EXISTENCE OF SMOOTH ATTRACTORS FOR THE NAVIER–STOKES-OMEGA MODEL OF TURBULENCE: EXPANDED VERSION

WILLIAM LAYTON*

Abstract. This is an expanded version of a report with the same title. It includes as appendices motivation, connections to related work and background material. This expanded version's purpose is to provide a fuller picture of the work presented in the shorter version submitted for publication.

We consider the Navier-Stokes-omega model, given by

$$u_t - u \times \overline{\omega} - \nu \triangle u + \nabla P = f, \omega = \nabla \times u, \nabla \cdot u = 0$$

subject to periodic boundary conditions with zero mean.The NS-omega model is an outgrowth of ideas in approximate deconvolution models and in NS-alpha models. Like the NS-alpha model, it is simple and conserves, in the appropriate context, kinetic energy and helicity (3d) or energy and enstrophy (2d). In first tests NS-omega appears to be accurate, robust and amenable to efficient numerical simulation. In this note we prove existence of a global attractor for the model.

Key words. attractor, filter, deconvolution, turbulence model

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A short summary of the report

1. Introduction,

This report considers the NSE regularization

$$u_t - u \times (\nabla \times D\overline{u}) - \nu \triangle u + \nabla P = f, \nabla \cdot u = 0, \overline{u} = (-\alpha^2 \triangle + 1)^{-1} u.$$

We prove under periodic boundary conditions:

Theorem [existence of an attractor).

Suppose D is a bounded, linear deconvolution operator that is smoothing in the sense that

$$||D(\overline{v})||_{\mathbf{H}^2_{\#}} \leq C||v||.$$

Suppose

$$u_0, f \in \mathbf{H}^1_{\#}(\mathbf{\Omega}), \nabla \cdot u_0 = \nabla \cdot f = 0, \int_\Omega f dx = \int_\Omega u_0 dx = 0.$$

Then the NS-omega deconvolution model (1.2), including the NS-omega model (1.1) when $D = I$) has a maximal global attractor in H.

2. Notation and preliminaries,

Basic notation is recalled.

2.1. Filtering and deconvolution,

Examples and properties of filtering and deconvolution operators recalled.

2.2. A priori bounds and two Gronwall type lemmas,

The uniform Gronwall lemma of Foias and Prodi [FP67] is recalled and a (new to our knowledge but the subject is old so it is likely to be known somewhere explicitly or implicitly) lemma of the same type given.

Lemma (uniform Gronwall lemma).

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA, wjl@pitt.edu, http://www.math.pitt.edu/~wjl, Partially supported by NSF Grant DMS 0508260 and 0810385

Assume that y, g, h are positive, locally integrable functions on (t_0, ∞), and that for $t \geq t_0$

$$y' \leq gy + h, \text{ with}$$
$$\int_t^{t+r} y(s)ds \leq k_1, \int_t^{t+r} g(s)ds \leq k_2, \int_t^{t+r} h(s)ds \leq k_3,$$

where k_1, k_2, k_3, and r are four positive constants. Then,

$$y(t+r) \leq (\frac{k_1}{r} + k_3)e^{k_2}, \text{ for all } t \geq t_0.$$

Lemma (another Gronwall like lemma).

Let $y(t)$ be a positive smooth function satisfying

$$y' + \nu\lambda_1 y \leq A + By^{\frac{3}{4}} \text{ , for } t > 0, y(0) = y_0,$$

where $A, B, y_0, \nu, \lambda_1$ are positive constants. Let $t_0 := \max\{\frac{2}{\nu\lambda_1}\ln(\frac{\nu\lambda_1 y_0}{2A}), 0\}$. Then,

$$y(t) \leq \max\{y_0, (\frac{2B}{\nu\lambda_1})^4, \frac{4A}{\nu\lambda_1}\}, \text{ for } t \geq 0,$$
$$y(t) \leq \max\{(\frac{2B}{\nu\lambda_1})^4, \frac{4A}{\nu\lambda_1}\}, \text{ for } t \geq t_0.$$

3. Existence of an attractor,

Existence of an attractor in $\mathbf{H}$ is proven.

3.1. Basic properties of attractors,

3.2. Proof of Theorem 1.1,

4. Smoothness of the attractor,

This section proves regularity of the maximal attractor by showing that an attractor of the NS-omega model exists in each space $\mathbf{H}^s_{\#}$. We first prove uniform boundedness of spacial derivatives.

Lemma (a priori bounds).

Consider the NS-omega model with $u_0 \in H^s_{\#}$, $f \in H^s_{\#}, s \geq 0$. Then there is a finite constant C such that

$$\sup_{[0,\infty)} ||u(t)||_s \leq \rho_s < \infty, .$$

Further, for every s there is a finite constant $C(||\nabla^s f||, \nu, \alpha, s)$ such that

$$\int_t^{t+r} ||\nabla\partial^s u(t')||^2 dt' \leq 2rC(||\nabla^s f||, \nu, \alpha, s) + \frac{2\rho_s^2}{\nu} < \infty. \tag{0.1}$$

The main result of this section is:

Theorem (smoothness of the attractor).

