi||_{H_p} ||u||_{H_f}) \\
&\leq \frac{n\rho g C_0 \tilde{C}_0 P_1^{1/2} P_2^{1/2}}{\sqrt{k_{min}}} (||\mathbf{K}^{1/2} \nabla\phi||_{L^2(\Omega_p)} ||v||_{H_f} + ||\mathbf{K}^{1/2} \nabla\psi||_{L^2(\Omega_p)} ||u||_{H_f}) \\
&\leq \varepsilon \rho g ||\mathbf{K}^{1/2} \nabla\phi||_{L^2(\Omega_p)}^2 + \frac{n^2 \rho g C_0^2 \tilde{C}_0^2 P_1 P_2}{4\varepsilon k_{min}} ||v||_{H_f}^2 + \varepsilon n\nu ||u||_{H_f}^2 + \frac{n\rho^2 g^2 C_0^2 \tilde{C}_0^2 P_1 P_2}{4\varepsilon k_{min} \nu} ||\mathbf{K}^{1/2} \nabla\psi||_{L^2(\Omega_p)}^2 \\
&\leq \varepsilon ||\mathbf{u}||_W^2 + \frac{n\rho g C_0^2 \tilde{C}_0^2 P_1 P_2}{4\varepsilon k_{min} \nu} ||\mathbf{v}||_W^2 \\
&\leq \varepsilon ||\mathbf{u}||_W^2 + \frac{n\rho g C_2}{4\varepsilon k_{min} \nu} ||\mathbf{v}||_W^2,
\end{aligned} \tag{2.7}$$

where $C_2 = C_0^2\tilde{C}_0^2 P_1 P_2$. If we don't use Poincaré inequality in (2.7), then

$$
\begin{aligned}
&|a_\Gamma(\mathbf{u},\mathbf{v})| = n\rho g|\int_\Gamma(\phi v\cdot\mathbf{n}_f - \psi u\cdot\mathbf{n}_f)| \\
&\le n\rho g||\phi||_{L^2(\Gamma)}||v\cdot\mathbf{n}_f||_{L^2(\Gamma)} + n\rho g||\psi||_{L^2(\Gamma)}||u\cdot\mathbf{n}_f||_{L^2(\Gamma)} \\
&\le n\rho g C_0\tilde{C}_0(||\phi||^{1/2}_{L^2(\Omega_p)}||\phi||^{1/2}_{H_p}||v||^{1/2}_{L^2(\Omega_f)}||v||^{1/2}_{H_f} + ||\psi||^{1/2}_{L^2(\Omega_p)}||\psi||^{1/2}_{H_p}||u||^{1/2}_{L^2(\Omega_f)}||u||^{1/2}_{H_f}) \\
&\le \varepsilon\sqrt{n\nu\rho g}(||\mathbf{K}^{1/2}\nabla\phi||_{L^2(\Omega_p)}||v||_{H_f} + ||\mathbf{K}^{1/2}\nabla\psi||_{L^2(\Omega_p)}||u||_{H_f}) \\
&\quad + \frac{(n\rho g)^{3/2}C_0^2\tilde{C}_0^2}{4\varepsilon\sqrt{k_{min}}}(||\phi||_{L^2(\Omega_p)}||v||_{L^2(\Omega_f)} + ||\psi||_{L^2(\Omega_p)}||u||_{L^2(\Omega_f)}) \\
&\le \frac{\varepsilon}{2}(||\mathbf{u}||_W^2 + ||\mathbf{v}||_W^2) + \frac{n\rho g C_0^2\tilde{C}_0^2}{4\varepsilon\sqrt{\nu S_0 k_{min}}}(||\mathbf{u}||_0^2 + ||\mathbf{v}||_0^2) \\
&\le \frac{\varepsilon}{2}(||\mathbf{u}||_W^2 + ||\mathbf{v}||_W^2) + \frac{n\rho g C_3}{4\varepsilon\sqrt{\nu S_0 k_{min}}}(||\mathbf{u}||_0^2 + ||\mathbf{v}||_0^2), \qquad (2.8)
\end{aligned}
$$

where $C_3 = C_0^2\tilde{C}_0^2$. If the finite element spaces satisfy the inverse inequality (2.2)-(2.3), then

$$
\begin{aligned}
&|a_\Gamma(\mathbf{u}_h,\mathbf{v}_h)| = n\rho g|\int_\Gamma(\phi_h v_h\cdot\mathbf{n}_f - \psi_h u_h\cdot\mathbf{n}_f)| \\
&\le n\rho g||\phi_h||_{L^2(\Gamma)}||v_h\cdot\mathbf{n}_f||_{L^2(\Gamma)} + n\rho g||\psi_h||_{L^2(\Gamma)}||u_h\cdot\mathbf{n}_f||_{L^2(\Gamma)} \\
&\le n\rho g C_0\tilde{C}_0(||\phi_h||^{1/2}_{L^2(\Omega_p)}||\phi_h||^{1/2}_{H_p}||v_h||^{1/2}_{L^2(\Omega_f)}||v_h||^{1/2}_{H_f} + ||\psi_h||^{1/2}_{L^2(\Omega_p)}||\psi_h||^{1/2}_{H_p}||u_h||^{1/2}_{L^2(\Omega_f)}||u_h||^{1/2}_{H_f}) \\
&\le \frac{n\rho g C_0\tilde{C}_0 P_2^{1/2}C_1^{1/2}h^{-1/2}}{\sqrt{k_{min}}}||\mathbf{K}^{1/2}\nabla\phi_h||_{L^2(\Omega_p)}||v_h||_{L^2(\Omega_f)} + n\rho g C_0\tilde{C}_0 P_1^{1/2}\tilde{C}_1^{1/2}h^{-1/2}||\psi_h||_{L^2(\Omega_p)}||u_h||_{H_f} \\
&\le \varepsilon||\mathbf{u}_h||_W^2 + \frac{n\rho g C_0^2\tilde{C}_0^2 P_2 C_1 h^{-1}}{4\varepsilon k_{min}}(n||v_h||^2_{L^2(\Omega_f)}) + \frac{n\rho g C_0^2\tilde{C}_0^2 P_1\tilde{C}_1 h^{-1}}{4\varepsilon\nu S_0}(\rho g S_0||\psi_h||^2_{L^2(\Omega_p)}) \\
&\le \varepsilon||\mathbf{u}_h||_W^2 + \frac{n\rho g C_0^2\tilde{C}_0^2}{4\varepsilon h}\max\{\frac{P_2C_1}{k_{min}}, \frac{P_1\tilde{C}_1}{\nu S_0}\}||\mathbf{v}_h||_0^2 \\
&\le \varepsilon||\mathbf{u}_h||_W^2 + \frac{n\rho g C_4}{4\varepsilon h}||\mathbf{v}_h||_0^2, \qquad (2.9)
\end{aligned}
$$

where $C_4 = C_0^2\tilde{C}_0^2\max\{\frac{P_2C_1}{k_{min}}, \frac{P_1\tilde{C}_1}{\nu S_0}\}$. □

We also introduce a subspace $\mathbf{V}_h$ of $\mathbf{W}_h$ defined by

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{W}_h : b(\mathbf{v}_h, q_h) = 0 \ \ \forall\ q_h \in Q_h\},$$

Following [17], we define a projection operator $P_h : (\mathbf{w}(t), p(t)) \in (\mathbf{W}, Q) \mapsto (P_h\mathbf{w}(t), P_h p(t)) \in (\mathbf{W}_h, Q_h)$, $\forall\ t \in [0,T]$ by

$$a(P_h\mathbf{w}(t),\mathbf{v}_h) + b(\mathbf{v}_h, P_h p(t)) = a(\mathbf{w}(t),\mathbf{v}_h) + b(\mathbf{v}_h, p(t)) \quad \forall\ \mathbf{v}_h \in W_h, \qquad (2.10)$$

$$b(P_h\mathbf{w}(t), q_h) = 0 \quad \forall\ q_h \in Q_h. \qquad (2.11)$$

Apparently, P_h is linear operator. Furthermore, under a certain smoothness assumption on $(\mathbf{w}(t), p(t))$, the following approximation properties hold:

$$
\begin{aligned}
||P_h\mathbf{w}(t) - \mathbf{w}(t)||_0 &\le Ch^2, \\
||P_h\mathbf{w}(t) - \mathbf{w}(t)||_W &\le Ch, \\
||P_h p(t) - p(t)||_0 &\le Ch.
\end{aligned}
$$

From now on, we always assume that $(u(t),\phi(t)) \in (H^2(\Omega_f)^d, H^2(\Omega_p))$, $(u_t(t),\phi_t(t)) \in (H^1(\Omega_f)^d, H^1(\Omega_p))$ and $(u_{tt}(t),\phi_{tt}(t)) \in (L^2(\Omega_f)^d, L^2(\Omega_p))$ for the solutions of (1.19).

2.1. The monolithically coupled, implicit method. In this section, we provide a monolithically coupled scheme which is used for comparison. Choose a uniform distribution of discrete time level,

$$\mathcal{Q} = \{0 = t^0, t^1, t^2, \cdots, t^N = T\},$$

where $t^m = m\triangle t, m = 0, 1, 2, \cdots, N$ for $\triangle t = \frac{T}{N}$. Here $(u^{h,m}, p^{h,m}, \phi^{h,m})$ denotes the discrete approximation to $(u(t^m), p(t^m), \phi(t^m))$.

Algorithm 2.1(Coupled scheme) Find $\mathbf{u}^{h,m+1} = (u^{h,m+1}, \phi^{h,m+1}) \in W_h$ and $p^{h,m+1} \in Q_h$ with $m = 0, \cdots, N-1$, such that

$$(\frac{\mathbf{u}^{h,m+1} - \mathbf{u}^{h,m}}{\triangle t}, \mathbf{v}) + a(\mathbf{u}^{h,m+1}, \mathbf{v}) + b(\mathbf{v}, p^{h,m+1}) = \mathbf{f}^{m+1}(\mathbf{v}) \quad \forall\ \mathbf{v} \in W_h, \tag{2.12}$$

$$b(\mathbf{u}^{h,m+1}, q_h) = 0 \qquad \forall\ q_h \in Q_h \tag{2.13}$$

$$\mathbf{u}^{h,0} = \mathbf{u}_0. \tag{2.14}$$

2.2. A Decoupled Scheme with Different Time Steps. To streamline our notation further, we shall suppress the subscript "h" and replace u_h^m, ϕ_h^m, p_h^m by u^m, ϕ^m, p^m, respectively. First, we choose discrete time levels

$$\mathcal{P} = \{0 = t^0, t^1, t^2, \cdots, t^N = T\},$$

where $t^m = m\triangle t,\ m = 0, 1, 2, \cdots, N$ for $\triangle t = \frac{T}{N}$. Denote by

$$\mathcal{S} = \{t^{m_0}, t^{m_1}, \cdots, t^{m_M}\} \subset \mathcal{P},$$

a subset satisfying $t^{m_k} = kr\triangle t$ such that $r \in \mathbb{N}$ is fixed and $Mr = N$. The time step size on Ω_p is given a separate notations hereafter, $\triangle s = r\triangle t$. For $t^m, t^{m_k} \in [0, T]$, (u^m, p^m, ϕ^{m_k}) will denote the discrete approximation to $(u(t^m), p(t^m), \phi(t^{m_k}))$. The approximations $(u^{m+1}, p^{m+1}) \in (H_{fh}, Q_h)$, for $m = m_0, m_0+1, \cdots, N-1$ and $\phi^{m_{k+1}} \in H_{ph}$ for $k = 0, 1, \cdots, M-1$ are calculated using Algorithm 2.2. In practice only the data at time t^0 would need to be provided. One important feature of Algorithm 2.2 is that (u^{m+1}, p^{m+1}) can be calculated for $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$ in parallel with $\phi^{m_{k+1}}$.

Algorithm 2.2(Decoupled scheme)

- Find $(u^{m+1}, p^{m+1}) \in (H_{fh}, Q_h)$, with $m = m_k, m_k+1, \cdots, m_{k+1}-1$, such that $\forall (v, q) \in (H_{fh}, Q_h)$:

$$n(\frac{u^{m+1} - u^m}{\triangle t}, v) + a_f(u^{m+1}, v) + b(v, p^{m+1}) = n(f_1^{m+1}, v) - n\rho g \int_\Gamma \phi^{m_k} v \cdot \mathbf{n}_f, \tag{2.15}$$

$$b(u^{m+1}, q) = 0, \tag{2.16}$$

$$u^0 = u_0, \tag{2.17}$$

  with the small time step size $\triangle t$.
- Set $S^{m_k} = \frac{1}{r}\sum_{i=m_k}^{m_{k+1}-1} u^i$,
- find $\phi^{m_{k+1}} \in H_{ph}$, such that $\psi \in H_{ph}$:

$$\rho g S_0(\frac{\phi^{m_{k+1}} - \phi^{m_k}}{\triangle s}, \psi) + a_p(\phi^{m_{k+1}}, \psi) = \rho g(f_2^{m_{k+1}}, \psi) + n\rho g \int_\Gamma \psi S^{m_k} \cdot \mathbf{n}_f, \tag{2.18}$$

$$\phi^{m_0} = \phi_0, \tag{2.19}$$

with the large time step size $\triangle s = r\triangle t$.

- Set $k = k + 1$ and repeat until $k = M - 1$.

3. Stability of the method. In this section, under a time step restriction of the form

$$\frac{4n\rho g C_3 \triangle t}{\sqrt{\nu S_0 k_{min}}} < 1, \tag{3.1}$$

where C_3 is a constant defined in (2.8), we prove the stability (possibly including terms like $C(T) \approx exp(aT)$) over bounded time intervals $[0, T]$ of the partitioned method Algorithm 2.2. It is also possible to prove stability under the alternate condition $\frac{\triangle t}{h} \leq C(physicalparameters)$, where C has a different dependency than in (3.1). We test sharpness of the restriction (3.1) in the numerical experiments section which indicates that (3.1) is not sharp with respect to its dependency on k_{min}.

THEOREM 3.1. *(Stability) Choose the initial data $\phi^{m_0} = \phi^0$, $u^{m_0} = u^0$, and $\phi^{m_{k+1}+J+1} = \phi^{m_{k+1}}$, $g^{m_{k+1}+J+1} = g^{m_{k+1}}$, $(-1 \leq J \leq r-2, 0 \leq k \leq l)$. Assume that $\triangle t$ satisfies (3.1), then for $-1 \leq l \leq M-1$, we have*

$$\begin{aligned}
n||u^{m_{l+1}+J+1}||^2_{L^2(\Omega_f)} &+ \frac{n\nu\triangle t}{2}\sum_{i=0}^{m_{l+1}+J}||u^{i+1}||^2_{H_f} + \rho g S_0||\phi^{m_{l+1}+J+1}||^2_{L^2(\Omega_p)} + \frac{\rho g\triangle t}{4r}\sum_{i=0}^{m_{l+1}+J}||\mathbf{K}^{1/2}\nabla\phi^{i+1}||^2_{L^2(\Omega_p)} \\
&\leq C(T)\{\frac{4nP_1^2\triangle t}{5\nu}\sum_{i=0}^{m_{l+1}+J}||f_1^{i+1}||^2_{L^2(\Omega_f)} + \frac{2\rho g P_2^2\triangle t}{k_{min}}\sum_{i=0}^{m_{l+1}+J}||f_2^{i+1}||^2_{L^2(\Omega_p)} \\
&+ \frac{\triangle t}{2}(n\nu||u^0||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)}) + n||u^0||^2_{L^2(\Omega_f)} + \rho g S_0||\phi^0||^2_{L^2(\Omega_p)}\},
\end{aligned} \tag{3.2}$$

where $C(T) \approx exp(aT)$, a constant depends on the finial time T.

Proof. Taking $v = 2\triangle t u^{m+1}$ in (2.15), using the divergence-free property, sum over $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$,

$$\begin{aligned}
n\{||u^{m_{k+1}}||^2_{L^2(\Omega_f)} &+ \sum_{i=m_k}^{m_{k+1}-1}||u^{i+1} - u^i||^2_{L^2(\Omega_f)} - ||u^{m_k}||^2_{L^2(\Omega_f)}\} + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_f(u^{i+1}, u^{i+1}) \\
&= 2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f_1^{i+1}, u^{i+1}) - 2n\rho g\triangle t\int_\Gamma\phi^{m_k}(\sum_{i=m_k}^{m_{k+1}-1}u^{i+1})\cdot\mathbf{n}_f.
\end{aligned} \tag{3.3}$$

Taking $\psi = 2\triangle s\phi^{m_{k+1}} = 2r\triangle t\phi^{m_{k+1}} = 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\phi^{m_{k+1}}$ in (2.18),

$$\begin{aligned}
\rho g S_0\{||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)} &+ ||\phi^{m_{k+1}} - \phi^{m_k}||^2_{L^2(\Omega_p)} - ||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)}\} + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}}) \\
&= 2\rho g\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f_2^{m_{k+1}}, \phi^{m_{k+1}}) + 2n\rho g\triangle t\int_\Gamma\phi^{m_{k+1}}(\sum_{i=m_k}^{m_{k+1}-1}u^i)\cdot\mathbf{n}_f.
\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned}
&n\{||u^{m_{k+1}}||^2_{L^2(\Omega_f)} + \sum_{i=m_k}^{m_{k+1}-1}||u^{i+1} - u^i||^2_{L^2(\Omega_f)} - ||u^{m_k}||^2_{L^2(\Omega_f)}\} + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_f(u^{i+1}, u^{i+1}) \\
&+ \rho g S_0\{||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + ||\phi^{m_{k+1}} - \phi^{m_k}||^2_{L^2(\Omega_p)} - ||\phi^{m_k}||^2_{L^2(\Omega_p)}\} + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_p(\phi^{m_{k+1}}, \phi^{m_{k+1}}) \\
&= 2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f_1^{i+1}, u^{i+1}) + 2\rho g\triangle t\sum_{i=m_k}^{m_{k+1}-1}(f_2^{m_{k+1}}, \phi^{m_{k+1}}) \\
&\quad - 2\triangle t a_\Gamma(\phi^{m_k}, \sum_{i=m_k}^{m_{k+1}-1}u^i; \phi^{m_{k+1}}, \sum_{i=m_k}^{m_{k+1}-1}u^{i+1}),
\end{aligned} \tag{3.5}$$

here and the following, we define $a_\Gamma(\phi, u; \psi, v) = n\rho g \int_\Gamma (\phi v \cdot \mathbf{n}_f - \psi u \cdot \mathbf{n}_f) d\Gamma$. The first two terms of RHS (right hand side) in (3.5) are bounded by Young and Hölder inequalities,

$$
\begin{aligned}
&2n\triangle t \sum_{i=m_k}^{m_{k+1}-1} (f_1^{i+1}, u^{i+1}) + 2\rho g \triangle t \sum_{i=m_k}^{m_{k+1}-1} (f_2^{m_{k+1}}, \phi^{m_{k+1}}) \\
&\leq 2nP_1 \triangle t \sum_{i=m_k}^{m_{k+1}-1} ||f_1^{i+1}||_{L^2(\Omega_f)} ||u^{i+1}||_{H_f} + \frac{2\rho g P_2 \triangle t}{\sqrt{k_{min}}} \sum_{i=m_k}^{m_{k+1}-1} ||f_2^{m_{k+1}}||_{L^2(\Omega_p)} ||\mathbf{K}^{1/2} \nabla \phi^{m_{k+1}}||_{L^2(\Omega_p)} \\
&\leq \frac{2nP_1^2 \triangle t}{\nu} \sum_{i=m_k}^{m_{k+1}-1} ||f_1^{i+1}||^2_{L^2(\Omega_f)} + \frac{2\rho g P_2^2 \triangle s}{k_{min}} ||f_2^{m_{k+1}}||^2_{L^2(\Omega_p)} \\
&\quad + \frac{n\nu \triangle t}{2} \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1}||^2_{H_f} + \frac{\rho g \triangle t}{2} \sum_{i=m_k}^{m_{k+1}-1} ||\mathbf{K}^{1/2} \nabla \phi^{m_{k+1}}||^2_{L^2(\Omega_p)}. \qquad (3.6)
\end{aligned}
$$

The remains of RHS in (3.5) have the following bound by (2.5)

$$
\begin{aligned}
&-2\triangle t a_\Gamma(\phi^{m_k}, \sum_{i=m_k}^{m_{k+1}-1} u^i; \phi^{m_{k+1}}, \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}) \\
&\leq \frac{\triangle t}{4} (n\nu || \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}||^2_{H_f} + \rho g ||\mathbf{K}^{1/2} \nabla \phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + n\nu || \sum_{i=m_k}^{m_{k+1}-1} u^i||^2_{H_f} + \rho g ||\mathbf{K}^{1/2} \nabla \phi^{m_k}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{2n\rho g C_3 \triangle t}{\sqrt{\nu S_0 k_{min}}} (n|| \sum_{i=m_k}^{m_{k+1}-1} u^{i+1}||^2_{L^2(\Omega_f)} + \rho g S_0 ||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + n|| \sum_{i=m_k}^{m_{k+1}-1} u^i||^2_{L^2(\Omega_f)} + \rho g S_0 ||\phi^{m_k}||^2_{L^2(\Omega_p)}) \\
&\leq \frac{\triangle t}{2} (\sum_{i=m_k}^{m_{k+1}-1} n\nu ||u^{i+1}||^2_{H_f} + n\nu ||u^{m_k}||^2_{H_f} + \rho g ||\mathbf{K}^{1/2} \nabla \phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + \rho g ||\mathbf{K}^{1/2} \nabla \phi^{m_k}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{4n\rho g C_3 \triangle t}{\sqrt{\nu S_0 k_{min}}} (\sum_{i=m_k}^{m_{k+1}} n||u^i||^2_{L^2(\Omega_f)} + \rho g S_0 ||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + \rho g S_0 ||\phi^{m_k}||^2_{L^2(\Omega_p)}). \qquad (3.7)
\end{aligned}
$$

Combining the above inequalities, using Holder's and Young's inequality, we obtain

$$
\begin{aligned}
&n\{||u^{m_{k+1}}||^2_{L^2(\Omega_f)} + \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1} - u^i||^2_{L^2(\Omega_f)} - ||u^{m_k}||^2_{L^2(\Omega_f)}\} + n\nu \triangle t \sum_{i=m_k}^{m_{k+1}-1} ||u^{i+1}||^2_{H_f} \\
&\quad - \frac{n\nu \triangle t}{2} ||u^{m_k}||^2_{H_f} + \rho g S_0 \{||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + ||\phi^{m_{k+1}} - \phi^{m_k}||^2_{L^2(\Omega_p)} - ||\phi^{m_k}||^2_{L^2(\Omega_p)}\} \\
&\quad + \rho g \triangle t \sum_{i=m_k}^{m_{k+1}-1} ||\mathbf{K}^{1/2} \nabla \phi^{m_{k+1}}||_{L^2(\Omega_p)} - \frac{\rho g \triangle t}{2} ||\mathbf{K}^{1/2} \nabla \phi^{m_k}||^2_{L^2(\Omega_p)} \\
&\leq \frac{2nP_1^2 \triangle t}{\nu} \sum_{i=m_k}^{m_{k+1}-1} ||f_1^{i+1}||^2_{L^2(\Omega_f)} + \frac{2\rho g P_2^2 \triangle s}{k_{min}} ||f_2^{m_{k+1}}||^2_{L^2(\Omega_p)} \\
&\quad + \frac{4n\rho g C_3 \triangle t}{\sqrt{\nu S_0 k_{min}}} (\sum_{i=m_k}^{m_{k+1}} n||u^i||^2_{L^2(\Omega_f)} + \rho g S_0 ||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)} + \rho g S_0 ||\phi^{m_k}||^2_{L^2(\Omega_p)}). \qquad (3.8)
\end{aligned}
$$

Sum over $k = 0, 1, \cdots, l$, with $0 \le l \le M-1$ we have

$$n||u^{m_{l+1}}||^2_{L^2(\Omega_f)} + \frac{n\nu\triangle t}{2}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||u^{i+1}||^2_{H_f} + \rho g S_0||\phi^{m_{l+1}}||^2_{L^2(\Omega_f)} + \frac{\rho g\triangle t}{2}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\mathbf{K}^{1/2}\nabla\phi^{m_{k+1}}||^2_{L^2(\Omega_p)}$$
$$\le \frac{4n\rho g C_3\triangle t}{\sqrt{\nu S_0 k_{min}}}\sum_{k=0}^{l}(\sum_{i=m_k}^{m_{k+1}} n||u^i||^2_{L^2(\Omega_f)} + \rho g S_0||\phi^{m_{k+1}}||^2_{L^2(\Omega_p)}) \tag{3.9}$$
$$+ \frac{2nP_1^2\triangle t}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||f_1^{i+1}||^2_{L^2(\Omega_f)} + \frac{2\rho g P_2^2\triangle s}{k_{min}}\sum_{k=0}^{l}||f_2^{m_{k+1}}||^2_{L^2(\Omega_p)}$$
$$+ \frac{\triangle t}{2}(n\nu||u^0||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)}) + n||u^0||^2_{L^2(\Omega_f)} + \rho g S_0||\phi^0||^2_{L^2(\Omega_p)}.$$

Taking $v = 2\triangle t u^{m+1}$ in (2.15), using the divergence-free property again, sum over $m = m_{l+1}, m_{l+1}+1, \cdots, m_{l+1}+J$, $(0 \le J \le r-2)$

$$n||u^{m_{l+1}+J+1}||^2_{L^2(\Omega_f)} + n\sum_{i=m_{l+1}}^{m_{l+1}+J}||u^{i+1}-u^i||^2_{L^2(\Omega_f)} - n||u^{m_{l+1}}||^2_{L^2(\Omega_f)} + 2\triangle t\sum_{i=m_{l+1}}^{m_{l+1}+J}a_f(u^{i+1}, u^{i+1})$$
$$= 2n\triangle t\sum_{i=m_{l+1}}^{m_{l+1}+J}(f_1^{i+1}, u^{i+1}) - 2n\rho g\triangle t\int_\Gamma \phi^{m_{l+1}}(\sum_{i=m_{l+1}}^{m_{l+1}+J}u^{i+1})\cdot\mathbf{n}_f$$
$$\le \frac{4nP_1^2\triangle t}{5\nu}\sum_{i=m_{l+1}}^{m_{l+1}+J}||f_1^{i+1}||^2_{L^2(\Omega_f)} + \frac{5n\nu\triangle t}{4}\sum_{i=m_{l+1}}^{m_{l+1}+J}||u^{i+1}||^2_{H_f} \tag{3.10}$$
$$+ \frac{\triangle t}{4}(\sum_{i=m_{l+1}}^{m_{l+1}+J}n\nu||u^{i+1}||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla\phi^{m_{l+1}}||^2_{L^2(\Omega_p)})$$
$$+ \frac{2n\rho g\triangle t}{\sqrt{\nu S_0 k_{min}}}(\sum_{i=m_{l+1}}^{m_{l+1}+J}n||u^{i+1}||^2_{L^2(\Omega_f)} + \rho g S_0||\phi^{m_{l+1}}||^2_{L^2(\Omega_p)}).$$

Rearrange the inequality, yield

$$n||u^{m_{l+1}+J+1}||^2_{L^2(\Omega_f)} + n\sum_{i=m_{l+1}}^{m_{l+1}+J}||u^{i+1}-u^i||^2_{L^2(\Omega_f)} - n||u^{m_{l+1}}||^2_{L^2(\Omega_f)} + \frac{n\nu\triangle t}{2}\sum_{i=m_{l+1}}^{m_{l+1}+J}||u^{i+1}||^2_{H_f}$$
$$\le \frac{2n\rho g\triangle t}{\sqrt{\nu S_0 k_{min}}}\sum_{i=m_{l+1}}^{m_{l+1}+J}(n||u^{i+1}||^2_{L^2(\Omega_f)} + \rho g S_0||\phi^{m_{l+1}}||^2_{L^2(\Omega_p)}) \tag{3.11}$$
$$+ \frac{4nP_1^2\triangle t}{5\nu}\sum_{i=m_{l+1}}^{m_{l+1}+J}||f_1^{i+1}||^2_{L^2(\Omega_f)} + \frac{\rho g\triangle t}{4}||\mathbf{K}^{1/2}\nabla\phi^{m_{l+1}}||^2_{L^2(\Omega_p)}.$$

Considering the special case, when $l = -1$, then $\phi^{m_{l+1}} = \phi^0$, $u^{m_{l+1}} = u^0$, the above equation can be written as follows:

$$n||u^{J+1}||^2_{L^2(\Omega_f)} + n\sum_{i=0}^{J}||u^{i+1}-u^i||^2_{L^2(\Omega_f)} + \frac{n\nu\triangle t}{2}\sum_{i=0}^{J}||u^{i+1}||^2_{H_f}$$
$$\le \frac{2n\rho g\triangle t}{\sqrt{\nu S_0 k_{min}}}\sum_{i=0}^{J}(n||u^{i+1}||^2_{L^2(\Omega_f)} + \rho g S_0||\phi^0||^2_{L^2(\Omega_p)}) + \frac{4nP_1^2\triangle t}{5\nu}\sum_{i=0}^{J}||f_1^{i+1}||^2_{L^2(\Omega_f)}$$
$$+ \frac{\rho g\triangle t}{4}||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)} + n||u^0||^2_{L^2(\Omega_f)}. \tag{3.12}$$

Add both sides by $\frac{\rho g\triangle t}{4}||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)}+\rho gS_0||\phi^0||^2_{L^2(\Omega_p)}$, and set $\phi^{J+1}=\phi^0,\ f_2^{J+1}=f_2^0,\ (0\leq J\leq r-2)$ since $\frac{\rho g\triangle t}{4r}\sum_{i=0}^{J}||\mathbf{K}^{1/2}\nabla\phi^{i+1}||^2_{L^2(\Omega_p)}\leq\frac{\rho g\triangle t}{4}||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)}$, then,

$$\begin{aligned}
n||u^{J+1}||^2_{L^2(\Omega_f)}&+n\sum_{i=0}^{J}||u^{i+1}-u^i||^2_{L^2(\Omega_f)}+\frac{n\nu\triangle t}{2}\sum_{i=0}^{J}||u^{i+1}||^2_{H_f}\\
&+\rho gS_0||\phi^{J+1}||^2_{L^2(\Omega_p)}+\frac{\rho g\triangle t}{4r}\sum_{i=0}^{J}||\mathbf{K}^{1/2}\nabla\phi^{i+1}||^2_{L^2(\Omega_p)}\\
&\leq\frac{2n\rho g\triangle t}{\sqrt{\nu S_0k_{min}}}\sum_{i=0}^{J}(n||u^{i+1}||^2_{L^2(\Omega_f)}+\rho gS_0||\phi^{i+1}||^2_{L^2(\Omega_p)})\\
&+\frac{4nP_1^2\triangle t}{5\nu}\sum_{i=0}^{J}||f_1^{i+1}||^2_{L^2(\Omega_f)}+\frac{2\rho gP_2^2\triangle t}{k_{min}}\sum_{i=0}^{J}||f_2^{i+1}||^2_{L^2(\Omega_p)}\\
&+\frac{\triangle t}{2}(n\nu||u^0||^2_{H_f}+\rho g||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)})+n||u^0||^2_{L^2(\Omega_f)}+\rho gS_0||\phi^0||^2_{L^2(\Omega_p)}.
\end{aligned}\tag{3.13}$$

Combine (3.9) and (3.11), and set $\phi^{m_{k+1}+J+1}=\phi^{m_{k+1}},\ f_2^{m_{k+1}+J+1}=f_2^{m_{k+1}},\ (-1\leq J\leq r-2,\ \forall\ l\geq -1)$, we arrive at

$$\begin{aligned}
n||u^{m_{l+1}+J+1}||^2_{L^2(\Omega_f)}&+\frac{n\nu\triangle t}{2}\sum_{i=0}^{m_{l+1}+J}||u^{i+1}||^2_{H_f}+\rho gS_0||\phi^{m_{l+1}+J+1}||^2_{L^2(\Omega_p)}+\frac{\rho g\triangle t}{4r}\sum_{i=0}^{m_{l+1}+J}||\mathbf{K}^{1/2}\nabla\phi^{i+1}||^2_{L^2(\Omega_p)}\\
&\leq\frac{4n\rho gC_3\triangle t}{\sqrt{\nu S_0k_{min}}}\sum_{i=0}^{m_{l+1}+J}(n||u^{i+1}||^2_{L^2(\Omega_f)}+\rho gS_0||\phi^{i+1}||^2_{L^2(\Omega_p)})\\
&+\frac{4nP_1^2\triangle t}{5\nu}\sum_{i=0}^{m_{l+1}+J}||f_1^{i+1}||^2_{L^2(\Omega_f)}+\frac{2\rho gP_2^2\triangle t}{k_{min}}\sum_{i=0}^{m_{l+1}+J}||f_2^{i+1}||^2_{L^2(\Omega_p)}\\
&+\frac{\triangle t}{2}(n\nu||u^0||^2_{H_f}+\rho g||\mathbf{K}^{1/2}\nabla\phi^0||^2_{L^2(\Omega_p)})+n||u^0||^2_{L^2(\Omega_f)}+\rho gS_0||\phi^0||^2_{L^2(\Omega_p)}.
\end{aligned}\tag{3.14}$$

Finally, choosing $\triangle t$, such that $\frac{4n\rho gC_3\triangle t}{\sqrt{\nu S_0k_{min}}}<1$, which is required to apply the discrete Gronwall inequality to (3.14), (which contributes a C(T) term). □

4. Convergence Analysis. In this section, we analyze the error in Algorithm 3.2. We will use the following notations. Define $u_c^m=u(t^m),\ \phi_c^m=\phi(t^m),\ p_c^m=p(t^m)$. Following (2.10)-(2.11), we define $u_m=P_hu(t^m),\ \phi_m=P_h\phi(t^m),\ p_m=P_hp(t^m)$, then we set $e_c^m=u_c^m-u_m,\ \epsilon_c^m=\phi_c^m-\phi_m,\ \eta_c^m=p_c^m-p_m$, and $e^m=u_m-u^m,\ \epsilon^m=\phi_m-\phi^m,\ \eta^m=p_m-p^m$. Obviously, we observe that $u(t^m)-u^m=e_c^m+e^m$ and $\phi(t^m)-\phi^m=\epsilon_c^m+\epsilon^m$, from approximation properties, we have $||e_c^m||_{L^2(\Omega_f)}+||\epsilon_c^m||_{L^2(\Omega_p)}\leq Ch^2$, $||e_c^m||_{H_f}+||\epsilon_c^m||_{H_p}\leq Ch$. Moreover, we suppose that $e^0=0,\epsilon^0=0$.

Then, by (1.19) and (2.10)-(2.11), for $(\mathbf{v},q)\in(\mathbf{W}_h,Q_h)$, we have

$$n(\frac{u_{m+1}-u_m}{\triangle t},v)+a_f(u_{m+1},v)+b(v,p_{m+1})=-n(w_{f,t}^{m+1},v)+n(f_1^{m+1},v)-n\rho g\int_\Gamma\phi_{m+1}v\cdot\mathbf{n}_f\tag{4.1}$$

$$b(u_{m+1},q)=0.\tag{4.2}$$

$$\rho gS_0(\frac{\phi_{m+1}-\phi_m}{\triangle t},\psi)+a_p(\phi_{m+1},\psi)=-\rho gS_0(w_{p,t}^{m+1},\psi)+\rho g(f_2^{m+1},\psi)+n\rho g\int_\Gamma\psi u_{m+1}\cdot\mathbf{n}_f,\tag{4.3}$$

where

$$\begin{aligned}
w_{f,t}^{m+1}&=\frac{u_{m+1}-u_m}{\triangle t}-u_t(t^{m+1})\\
&=[\frac{u_{m+1}-u_m}{\triangle t}-\frac{u(t^{m+1})-u(t^m)}{\triangle t}]+[\frac{u(t^{m+1})-u(t^m)}{\triangle t}-u_t(t^{m+1})]\\
&=w_{f,t,1}^{m+1}+w_{f,t,2}^{m+1},
\end{aligned}$$

and

$$\begin{aligned} w_{p,t}^{m+1} &= \frac{\phi_{m+1}-\phi_m}{\triangle t} - \phi_t(t^{m+1}) \\ &= [\frac{\phi_{m+1}-\phi_m}{\triangle t} - \frac{\phi(t^{m+1})-\phi(t^m)}{\triangle t}] + [\frac{\phi(t^{m+1})-\phi(t^m)}{\triangle t} - \phi_t(t^{m+1})] \\ &= w_{p,t,1}^{m+1} + w_{p,t,2}^{m+1}. \end{aligned}$$

It is easy to verify that the following properties of $w_{f,t,1}^{m+1}, w_{f,t,2}^{m+1}, w_{p,s,1}^{m+1}$ and $w_{p,s,2}^{m+1}$ hold: from the definition

$$w_{f,t,1}^{m+1} = (P_h - I)\frac{u(t^{m+1})-u(t^m)}{\triangle t} = \frac{1}{\triangle t}\int_{t^m}^{t^{m+1}} (P_h - I)u_t(t)dt,$$

then we have

$$\begin{aligned} ||w_{f,t,1}^{m+1}||_{L^2(\Omega_f)}^2 &= \frac{1}{\triangle t^2}\int_\Omega (\int_{t^m}^{t^{m+1}} (P_h - I)u_t(t)dt)^2 dx \\ &\leq \frac{1}{\triangle t^2}\int_\Omega \int_{t^m}^{t^{m+1}} ((P_h - I)u_t(t))^2 dt \int_{t^m}^{t^{m+1}} 1^2 dtdx \\ &\leq \frac{1}{\triangle t}\int_{t^m}^{t^{m+1}} ||(P_h - I)u_t(t)||_{L^2(\Omega_f)}^2 dt. \end{aligned} \quad (4.4)$$

Similarly,

$$\triangle t w_{f,t,2}^{m+1} = u(t^{m+1}) - u(t^m) - \triangle t u_t(t^{m+1}) = -\int_{t^m}^{t^{m+1}} (t - t^m)u_{tt}(t)dt,$$

which means

$$\begin{aligned} ||w_{f,t,2}^{m+1}||_{L^2(\Omega_f)}^2 &= \frac{1}{\triangle t^2}\int_\Omega (\int_{t^m}^{t^{m+1}} (t-t^m)u_{tt}(t)dt)^2 dx \\ &\leq \frac{1}{\triangle t^2}\int_\Omega \int_{t^m}^{t^{m+1}} (u_{tt}(t))^2 dt \int_{t^m}^{t^{m+1}} (t-t^m)^2 dtdx \leq \triangle t \int_{t^m}^{t^{m+1}} ||u_{tt}||_{L^2(\Omega_f)}^2 dt. \end{aligned} \quad (4.5)$$

The same as $w_{p,t,1}^{m+1}$, $w_{p,t,2}^{m+1}$, while consider the large time step size $\triangle s$, then,

$$||w_{p,s,1}^{m_{k+1}}||_{L^2(\Omega_p)}^2 \leq \frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}} ||(P_h - I)\phi_s(s)||_{L^2(\Omega_p)}^2 ds, \quad (4.6)$$

and

$$||w_{p,s,2}^{m_{k+1}}||_{L^2(\Omega_p)}^2 \leq \triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_{ss}||_{L^2(\Omega_p)}^2 ds. \quad (4.7)$$

By the equivalence between $||u||_{H_f}$ and $||\nabla u||_{L^2(\Omega_f)}$, $||\phi||_{H_p}$ and $||\nabla \phi||_{L^2(\Omega_p)}$,

$$\begin{aligned} ||u_{m+1} - u_m||_{H_f}^2 &= ||P_h(u(t^{m+1}) - u(t^m))||_{H_f}^2 \leq C||u(t^{m+1}) - u(t^m)||_{H_f}^2 \\ &\leq C\int_{\Omega_f} (\nabla(u(t^{m+1}) - u(t^m)))^2 dx \leq C\int_{\Omega_f} (\int_{t^m}^{t^{m+1}} \nabla u_t dt)^2 dx \\ &\leq C\int_{\Omega_f} \int_{t^m}^{t^{m+1}} |\nabla u_t|^2 dt \int_{t^m}^{t^{m+1}} 1 dtdx \leq C\triangle t \int_{t^m}^{t^{m+1}} ||u_t||_{H_f}^2 dt. \end{aligned} \quad (4.8)$$

Do the same as (4.8), we have

$$||\phi_{m+1} - \phi_m||^2_{H_p} \leq C\triangle t \int_{t^m}^{t^{m+1}} ||\phi_t||^2_{H_p} dt, \tag{4.9}$$

$$||u_{m_{k+1}} - u_{m_k}||^2_{H_f} \leq C\triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||u_s||^2_{H_f} ds, \tag{4.10}$$

$$||\phi_{m_{k+1}} - \phi_{m_k}||^2_{H_p} \leq C\triangle s \int_{t^{m_k}}^{t^{m_{k+1}}} ||\phi_s||^2_{H_p} ds. \tag{4.11}$$

Considering small time step size $\triangle t$ and subtracting (4.1) from (2.15) gives

$$\begin{aligned}
&n(\frac{e^{m+1}-e^m}{\triangle t}, v) + a_f(e^{m+1}, v) + b(v, \eta^{m+1}) \\
&\qquad = -n(w^{m+1}_{f,t}, v) - n\rho g \int_\Gamma (\phi_{m+1} - \phi_m) v \cdot \mathbf{n}_f - n\rho g \int_\Gamma (\phi_m - \phi^{m_k}) v \cdot \mathbf{n}_f, \\
&b(e^{m+1}, q) = 0.
\end{aligned} \tag{4.12}$$

Considering larger time step size $\triangle s = r\triangle t$ and subtracting (4.3) from (2.18), we obtain

$$\begin{aligned}
&\rho g S_0(\frac{\epsilon^{m_{k+1}} - \epsilon^{m_k}}{\triangle s}, \psi) + a_p(\epsilon^{m_{k+1}}, \psi) \\
&\quad = -\rho g S_0(w^{m_{k+1}}_{p,s}, \psi) + n\rho g \int_\Gamma \psi(u_{m_{k+1}} - u_{m_k}) \cdot \mathbf{n}_f + n\rho g \int_\Gamma \psi(u_{m_k} - S^{m_k}) \cdot \mathbf{n}_f.
\end{aligned} \tag{4.13}$$

For the error estimate we impose a timestep restriction of the form $\triangle t \leq Ch$ that is different than (3.1). Since convergence implies stability, Theorem 4.1 also gives a stability condition with different dependencies on the physical parameters than (3.1).

Theorem 4.1. *Suppose the true solution is smooth, the initial approximations are sufficiently accurate and that the time step and mesh width $\triangle t, h$ satisfy*

$$4rn\rho g C_4 \triangle t h^{-1} \leq 1, \tag{4.14}$$

then the following estimate for the error at the larger time steps (the synchronization points) holds:

$$\begin{aligned}
n||e^{m_{l+1}}||^2_{L^2(\Omega_f)} + n\nu\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||e^{i+1}||^2_{H_f} + \rho g S_0 ||\epsilon^{m_{l+1}}||^2_{L^2(\Omega_p)} + \rho g \triangle s \sum_{k=0}^{l} ||\mathbf{K}^{1/2} \nabla \epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} \\
\leq C_5(\triangle t^2 + h^4).
\end{aligned} \tag{4.15}$$

Proof. Taking $v = 2\triangle t e^{m+1}$ in (4.12), using the divergence-free property, sum over $m = m_k, m_k + 1, \cdots, m_{k+1} - 1$, yield

$$\begin{aligned}
&n||e^{m_{k+1}}||^2_{L^2(\Omega_f)} + n \sum_{i=m_k}^{m_{k+1}-1} ||e^{i+1} - e^i||^2_{L^2(\Omega_f)} - n||e^{m_k}||^2_{L^2(\Omega_f)} + 2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_f(e^{i+1}, e^{i+1}) \\
&\quad = -2n\triangle t \sum_{i=m_k}^{m_{k+1}-1} (w^{i+1}_{f,t}, e^{i+1}) - 2n\rho g \triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_{i+1} - \phi_i) e^{i+1} \cdot \mathbf{n}_f \\
&\qquad - 2n\rho g \triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_i - \phi^{m_k}) e^{i+1} \cdot \mathbf{n}_f.
\end{aligned} \tag{4.16}$$

Taking $\psi = 2r\triangle t\epsilon^{m_{k+1}} = 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\epsilon^{m_{k+1}}$ in (4.13),

$$\begin{aligned}
&\rho g S_0||\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + \rho g S_0||\epsilon^{m_{k+1}} - \epsilon^{m_k}||^2_{L^2(\Omega_p)} - \rho g S_0||\epsilon^{m_k}||^2_{L^2(\Omega_p)} + 2\triangle s a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&= -2\rho g S_0\triangle s(w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) + 2n\rho g\triangle t\int_\Gamma \sum_{i=m_k}^{m_{k+1}-1}\epsilon^{m_{k+1}}(u_{m_{k+1}} - u_{m_k})\cdot \mathbf{n}_f \\
&\quad + 2n\rho g\triangle t\int_\Gamma \sum_{i=m_k}^{m_{k+1}-1}\epsilon^{m_{k+1}}(u_{m_k} - u^i)\cdot \mathbf{n}_f.
\end{aligned} \tag{4.17}$$

Combining the above equalities (4.16) and (4.17), we obtain

$$\begin{aligned}
&n\{||e^{m_{k+1}}||^2_{L^2(\Omega_f)} + \sum_{i=m_k}^{m_{k+1}-1}||e^{i+1} - e^i||^2_{L^2(\Omega_f)} - ||e^{m_k}||^2_{L^2(\Omega_f)}\} + 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_f(e^{i+1}, e^{i+1}) \\
&+ \rho g S_0\{||\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + ||\epsilon^{m_{k+1}} - \epsilon^{m_k}||^2_{L^2(\Omega_p)} - ||\epsilon^{m_k}||^2_{L^2(\Omega_p)}\} + 2\triangle s a_p(\epsilon^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&= -2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, e^{i+1}) - 2\rho g S_0\triangle s(w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&\quad - 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_{i+1} - \phi_i, u_{m_{k+1}} - u_{m_k}; \epsilon^{m_{k+1}}, e^{i+1}) \\
&\quad - 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_i - \phi^{m_k}, u_{m_k} - u^i; \epsilon^{m_{k+1}}, e^{i+1}).
\end{aligned} \tag{4.18}$$

The first term of the RHS in (4.18) is bounded by Young, Poincaré and Hölder inequalities:

$$\begin{aligned}
&- 2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, e^{i+1}) - 2\rho g S_0\triangle s(w_{p,s}^{m_{k+1}}, \epsilon^{m_{k+1}}) \\
&\le \frac{n\nu\triangle t}{4}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||^2_{H_f} + \frac{\rho g\triangle s}{4}||\mathbf{K}^{1/2}\nabla\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} \\
&\quad + \triangle t\sum_{i=m_k}^{m_{k+1}-1}\frac{4nP_1^2}{\nu}||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)} + \frac{4\rho g P_2^2 S_0^2\triangle s}{k_{min}}||w_{p,s}^{m_{k+1}}||^2_{L^2(\Omega_p)}.
\end{aligned} \tag{4.19}$$

The second term of the RHS in (4.18) is bounded using (2.4) by:

$$\begin{aligned}
&- 2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_{i+1} - \phi_i, u_{m_{k+1}} - u_{m_k}; \epsilon^{m_{k+1}}, e^{i+1}) \\
&\le \frac{n\nu\triangle t}{4}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||^2_{H_f} + \frac{\rho g\triangle s}{4}||\mathbf{K}^{1/2}\nabla\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} \\
&\quad + \frac{4n\rho^2 g^2 k_{max}C_2\triangle t}{k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1} - \phi_i||^2_{H_p} + \frac{4n^2\nu\rho g C_2\triangle s}{k_{min}}||(u_{m_{k+1}} - u_{m_k})||^2_{H_f}.
\end{aligned} \tag{4.20}$$

The third term of RHS in (4.18) is bounded by (2.5)-(2.6)

$$
\begin{aligned}
&-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\phi_i-\phi^{m_k},u_{m_k}-u^i;\epsilon^{m_{k+1}},e^{i+1})\\
&=-2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\{a_\Gamma(\phi_i-\phi_{m_k},u_{m_k}-u_i;\epsilon^{m_{k+1}},e^{i+1})+a_\Gamma(\epsilon^{m_k},e^i;\epsilon^{m_{k+1}},e^{i+1})\}\\
&=2\triangle t\sum_{i=m_k}^{m_{k+1}-1}\{a_\Gamma(\epsilon^{m_{k+1}}-\epsilon^{m_k},e^{i+1}-e^i;\epsilon^{m_{k+1}},e^{i+1})-a_\Gamma(\phi_i-\phi_{m_k},u_{m_k}-u_i;\epsilon^{m_{k+1}},e^{i+1})\}\\
&\le\frac{n\nu\triangle t}{2}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||^2_{H_f}+\frac{\rho g\triangle s}{2}||\mathbf{K}^{1/2}\nabla\epsilon^{m_{k+1}}||^2_{L^2(\Omega)}\\
&\quad+\frac{4rn\rho gC_4\triangle t}{h}(\sum_{i=m_k}^{m_{k+1}-1}n||e^{i+1}-e^i||^2_{L^2(\Omega_f)}+\rho gS_0||\epsilon^{m_{k+1}}-\epsilon^{m_k}||^2_{L^2(\Omega_p)})\\
&\quad+\frac{4n\rho gC_2\triangle t}{k_{min}}\sum_{i=m_k}^{m_{k+1}-1}(n\nu||u_{m_k}-u_i||^2_{H_f}+\rho g||\mathbf{K}^{1/2}\nabla(\phi_i-\phi_{m_k})||^2_{L^2(\Omega_p)}). \qquad (4.21)
\end{aligned}
$$

Note that $\frac{4rn\rho gC_4\triangle t}{h}\le 1$. Combine the above inequalities and sum over $k=0,1,\cdots,l$. We arrive at

$$
\begin{aligned}
&n||e^{m_{l+1}}||^2_{L^2(\Omega_f)}+n\nu\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||^2_{H_f}+\rho gS_0||\epsilon^{m_{l+1}}||^2_{L^2(\Omega_p)}+\rho g\triangle s\sum_{k=0}^{l}||\mathbf{K}^{1/2}\nabla\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)}\\
&\le\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\frac{4nP_1^2}{\nu}||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)}+\frac{4\rho gP_2^2S_0^2\triangle s}{k_{min}}\sum_{k=0}^{l}||w_{p,s}^{m_{k+1}}||^2_{L^2(\Omega_p)}\\
&\quad+\frac{4\nu n^2\rho gC_2\triangle s}{k_{min}}\sum_{k=0}^{l}||u_{m_{k+1}}-u_{m_k}||^2_{H_f}+\frac{4n\rho^2g^2k_{max}C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1}-\phi_i||^2_{H_p}\\
&\quad+\frac{4\nu n^2\rho gC_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||u_{m_k}-u_i||^2_{H_f}+\frac{4n\rho^2g^2k_{max}C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_i-\phi_{m_k}||^2_{H_p}. \qquad (4.22)
\end{aligned}
$$

By using (4.4)-(4.11) and the approximate properties of P_h, the first term of RHS in (4.22) is bounded by

$$
\begin{aligned}
&\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\frac{4nP_1^2}{\nu}||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)}+\frac{4\rho gP_2^2S_0^2\triangle s}{k_{min}}\sum_{k=0}^{l}||w_{p,s}^{m_{k+1}}||^2_{L^2(\Omega_p)}\\
&\le\frac{4nP_1^2\triangle t}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}(\frac{1}{\triangle t}\int_{t^i}^{t^{i+1}}||(P_h-I)u_t(t)||^2_{L^2(\Omega_f)}dt+\triangle t\int_{t^i}^{t^{i+1}}||u_{tt}||^2_{L^2(\Omega_f)}dt)\\
&\quad+\frac{4\rho gP_2^2S_0^2\triangle s}{k_{min}}\sum_{k=0}^{l}(\frac{1}{\triangle s}\int_{t^{m_k}}^{t^{m_{k+1}}}||(P_h-I)\phi_s(s)||^2_{L^2(\Omega_p)}ds+\triangle s\int_{t^{m_k}}^{t^{m_{k+1}}}||\phi_{ss}||^2_{L^2(\Omega_p)}ds)\\
&\le\frac{4nP_1^2}{\nu}(\int_0^T||(P_h-I)u_t(t)||^2_{L^2(\Omega_f)}dt+\triangle t^2\int_0^T||u_{tt}||^2_{L^2(\Omega_f)}dt)\\
&\quad+\frac{4\rho gP_2^2S_0^2}{k_{min}}(\int_0^T||(P_h-I)\phi_s(s)||^2_{L^2(\Omega_p)}ds+\triangle s^2\int_0^T||\phi_{ss}||^2_{L^2(\Omega_p)}ds)\\
&\le C_5(\triangle t^2+h^4). \qquad (4.23)
\end{aligned}
$$

Here and afterwards, C_5 denotes a constant depending on $\nu,n,\rho,g,S_0,k_{min},k_{max},r,T,P_1$ and C_2. The second term of RHS in (4.22) is bounded by

$$\frac{4\nu n^2\rho g C_2\triangle s}{k_{min}}\sum_{k=0}^{l}||u_{m_{k+1}}-u_{m_k}||_{H_f}^2 \le \frac{4\nu n^2\rho g C_2\triangle s^2}{k_{min}}\int_0^T ||u_s(s)||_{H_f}^2 ds \le C_5\triangle t^2. \tag{4.24}$$

The third term of RHS in (4.22) is bounded by

$$\frac{4n\rho^2 g^2 k_{max}C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1}-\phi_i||_{H_p}^2 \le \frac{4n\rho^2 g^2 k_{max}C_2\triangle t^2}{k_{min}}\int_0^T ||\phi_t(t)||_{H_p}^2 dt \le C_5\triangle t^2. \tag{4.25}$$

The remaining terms in (4.22) are bounded by

$$\begin{aligned}
&\frac{4\nu n^2\rho g C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||u_{m_k}-u_i||_{H_f}^2 + \frac{4n\rho^2 g^2 k_{max}C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_i-\phi_{m_k}||_{H_p}^2 \\
&\le \frac{4\nu n^2\rho g r C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||u_{i+1}-u_i||_{H_f}^2 + \frac{4n\rho^2 g^2 k_{max} r C_2\triangle t}{k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||\phi_i-\phi_{i+1}||_{H_p}^2 \\
&\le \frac{4\nu n^2\rho g r C_2\triangle t^2}{k_{min}}\int_0^T ||u_t(t)||_{H_f}^2 dt + \frac{4n\rho^2 g^2 k_{max} r C_2\triangle t^2}{k_{min}}\int_0^T ||\phi_t(t)||_{H_p}^2 dt \\
&\le C_5\triangle t^2.
\end{aligned} \tag{4.26}$$

Combine the above bounds and add the initial data. This yields the final result,

$$\begin{aligned}
n||e^{m_{l+1}}||_{L^2(\Omega_f)}^2 + n\nu\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||e^{i+1}||_{H_f}^2 + \rho g S_0||\epsilon^{m_{l+1}}||_{L^2(\Omega_p)}^2 + \rho g\triangle s\sum_{k=0}^{l}||\mathbf{K}^{1/2}\nabla\epsilon^{m_{k+1}}||_{L^2(\Omega_p)}^2 \\
\le C_5(\triangle t^2 + h^4).
\end{aligned} \tag{4.27}$$

□

For the error in time derivatives, we have the following error estimate. Here we use the following notations:

$$d_t e^{m+1} \triangleq \frac{e^{m+1}-e^m}{\triangle t}, \quad \lambda(u,v) \triangleq \sum_{i=1}^{d-1}\int_\Gamma \frac{n\alpha}{\sqrt{\tau_i\cdot\mathbf{K}\tau_i}}(u\cdot\tau_i)(v\cdot\tau_i).$$

THEOREM 4.2. *Under the assumptions of the Theorem 4.1, the following error estimate holds:*

$$\begin{aligned}
n\triangle t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}||d_t e^{i+1}||_{L^2(\Omega_f)}^2 + n\nu||e^{m_{l+1}}||_{H_f}^2 + \lambda(e^{m_{l+1}}, e^{m_{l+1}}) + \rho g S_0\triangle t\sum_{k=0}^{l}||d_t\epsilon^{m_{k+1}}||_{L^2(\Omega_p)}^2 \\
+\rho g\sum_{i=m_k}^{m_{k+1}-1}||\mathbf{K}^{1/2}\nabla\epsilon^{m_{l+1}}||_{L^2(\Omega_p)}^2 \le C_5(\triangle t + h^4 + \triangle t^{-1}h^4).
\end{aligned} \tag{4.28}$$

Proof. Taking $v = 2\triangle t d_t e^{m+1} = 2(e^{m+1}-e^m)$ in (4.12), using the divergence-free property, sum over $m = m_k, m_k+1, \cdots, m_{k+1}-1$, we get

$$\begin{aligned}
2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_t e^{i+1}||_{L^2(\Omega_f)}^2 + a_f(e^{m_{k+1}}, e^{m_{k+1}}) - a_f(e^{m_k}, e^{m_k}) + \triangle t^2\sum_{i=m_k}^{m_{k+1}-1}a_f(d_t e^{i+1}, d_t e^{i+1}) \\
= -2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, d_t e^{i+1}) - 2n\rho g\triangle t\int_\Gamma\sum_{i=m_k}^{m_{k+1}-1}(\phi_{i+1}-\phi_i)d_t e^{i+1}\cdot\mathbf{n}_f \\
- 2n\rho g\triangle t\int_\Gamma\sum_{i=m_k}^{m_{k+1}-1}(\phi_i-\phi^{m_k})d_t e^{i+1}\cdot\mathbf{n}_f.
\end{aligned} \tag{4.29}$$

Taking $\psi = 2r\triangle t d_t\epsilon^{m_{k+1}} = 2r(\epsilon^{m_{k+1}} - \epsilon^{m_k}) = 2\sum_{i=m_k}^{m_{k+1}-1}(\epsilon^{m_{k+1}} - \epsilon^{m_k})$ in (4.13) yield

$$
\begin{aligned}
&2\rho g S_0 \triangle t||d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + \sum_{i=m_k}^{m_{k+1}-1}\left\{a_p(\epsilon^{m_{k+1}},\epsilon^{m_{k+1}}) - a_p(\epsilon^{m_k},\epsilon^{m_k}) + \triangle t^2 a_p(d_t\epsilon^{m_{k+1}}, d_t\epsilon^{m_{k+1}})\right\} \\
&= -2\rho g S_0 \triangle s(w_{p,s}^{m_{k+1}}, d_t\epsilon^{m_{k+1}}) + 2n\rho g\triangle t\int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} d_t\epsilon^{m_{k+1}}(u_{m_{k+1}} - u_{m_k})\cdot\mathbf{n}_f \\
&\quad + 2n\rho g\triangle t\int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} d_t\epsilon^{m_{k+1}}(u_{m_k} - u^i)\cdot\mathbf{n}_f.
\end{aligned} \tag{4.30}
$$

Combining the above two equalities (4.29) and (4.30), we have

$$
\begin{aligned}
&2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_te^{i+1}||^2_{L^2(\Omega_f)} + a_f(e^{m_{k+1}}, e^{m_{k+1}}) - a_f(e^{m_k}, e^{m_k}) + \triangle t^2\sum_{i=m_k}^{m_{k+1}-1} a_f(d_te^{i+1}, d_te^{i+1}) \\
&+ 2\rho g S_0\triangle t||d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + \sum_{i=m_k}^{m_{k+1}-1}\left\{a_p(\epsilon^{m_{k+1}},\epsilon^{m_{k+1}}) - a_p(\epsilon^{m_k},\epsilon^{m_k}) + \triangle t^2 a_p(d_t\epsilon^{m_{k+1}}, d_t\epsilon^{m_{k+1}})\right\} \\
&= -2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, d_te^{i+1}) - 2\rho g S_0\triangle s(w_{p,s}^{m_{k+1}}, d_t\epsilon^{m_{k+1}}) \\
&\quad - 2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_{i+1} - \phi_i, u_{m_{k+1}} - u_{m_k}; d_t\epsilon^{m_{k+1}}, d_te^{i+1}) \\
&\quad - 2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_i - \phi^{m_k}, u_{m_k} - u^i; d_t\epsilon^{m_{k+1}}, d_te^{i+1}).
\end{aligned} \tag{4.31}
$$

The first term of RHS in (4.31) is bounded by Young and Hölder inequalities

$$
\begin{aligned}
&- 2n\triangle t\sum_{i=m_k}^{m_{k+1}-1}(w_{f,t}^{i+1}, d_te^{i+1}) - 2\rho g S_0\triangle s(w_{p,s}^{m_{k+1}}, d_t\epsilon^{m_{k+1}}) \\
&\le n\triangle t\sum_{i=m_k}^{m_{k+1}-1}||d_te^{i+1}||^2_{L^2(\Omega_f)} + n\triangle t\sum_{i=m_k}^{m_{k+1}-1}||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)} \\
&\quad + \rho g S_0\triangle t||d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + r\rho g S_0\triangle s||w_{p,s}^{m_{k+1}}||^2_{L^2(\Omega_p)}.
\end{aligned} \tag{4.32}
$$

The second term of the RHS in (4.31) is bounded by (2.4)

$$
\begin{aligned}
&-2\triangle t\sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_{i+1} - \phi_i, u_{m_{k+1}} - u_{m_k}; d_t\epsilon^{m_{k+1}}, d_te^{i+1}) \\
&\le \frac{\triangle t^2}{3}\sum_{i=m_k}^{m_{k+1}-1}(n\nu||d_te^{i+1}||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{3n\rho^2 g^2 k_{max} C_2}{k_{min}}\sum_{i=m_k}^{m_{k+1}-1}||\phi_{i+1} - \phi_i||^2_{H_p} + \frac{3rn^2\nu\rho g C_2}{k_{min}}||u_{m_{k+1}} - u_{m_k}||^2_{H_f}.
\end{aligned} \tag{4.33}
$$

The third term of the RHS in (4.31) is bounded by

$$
\begin{aligned}
& -2\triangle t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\phi_i - \phi^{m_k}, u_{m_k} - u^i; d_t\epsilon^{m_{k+1}}, d_t e^{i+1}) \\
&= -2\triangle t \sum_{i=m_k}^{m_{k+1}-1} \{a_\Gamma(\phi_i - \phi_{m_k}, u_{m_k} - u_i; d_t\epsilon^{m_{k+1}}, d_t e^{i+1}) + a_\Gamma(\epsilon^{m_k}, e^i; d_t\epsilon^{m_{k+1}}, d_t e^{i+1})\} \\
&\leq \frac{2\triangle t^2}{3} \sum_{i=m_k}^{m_{k+1}-1} (n\nu||d_t e^{i+1}||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{3n\rho g C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} (n\nu||e^i||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla\epsilon^{m_k}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{3n^2\nu\rho g C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} ||u_{m_k} - u_i||^2_{H_f} + \frac{3n\rho^2 g^2 k_{max} C_2}{k_{min}} ||\phi_i - \phi_{m_k}||^2_{H_p} \\
&\leq \frac{2\triangle t^2}{3} \sum_{i=m_k}^{m_{k+1}-1} (n\nu||d_t e^{i+1}||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{3n\rho g C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} (n\nu||e^i||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla\epsilon^{m_k}||^2_{L^2(\Omega_p)}) \\
&\quad + \frac{3rn^2\nu\rho g C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} ||u_{i+1} - u_i||^2_{H_f} + \frac{3rn\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_i - \phi_{i+1}||^2_{H_p}. \qquad (4.34)
\end{aligned}
$$

Then, by using (4.31)-(4.34), we have

$$
\begin{aligned}
& n\triangle t \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + n\nu\{||e^{m_{k+1}}||^2_{H_f} - ||e^{m_k}||^2_{H_f}\} \\
&\quad + \lambda(e^{m_{k+1}}, e^{m_{k+1}}) - \lambda(e^{m_k}, e^{m_k}) + \triangle t^2 \sum_{i=m_k}^{m_{k+1}-1} \lambda(d_t e^{i+1}, d_t e^{i+1}) \\
&\quad + \rho g S_0 \triangle t||d_t\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + \rho g \sum_{i=m_k}^{m_{k+1}-1} \Big\{||\mathbf{K}^{1/2}\nabla\epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} - ||\mathbf{K}^{1/2}\nabla\epsilon^{m_k}||^2_{L^2(\Omega_p)}\Big\} \\
&\leq n\triangle t \sum_{i=m_k}^{m_{k+1}-1} ||w^{i+1}_{f,t}||^2_{L^2(\Omega_f)} + r\rho g S_0 \triangle s||w^{m_{k+1}}_{p,s}||^2_{L^2(\Omega_p)} \\
&\quad + \frac{3n\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_i||^2_{H_p} + \frac{3rn^2\nu\rho g C_2}{k_{min}} ||u_{m_{k+1}} - u_{m_k}||^2_{H_f} \\
&\quad + \frac{3rn^2\nu\rho g C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} ||u_{i+1} - u_i||^2_{H_f} + \frac{3rn\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_i - \phi_{i+1}||^2_{H_p} \\
&\quad + \frac{3n\rho g C_2}{k_{min}} \sum_{i=m_k}^{m_{k+1}-1} (n\nu||e^i||^2_{H_f} + \rho g||\mathbf{K}^{1/2}\nabla\epsilon^{m_k}||^2_{L^2(\Omega_p)}). \qquad (4.35)
\end{aligned}
$$

Sum over $k = 0, 1, \cdots, l$, since $\lambda(u, u) \geq 0$, we arrive at

$$
\begin{aligned}
& n\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + n\nu ||e^{m_{l+1}}||^2_{H_f} + \lambda(e^{m_{l+1}}, e^{m_{l+1}}) + \triangle t^2 \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \lambda(d_t e^{i+1}, d_t e^{i+1}) \\
& \quad + \rho g S_0 \triangle t \sum_{k=0}^{l} ||d_t \epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} + \rho g \sum_{i=m_k}^{m_{k+1}-1} ||\mathbf{K}^{1/2} \nabla \epsilon^{m_{l+1}}||^2_{L^2(\Omega_p)} \\
& \leq n\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||w^{i+1}_{f,t}||^2_{L^2(\Omega_f)} + r\rho g S_0 \triangle s \sum_{k=0}^{l} ||w^{m_{k+1}}_{p,s}||^2_{L^2(\Omega_p)} \\
& \quad + \frac{3n\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_i||^2_{H_p} + \frac{3rn^2 \nu \rho g C_2}{k_{min}} \sum_{k=0}^{l} ||u_{m_{k+1}} - u_{m_k}||^2_{H_f} \\
& \quad + \frac{3rn^2 \nu \rho g C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||u_{i+1} - u_i||^2_{H_f} + \frac{3rn\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_i - \phi_{i+1}||^2_{H_p} \\
& \quad + \frac{3n\rho g C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} (n\nu ||e^i||^2_{H_f} + \rho g ||\mathbf{K}^{1/2} \nabla \epsilon^{m_k}||^2_{L^2(\Omega_p)}). \qquad (4.36)
\end{aligned}
$$

From (4.4)-(4.11), we have

$$n\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||w^{i+1}_{f,t}||^2_{L^2(\Omega_f)} + r\rho g S_0 \triangle s \sum_{k=0}^{l} ||w^{m_{k+1}}_{p,s}||^2_{L^2(\Omega_p)} \leq C_5(\triangle t^2 + h^4), \qquad (4.37)$$

$$\frac{3n\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_{i+1} - \phi_i||^2_{H_p} \leq \frac{3n\rho^2 g^2 k_{max} C_2 \triangle t}{k_{min}} \int_0^T ||\phi_t(t)||^2_{H_p} dt \leq C_5 \triangle t, \qquad (4.38)$$

$$\frac{3rn^2 \nu \rho g C_2}{k_{min}} \sum_{k=0}^{l} ||u_{m_{k+1}} - u_{m_k}||^2_{H_f} \leq \frac{3rn^2 \nu \rho g C_2 \triangle s}{k_{min}} \int_0^T ||u_s(s)||^2_{H_f} ds \leq C_5 \triangle t, \qquad (4.39)$$

$$\frac{3rn^2 \nu \rho g C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||u_{i+1} - u_i||^2_{H_f} + \frac{3rn\rho^2 g^2 k_{max} C_2}{k_{min}} \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||\phi_i - \phi_{i+1}||^2_{H_p} \leq C_5 \triangle t. \qquad (4.40)$$

From Theorem 4.1, we have,

$$\frac{3n\rho g C_2}{k_{min}} \{\sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} n\nu ||e^i||^2_{H_f} + r\rho g \sum_{k=0}^{l} ||\mathbf{K}^{1/2} \nabla \epsilon^{m_k}||^2_{L^2(\Omega_p)}\} \leq C_5(\triangle t + \triangle t^{-1} h^4). \qquad (4.41)$$

Combining (4.36)-(4.41) yields

$$
\begin{aligned}
& n\triangle t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + n\nu ||e^{m_{l+1}}||^2_{H_f} + \lambda(e^{m_{l+1}}, e^{m_{l+1}}) + \rho g S_0 \triangle t \sum_{k=0}^{l} ||d_t \epsilon^{m_{k+1}}||^2_{L^2(\Omega_p)} \\
& \qquad + \rho g \sum_{i=m_k}^{m_{k+1}-1} ||\mathbf{K}^{1/2} \nabla \epsilon^{m_{l+1}}||^2_{L^2(\Omega_p)} \leq C_5(\triangle t + h^4 + \triangle t^{-1} h^4). \qquad (4.42)
\end{aligned}
$$

□

At the smaller time steps used for the faster problem we have the following error estimate. Recall that

$$\lambda(u, v) \triangleq \sum_{i=1}^{d-1} \int_\Gamma \frac{n\alpha}{\sqrt{\tau_i \cdot \mathbf{K}\tau_i}} (u \cdot \tau_i)(v \cdot \tau_i).$$

Theorem 4.3. *Under the assumptions of Theorem 4.1, then the following error estimate holds: for $J = 1, 2, \cdots, r-1$, and $k = 0, 1, \cdots, l$,*

$$n||e^{m_k+J+1}||^2_{L^2(\Omega_f)} + n\nu\triangle t \sum_{i=m_k}^{m_k+J} ||e^{i+1}||^2_{H_f} \leq C_5(\triangle t^2 + h^4).$$

Proof. Taking $v = 2\triangle t e^{m+1}$ in (4.12), using the divergence-free property, sum over $m = m_k, m_k + 1, \cdots, m_k + J$, yield

$$\begin{aligned}
& n\{||e^{m_k+J+1}||^2_{L^2(\Omega_f)} + \sum_{i=m_k}^{m_k+J} ||e^{i+1} - e^i||^2_{L^2(\Omega_f)} - ||e^{m_k}||^2_{L^2(\Omega_f)}\} + 2\triangle t \sum_{i=m_k}^{m_k+J} a_f(e^{i+1}, e^{i+1}) \\
&= -2n\triangle t \sum_{i=m_k}^{m_k+J} (w^{i+1}_{f,t}, e^{i+1}) - 2n\rho g \triangle t \int_\Gamma \sum_{i=m_k}^{m_k+J} (\phi_{i+1} - \phi_i) e^{i+1} \cdot \mathbf{n}_f \\
&\quad - 2n\rho g \triangle t \int_\Gamma \sum_{i=m_k}^{m_{k+1}-1} (\phi_i - \phi^{m_k}) e^{i+1} \cdot \mathbf{n}_f \\
&\leq \frac{n\nu\triangle t}{3} \sum_{i=m_k}^{m_k+J} ||e^{i+1}||^2_{H_f} + \frac{3nP_1^2\triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||w^{i+1}_{f,t}||^2_{L^2(\Omega_f)} + \frac{n\nu\triangle t}{3} \sum_{i=m_k}^{m_k+J} ||e^{i+1}||^2_{H_f} \\
&\quad + \frac{3n\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||^2_{H_p} + \frac{n\nu\triangle t}{3} \sum_{i=m_k}^{m_k+J} ||e^{i+1}||^2_{H_f} + \frac{3n\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi^{m_k}||^2_{H_p} \\
&\leq n\nu\triangle t \sum_{i=m_k}^{m_k+J} ||e^{i+1}||^2_{H_f} + \frac{3nP_1^2\triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||w^{i+1}_{f,t}||^2_{L^2(\Omega_f)} \\
&\quad + \frac{3n\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} (||\phi_{i+1} - \phi_i||^2_{H_p} + ||\phi_i - \phi_{m_k}||^2_{H_p} + ||\phi_{m_k} - \phi^{m_k}||^2_{H_p}).
\end{aligned} \tag{4.43}$$

From (4.4), (4.5), (4.9), (4.11), and Theorem 4.1, we have

$$\begin{aligned}
\frac{3nP_1^2\triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||w^{i+1}_{f,t}||^2_{L^2(\Omega_f)} &\leq C_5(\triangle t^2 + h^4), \\
\frac{3n\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||^2_{H_p} &\leq C_5 \triangle t^2, \\
\frac{3n\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi_{m_k}||^2_{H_p} \leq \frac{3rn\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi_{i+1}||^2_{H_p} &\leq C_5 \triangle t^2, \\
\frac{3n\rho^2 g^2 C_2 \triangle t}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_{m_k} - \phi^{m_k}||^2_{H_p} \leq \frac{3n\rho^2 g^2 C_2 \triangle s}{\nu\sqrt{k_{min}}} ||\mathbf{K}^{1/2}\nabla\epsilon^{m_k}||^2_{L^2(\Omega_p)} &\leq C_5(\triangle t^2 + h^4).
\end{aligned}$$

Note that the last two inequalities above, we used the fact $m_k + J - m_k \leq r$ for $J = 1, 2, \cdots, r-1$ and the general triangle inequality. Combine the above bounds, the final result follows by Theorem 4.1,

$$\begin{aligned}
n||e^{m_k+J+1}||^2_{L^2(\Omega_f)} + \sum_{i=m_k}^{m_k+J} n\nu||e^{i+1} - e^i||^2_{L^2(\Omega_f)} + n\nu\triangle t \sum_{i=m_k}^{m_k+J} ||e^{i+1}||^2_{H_f} \\
\leq C_5(\triangle t^2 + h^4) + n||e^{m_k}||^2_{L^2(\Omega_f)} \leq C_5(\triangle t^2 + h^4).
\end{aligned}$$

□

For the error in time derivatives on smaller time steps, we have the following error estimate.

THEOREM 4.4. *Based on the smoothness assumption on the true solution, $J = 1, 2, \cdots, r-1$, and k can be $0, 1, \cdots, l$, the following estimate holds:*

$$n\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + n\nu||e^{m_k+J+1}||^2_{H_f} + \lambda(e^{m_k+J+1}, e^{m_k+J+1}) \leq C_5(\triangle t + h^4 + \triangle t^{-1}h^4). \quad (4.44)$$

Proof. Taking $2\triangle t d_t e^{m+1} = 2(e^{m+1} - e^m)$ in (4.12), using the divergence-free property, sum over $m = m_k, m_k+1, \cdots, m_k+J$, we obtain

$$\begin{aligned}
&2n\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + a_f(e^{m_k+J+1}, e^{m_k+J+1}) - a_f(e^{m_k}, e^{m_k}) + \triangle t^2 \sum_{i=m_k}^{m_k+J} a_f(d_t e^{i+1}, d_t e^{i+1}) \\
&= -2n\triangle t \sum_{i=m_k}^{m_k+J} (w_{f,t}^{i+1}, d_t e^{i+1}) - 2n\rho g\triangle t \int_\Gamma \sum_{i=m_k}^{m_k+J} (\phi_{i+1} - \phi_i) d_t e^{i+1} \cdot \mathbf{n}_f \\
&\quad - 2n\rho g\triangle t \int_\Gamma \sum_{i=m_k}^{m_k+J} (\phi_i - \phi^{m_k}) d_t e^{i+1} \cdot \mathbf{n}_f \\
&\leq n\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + n\triangle t \sum_{i=m_k}^{m_k+J} ||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)} + \frac{\triangle t^2}{2} \sum_{i=m_k}^{m_k+J} n\nu||d_t e^{i+1}||^2_{H_f} \\
&\quad + \frac{2n\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||^2_{H_p} + \frac{\triangle t^2}{2} \sum_{i=m_k}^{m_k+J} n\nu||d_t e^{i+1}||^2_{H_f} + \frac{2n\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi^{m_k}||^2_{H_p} \\
&\leq n\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} + \triangle t^2 \sum_{i=m_k}^{m_k+J} n\nu||d_t e^{i+1}||^2_{H_f} + n\triangle t \sum_{i=m_k}^{m_k+J} ||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)} \\
&\quad + \frac{2n\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} (||\phi_{i+1} - \phi_i||^2_{H_p} + ||\phi_i - \phi_{m_k}||^2_{H_p} + ||\phi_{m_k} - \phi^{m_k}||^2_{H_p}).
\end{aligned} \quad (4.45)$$

Just as the proof of the Theorem 4.3, using (4.4), (4.5), (4.9), (4.11) and Theorem 4.1, as well as the general triangle inequality and the fact $m_k + J - m_k \leq r$ for $J = 1, 2, \cdots, r-1$, we obtain

$$n\triangle t \sum_{i=m_k}^{m_k+J} ||w_{f,t}^{i+1}||^2_{L^2(\Omega_f)} \leq C_5(\triangle t^2 + h^4),$$

$$\frac{2n\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_{i+1} - \phi_i||^2_{H_p} \leq C_5\triangle t,$$

$$\frac{2n\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi_{m_k}||^2_{H_p} \leq \frac{2rn\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_i - \phi_{i+1}||^2_{H_p} \leq C_5\triangle t,$$

$$\frac{2n\rho^2 g^2 C_2}{\nu} \sum_{i=m_k}^{m_k+J} ||\phi_{m_k} - \phi^{m_k}||^2_{H_p} \leq \frac{2rn\rho^2 g^2 C_2}{\nu\sqrt{k_{min}}} ||\mathbf{K}^{1/2}\nabla\epsilon^{m_k}||^2_{L^2(\Omega_p)} \leq C_5(\triangle t + \triangle t^{-1}h^4).$$

Combining the above bounds and using Theorem 4.2 yields

$$\begin{aligned}
n\triangle t \sum_{i=m_k}^{m_k+J} ||d_t e^{i+1}||^2_{L^2(\Omega_f)} &+ n\nu||e^{m_k+J+1}||^2_{H_f} + \lambda(e^{m_k+J+1}, e^{m_k+J+1}) \\
&\leq C_5(\triangle t + h^4 + \triangle t^{-1}h^4) + n\nu||e^{m_k}||^2_{H_f} + \lambda(e^{m_k}, e^{m_k}) \\
&\leq C_5(\triangle t + h^4 + \triangle t^{-1}h^4).
\end{aligned}$$

□

COROLLARY 4.5. *Under the assumptions of the Theorem 4.1, then for $k = 0, 1, \cdots, M-1$, and $m = 1, 2, \cdots, N$, the following estimates hold:*

$$||u_h^m - u(t^m)||_{L^2(\Omega_f)} \leq C_5(\triangle t + h^2), \tag{4.46}$$

$$||\phi_h^{m_{k+1}} - \phi(t^{m_{k+1}})||_{L^2(\Omega_p)} \leq C_5(\triangle t + h^2), \tag{4.47}$$

$$||u_h^m - u(t^m)||_{H_f} \leq C_5(\triangle t^{1/2} + h + \triangle t^{-1/2} h^2), \tag{4.48}$$

$$||\phi_h^{m_{k+1}} - \phi(t^{m_{k+1}})||_{H_p} \leq C_5(\triangle t^{1/2} + h + \triangle t^{-1/2} h^2). \tag{4.49}$$

Proof. By using the triangle inequality, combine the approximation properties and Theorem 4.1-4.4, the claim of this theorem follows. □

Remark: In this paper, different conditions are needed for stability and error estimation, for stability, we need $\triangle t$ satisfies that $\frac{4n\rho g C_3 \triangle t}{\sqrt{\nu S_0 k_{min}}} < 1$ with C_3 is a constant. For the error estimation, we assume that $\triangle t$ satisfies that $\frac{4rn\rho g C_4 \triangle t}{h} \leq 1$ with C_4 is a constant. Which condition is better is still an open question, it depends on the problem and many other factors.

5. Numerical tests. This section gives two numerical tests. The first one is adapted from [17]. It has $O(1)$ material parameters and confirms both the predicted convergence rates and the efficiency of using different time steps. The second one is a test of stability for k_{min} very small. It reveals that the methods are stable for beyond the range of $\triangle t$ given by (3.1) in our (worse case) analysis.

5.1. Test 1. Assume $\Omega_f = [0, 1] \times [1, 2]$ and $\Omega_p = [0, 1] \times [0, 1]$ with interface $\Gamma = (0, 1) \times \{1\}$. The exact solution is given by

$$\begin{aligned}
&(u1, u2) = ([x^2(y-1)^2 + y]cos(\omega t), [-\frac{2}{3}x(y-1)^3]cos(\omega t) + [2 - \pi sin(\pi x)]cos(t)), \\
&p = [2 - \pi sin(\pi x)]sin(0.5\pi y)cos(t), \\
&\phi = [2 - \pi sin(\pi x)][1 - y - cos(\pi y)]cos(t).
\end{aligned}$$

Here $\omega = 5$, and the initial conditions, boundary conditions, and the forcing terms follows the solution.

The finite element spaces are constructed by using the well-known MINI elements ($P1b - P1$) for the Stokes problem and the linear Lagrangian elements ($P1$) for the Darcy flow. The code was implemented using the software package FreeFEM++[12]. For the monolithically coupled scheme, the GMRES routine is used to solve the (non-symmetric) coupled system. For the uncoupled scheme, a multi-frontal Gauss LU factorization implemented to solve the SPD sub-systems.

We define some notations first, for coupled scheme, we denote

$$e_u^{h,m} = u^{h,m} - u(t^m), \; e_p^{h,m} = p^{h,m} - p(t^m), \; e_\phi^{h,m} = \phi^{h,m} - \phi(t^m).$$

For the decoupled scheme, we denote

$$e_{h,u}^m = u_h^m - u(t^m), \; e_{h,p}^m = p_h^m - p(t^m), \; e_{h,\phi}^m = \phi_h^m - \phi(t^m).$$

First, we compare the errors, convergence rates and CPU times for both the coupled scheme and the decoupled scheme. In Table 5.1-5.2, we consider both schemes at time $t^m = 1.0$, with varying mesh h but fixed time step $\triangle t$ and $\triangle s = \omega \triangle t$. The two schemes achieve similar precision, although the monolithically coupled scheme is slightly more accurate than the decoupled scheme. However, the coupled scheme required much more CPU time than the decoupled scheme. The relative advantage of the decoupled scheme increased as the mesh was decreased. In Table 5.3-5.4, at the same time $t^m = 1.0$, with varying time step $\triangle t$ and $\triangle s = \omega \triangle t$ but fixed mesh $h = \frac{1}{8}$ are tested for both schemes. The two schemes almost get the same accuracy, but the coupled scheme needs much more CPU time than the decoupled scheme. In all, the decoupled scheme is comparable with the coupled scheme, and cheaper and more efficient.

Next, we will focus on the decoupled scheme, and examine the orders of convergence with respect to the spacing h or the time step $\triangle t$. Following [17], we introduce a more accurate approach to examine the orders of convergence with respect to the time step $\triangle t$ or the mesh size h due to the approximation errors $O(\triangle t^{\gamma})+O(h^{\mu})$. For example, assuming

$$v_h^{\triangle t}(x,t^m)\approx v(x,t^m)+C_1(x,t^m)\triangle t^{\gamma}+C_2(x,t^m)h^{\mu}.$$

Thus,

$$\rho_{v,h,i}=\frac{||v_h^{\triangle t}(x,t^m)-v_{\frac{h}{2}}^{\triangle t}(x,t^m)||_i}{||v_{\frac{h}{2}}^{\triangle t}(x,t^m)-v_{\frac{h}{4}}^{\triangle t}(x,t^m)||_i}\approx\frac{4^{\mu}-2^{\mu}}{2^{\mu}-1}.$$

$$\rho_{v,\triangle t,i}=\frac{||v_h^{\triangle t}(x,t^m)-v_h^{\frac{\triangle t}{2}}(x,t^m)||_i}{||v_h^{\frac{\triangle t}{2}}(x,t^m)-v_h^{\frac{\triangle t}{4}}(x,t^m)||_i}\approx\frac{4^{\gamma}-2^{\gamma}}{2^{\gamma}-1}.$$

Here, v can be $u,\ p,\ \phi$ and i can be $0,\ 1$. While $\rho_{v,h,i},\ \rho_{v,\triangle t,i}$ approach 4.0 or 2.0, the convergence order will be 2.0 or 1.0, respectively.

In Table 5.5, we study the convergence order with a fixed time step $\triangle t=0.01$ and $\triangle s=\omega\triangle t$ and varying spacing $h=1/2,1/4,1/8,1/16,1/32$. Observe that, $\rho_{u,h,0},\ \rho_{\phi,h,0}$ is a little larger than 4.0, and $\rho_{u,h,1},\ \rho_{p,h,0},\ \rho_{\phi,h,1}$ approach 2.0, which suggest that the error estimates $O(h^2)$ for the L^2-norm of u and ϕ, $O(h)$ for the H^1-norm of u and ϕ and the L^2-norm of p is optimal in space for the decoupled scheme. However, in Table 5.5, we study the convergence order with a fixed spacing $h=1/8$ and varying time step $\triangle t=0.02,\ 0.01,\ 0.005,\ 0.0025,\ 0.00125$ and $\triangle s=\omega\triangle t$. The numerical experiments strongly suggest that the orders of convergence in time for all should be $O(\triangle t)$, which implies that the error estimates for the L^2-norm of u and ϕ is optimal, however, the error estimates for the H^1-norm of u and ϕ might not be optimal for the decoupled scheme, and may be further improved from $O(\triangle t^{1/2})$ to $O(\triangle t)$ by a finer analysis- an open problem for further work.

Table 5.1

The convergence performance and CPU time of coupled scheme at time $t^m=1.0$, with varying mesh h but fixed time step $\triangle t=0.01$.

h	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
$\frac{1}{2}$	0.260588	1.50020	0.84932	0.154474	1.37573	4.428
$\frac{1}{4}$	0.073905	1.03481	0.82981	0.058474	0.86908	8.741
$\frac{1}{8}$	0.017644	0.40179	0.20873	0.010962	0.38724	32.081
$\frac{1}{16}$	0.004265	0.19129	0.07193	0.002688	0.19679	149.358
$\frac{1}{32}$	0.001120	0.09931	0.03493	0.000756	0.10059	698.809

Table 5.2

The convergence performance and CPU time of decoupled scheme at time $t^m=1.0$, with varying mesh h but fixed small time step $\triangle t=0.01$ and fixed large time step $\triangle s=\omega\triangle t$.

h	$\|\|e_{h,u}^m\|\|_0$	$\|\|e_{h,u}^m\|\|_1$	$\|\|e_{h,p}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	CPU
$\frac{1}{2}$	0.260588	1.50020	0.85337	0.154915	1.37554	0.856
$\frac{1}{4}$	0.070324	0.80750	0.47382	0.047873	0.79309	3.020
$\frac{1}{8}$	0.017953	0.41543	0.224210	0.013647	0.40958	10.038
$\frac{1}{16}$	0.004287	0.18950	0.07584	0.003879	0.19556	38.423
$\frac{1}{32}$	0.001185	0.09608	0.03781	0.002168	0.10105	143.963

Table 5.3

The convergence performance and CPU time of coupled scheme at time $t^m = 1.0$, with varying time step $\triangle t$ but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|e_u^{h,m}\|\|_0$	$\|\|e_u^{h,m}\|\|_1$	$\|\|e_p^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	$\|\|e_\phi^{h,m}\|\|_0$	CPU
0.02	0.017658	0.401804	0.209185	0.010998	0.387225	19.656
0.01	0.017644	0.401971	0.208733	0.010962	0.387235	31.839
0.005	0.017638	0.401786	0.208770	0.010944	0.387240	55.723
0.0025	0.017639	0.401786	0.208897	0.010935	0.387242	103.725
0.00125	0.017639	0.401786	0.208942	0.010930	0.387242	215.046

Table 5.4

The convergence performance and CPU of decoupled scheme at time $t^m = 1.0$, with varying small time step $\triangle t$ and large time step $\triangle s = \omega \triangle t$ and but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|e_{h,u}^m\|\|_0$	$\|\|e_{h,u}^m)\|\|_1$	$\|\|e_{h,p}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	$\|\|e_{h,\phi}^m\|\|_0$	CPU
0.02	0.017849	0.415564	0.226369	0.014735	0.409701	5.429
0.01	0.017953	0.415431	0.224210	0.013647	0.409579	10.639
0.005	0.018010	0.415403	0.223604	0.013128	0.409577	21.435
0.0025	0.018038	0.415398	0.223444	0.012877	0.409592	41.262
0.00125	0.018050	0.415397	0.223404	0.012753	0.409603	76.190

Table 5.5

Convergence orders of $O(h^\mu)$ of Uncouple scheme at time $t^m = 1.0$, with varying mesh h but fixed small time step $\triangle t = 0.01$ and fixed large time step $\triangle s = \omega \triangle t$.

h	$\|\|u_h^m - u_{\frac{h}{2}}^m\|\|_0$	$\rho_{u,h,0}$	$\|\|u_h^m - u_{\frac{h}{2}}^m\|\|_1$	$\rho_{u,h,1}$	$\|\|p_h^m - p_{\frac{h}{2}}^m\|\|_0$	$\rho_{p,h,0}$
$\frac{1}{2}$	0.210264	3.74520	1.60993	1.94293	0.71638	1.48895
$\frac{1}{4}$	0.056142	3.83200	0.82861	1.93881	0.48113	2.15270
$\frac{1}{8}$	0.014651	4.23579	0.42738	2.14606	0.22350	2.89976
$\frac{1}{16}$	0.003458		0.19915		0.07708	
h	$\|\|\phi_h^m - \phi_{\frac{h}{2}}^m\|\|_0$	$\rho_{\phi,h,0}$	$\|\|\phi_h^m - \phi_{\frac{h}{2}}^m\|\|_0$	$\rho_{\phi,h,1}$		
$\frac{1}{2}$	0.134538	3.38510	1.30491	1.67120		
$\frac{1}{4}$	0.039744	3.56065	0.78083	1.87755		
$\frac{1}{8}$	0.011162	4.81406	0.41587	2.05836		
$\frac{1}{16}$	0.002319		0.20204			

Table 5.6

Convergence orders of $O(\triangle t^\gamma)$ of Uncouple at time $t^m = 1.0$, with varying small time step $\triangle t$ and large time step $\triangle s = \omega \triangle t$ and but fixed mesh $h = \frac{1}{8}$.

$\triangle t$	$\|\|u_{\triangle t}^m - u_{\frac{\triangle t}{2}}^m\|\|_0$	$\rho_{u,\triangle t,0}$	$\|\|u_{\triangle t}^m - u_{\frac{\triangle t}{2}}^m\|\|_1$	$\rho_{u,\triangle t,1}$	$\|\|p_{\triangle t}^m - p_{\frac{\triangle t}{2}}^m\|\|_0$	$\rho_{p,\triangle t,0}$
0.02	6.49961e-4	2.03698	6.52832e-3	2.05855	1.68035e-2	1.91948
0.01	3.19081e-4	2.15518	3.17132e-3	2.18070	8.75420e-3	1.99190
0.005	1.48053e-4	2.17448	1.45427e-3	2.21398	4.39490e-3	2.01493
0.0025	6.80866e-5		6.656858e-4		2.18117e-3	
$\triangle s$	$\|\|\phi_{\triangle s}^m - \phi_{\frac{\triangle s}{2}}^m\|\|_0$	$\rho_{\phi,\triangle s,0}$	$\|\|\phi_{\triangle s}^m - \phi_{\frac{\triangle s}{2}}^m\|\|_0$	$\rho_{\phi,\triangle s,1}$		
0.1	1.51669e-3	1.96730	7.98752e-3	1.96671		
0.05	7.70949e-4	1.98387	4.06136e-3	1.98326		
0.025	3.88608e-4	1.99199	2.04781e-4	1.99160		
0.0125	1.95085e-4		1.02822e-4			

5.2. Test 2. Stability for $k_{min} = 1, 1.0e-4$ and $1.0e-8$. We do another experiment with k_{min} very small to test the stability restriction (3.1). In this test, we set $f_1 = f_2 = 0$, and for simplicity, we choose $\mathbf{u}^0 = \phi^0 = 1$ and $\mathbf{u}|_{\partial\Omega_f \setminus \Gamma} = 0, \phi|_{\partial\Omega_p \setminus \Gamma} = 0$, the small time step $\triangle t = 0.1$ and the large time step $\triangle s = 0.5$, the Figure 5.1 displays the quantity of energy $n\nu||\mathbf{u}_h||^2_{L^2(\Omega_f)} + \rho g S_0||\phi_h||^2_{L^2(\Omega_p)}$ on large time step size, and Figure 5.2 displays the counterpart on the small time step size. The partitioned method is clearly stable for k_{min} much smaller (with respect to $\triangle t$) than predicted by our stability analysis.

Remark: Note that, in Figure 5.2, there is a energy jump at the first large time point $t = 0.5$, that is because we only calculate ϕ_h on the large time point, which means before $t = 0.5$, we set $\phi_h = \phi^0 = 1$.

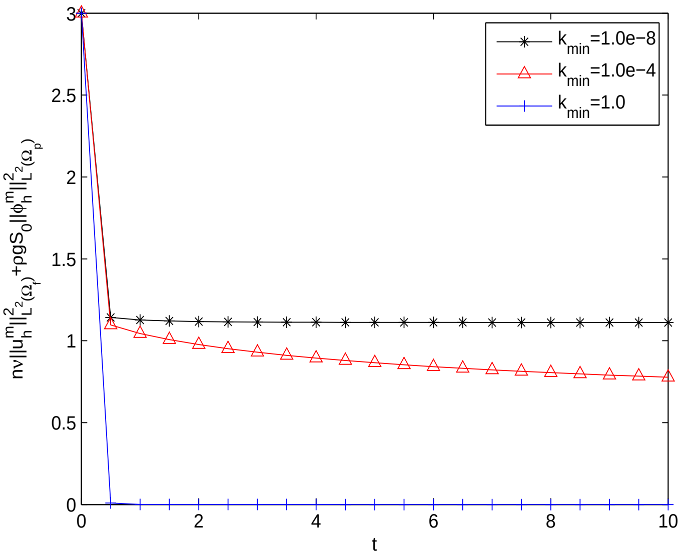

FIG. 5.1. *Energy vs. Time with the large time step* $\triangle s = 0.5$.

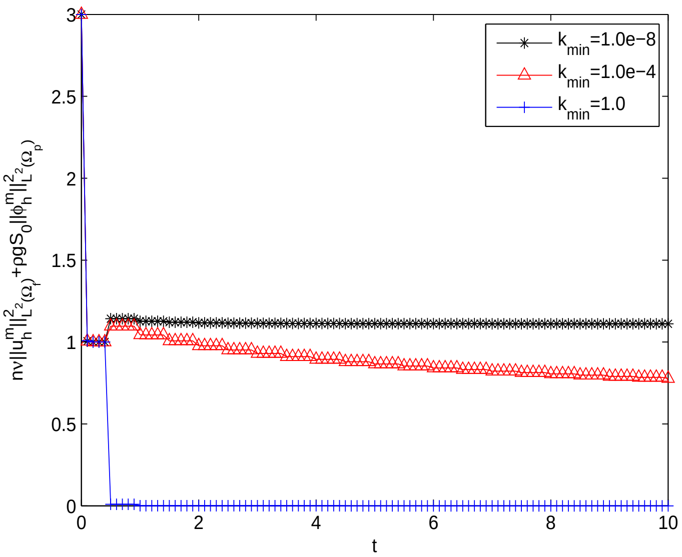

FIG. 5.2. *Energy vs. Time with the small time step* $\triangle t = 0.1$.

6. Conclusions. A decoupled method with different time steps in each sub-domain for the mixed Stokes-Darcy problem is proposed and analyzed in this paper. Under a time step restriction of the form $\triangle t \leq C(physicalparameters)$ we prove stability over bounded time intervals of the method. An analysis of the asymptotic stability over infinite time intervals and the possibly uniformity of the error in time is an important open problem. An error estimation is presented and numerical experiments are conducted to demonstrate the computational effectiveness or the decoupling approach.

In our analysis, we have made several choices to offset the notational complexity of asynchronous time stepping methods. In particular we have studied a formulation of the porous media problem as one second order problem for the Darcy pressure instead of as a mixed system for the pressure and Darcy velocity. Extension to a mixed discretization in the porous media region is also an important open problem. The boundary condition on $\partial\Omega_{f/p}\backslash\Gamma$ were also chosen for simplicity and can be modified. At this early stage of development, it does seem like uncoupled, partitioned methods are very promising for solving coupled surface water-ground water flow problems. They are very efficient, can be accurate and do not require reference to any monolithically coupled system of even iteration between sub-problems.

Open problems abound in partitioned methods for the Stokes-Darcy problems. Important one include expanding the partitioned methods available and analyzing and testing their stability, efficiency and accuracy for large T, small k_{min}, small S_0, small n, generic large domains, different spacial discretization and large but thin porous media regimes.

REFERENCES

[1] M. Anitescu, W. J. Layton and F. Pahlevani, Implicit for local effects and explicit for nonlocal effects is unconditionally stable, ETNA 18 (2004) 174–187.

[2] G. Beavers and D. Joseph, Boundary conditions at a naturally impermeable wall, J. Fluid. Mech, 30 (1967), 197-207.

[3] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang and W. Zhao, Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions, SINUM 47(2010) 4239-4256.

[4] Y. Cao, M. Gunzburger, F. Hua and X. Wang, Coupled Stokes-Darcy Model with Beavers-Joseph Interface Boundary Condition, Comm. Math. Sci., 8, (2010), 1-25.

[5] Y. Cao, M. Gunzburger, X. He and X. Wang, Parallel, non-iterative, multi-physics, domain decomposition methods for the time-dependedt Stokes-Darcy model, tech report 2011.

[6] J. M. Connors, Partitioned time discretization for atmosphere-ocean interaction, PhD dissertation, University of Pittsburgh, 2010.

[7] J. M. Connors and J. S. Howell, Variable time stepping for decoupled computation of fluid-fluid interaction, technical report 2010.

[8] M. Discacciati, Domain decomposition methods for the coupling of surface and groundwater flows, Ph.D. dissertation, École Polytechnique Fédérale de Lausanne, 2004.

[9] M. Discacciati, E. Miglio and A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math., 43 (2002), 57-74.

[10] M. Discacciati and A. Quarteroni, Navier-Stokes/Darcy Coupling: Modeling, Analysis, and Numerical Approximation, Revista Matemática Complutense, 2009

[11] H. I. Ene and E. Sanchez-Palencia, Equations et ph´enomen‘es de surface pour l’´ecoulement dans un mod‘ele de milieu poreux, J. M´ecanique, 14(1):73–108, 1975.

[12] F. Hecht, O. Pironneau, and K. Ohtsuka, FreeFEM++, http://www.freefem.org/ff++/ftp/ (2010).

[13] W. Jaeger and A. Mikelic, On the interface boundary conditions of Beavers, Joseph and Saffman, SIAM J Applied Math. 60(2000) 1111-1127.

[14] V. John, W. J. Layton and C. Manica, Time Averaged convergence of algorithms for flow problems, SINUM, 46, 2007, 151-179.

[15] W. J. Layton, F. Schieweck and I. Yotov, Coupling fluid flow with porous media flow, SIAM J. Numer. Anal. 40(2003) 2195-2218.

[16] M. Mu and J. Xu, A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow, SIAM J. Numer. Anal., 45 (2007), 1801-1813.

[17] M. Mu and X. H. Zhu, Decoupled schemes for a non-stationary mixed Stokes-Darcy model,Mathematics of Computation, 79 (2010), 707-731.

[18] L. E. Payne, J .C. Song and B. Straughan, Continuous dependence and convergence results for Brinkman and Forcheimer models with variable viscosity, Proc. Royal Soc. London, A 455(1999) 2173-2190.

[19] L. E. Payne and B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions, J Math. Pure Appl. 77(1998) 1959-1977.

[20] P. G. Saffman, On the boundary condition at the interface of a porous medium, Studies in Appl. Math. 1(1971) 93-101.