Suppose

$$u_0, f \in \mathbf{H}^s_{\#}(\mathbf{\Omega}), \nabla \cdot u_0 = \nabla \cdot f = 0, \int_\Omega f dx = \int_\Omega u_0 dx = 0.$$

Then A is an attractor of the NS-omega model (1.1) in each $H^s_{\#}(\Omega)$ and thus consists of $C^\infty_{\#}(\Omega)$ functions.

References,

Appendices:

The appendices give background material about filtering, the motivation for the regularization and deconvolution.

5. Motivation for the NS-omega model,

As one example, the NS-omega model can be solved much more economically by, simplifying and suppressing spatial discretizations,

$$\frac{1}{\Delta t}(u_{n+1}-u_n)+\overline{(\frac{3}{2}\omega_n-\frac{1}{2}\omega_{n-1})}\times u_{n+1/2}+\nabla P_{n+1/2}-\nu\triangle^h u_{n+1/2}=f_{n+1/2}, \tag{0.2}$$

$$\text{where } \omega_n=\nabla\times u_n \tag{0.3}$$

$$\text{and } (\nabla\cdot u^h_{n+1},\chi^h)=0\ ,\forall\chi^h\in Q^h. \tag{0.4}$$

This method filters an explicitly known (from previous time levels) variable. This simple method is second order accurate, unconditionally and nonlinearly stable and requires only the solution of one smaller ($N_{velocity}+N_{pressure}$ unknowns) *linear* system at every time step

6. More about averaging or filtering,

The sections recalls the 1984 result of Koenderink. It begins by assuming the filter is

CONDITION 0.1. *linearity:*

$$\overline{\alpha u+\beta v}=\alpha\overline{u}+\beta\overline{v}$$

CONDITION 0.2. *spacial invariance (no preferred location):*

$$\overline{u(x-a)}=\overline{u}(x-a)$$

CONDITION 0.3. *isotropy (no preferred orientation): for all rotations R :*

$$\overline{u}(x)=R^*\overline{u(Rx)}$$

CONDITION 0.4. *scale invariance, or the semigroup property (no preferred size): if $\overline{u}(x,t):=g_\alpha\star u(x,t)$ (so $\overline{\overline{u}}:=g_\alpha\star g_\alpha\star u$) then*

$$\overline{\overline{u}}=g_{\sqrt{2}\alpha}\star u.$$

Theorem (Koenderink's theorem on uniqueness of the filter).

The only filter satisfying C1, C2, C3 and C4 is the Gaussian filter.

Thus, there is really only one mathematical correct filter: the Gaussian. Convolution with the Gaussian is quite expensive computationally so for large eddy simulation the differential filter is assessed as approximations to the Gaussian.

7. More on deconvolution.

This section recalls that the deconvolution problem

$$\text{given } \overline{u}\ (+\ noise) \text{ solve } \overline{u}=Gu \text{ for } u,$$

is ill-posed. Indeed:

Theorem (deconvolution is ill-posed).

Let $G\phi = \overline{\phi}$ denote the filtering operator. If G is smoothing so $G : L^2(\Omega) \to H^s(\Omega)$ is bounded for some $s > 0$ then G cannot have an exact inverse that is a bounded linear operator $: L^2(\Omega) \to L^2(\Omega)$.

Next some commonly used deconvolution operators are presents from the point of view as approximations to this ill-posed problem.

1. Introduction. The problem of inverting the filter G is is ill-posed and thus the closure problem itself must be ill posed. This does not mean that accurate approximate closure is impossible. There are after all many good methods for approximate solution of ill posed problems!

The Navier-Stokes-omega model, [LST08] [LMNR08b], is a development from the NS-alpha circle of ideas, e.g., [FHT02], and approximate deconvolution large eddy simulation models, e.g., [SA99], [SAK01], [BIL06], [Geu97], [LL03], [Dunca04], [R08]. In rotational form, it is to find a velocity u and Bernoulli or dynamic pressure P satisfying

$$u_t - u \times \overline{\omega} - \nu \triangle u + \nabla P = f, \omega = \nabla \times u, \nabla \cdot u = 0. \tag{1.1}$$

In (1.1) $\overline{\omega} = \nabla \times \overline{u}$ denotes the filtered / averaged / smoothed vorticity, where the filter is defined precisely in Section 2 in (2.1). We consider (1.1) in $\Omega = (0, 2\pi)^3$ subject to periodic with zero mean boundary conditions on u, P, f and the initial condition $u(x, 0) = u_0(x)$. Here u_0, f are smooth, zero-mean periodic functions. Minimally we suppose

$$u_0, f \in L_0^2(\Omega), \nabla \cdot u_0 = \nabla \cdot f = 0, \text{ and } \int_\Omega f dx = \int_\Omega u_0 dx = 0.$$

The NS-omega model (1.1) (nonlinearity $-u \times \nabla \times \overline{u}$) is a basic regularization of the NSE similar in spirit to the Leray regularization (nonlinearity $+\overline{u} \cdot \nabla u$, [L34a], [L34b], [CHOT05]) and the NS-alpha model (nonlinearity $-\overline{u} \times \nabla \times u$, [FHT01]). It also can be extended to a family of NS-omega-deconvolution models of arbitrary high accuracy. The idea of using deconvolution operators to obtain high order accurate regularizations is an idea of A. Dunca (private communication) developed for the Leray model in [LL08], [LMNR08a], the NS–α model in [R08] and the NS-omega model in [LMNR08a]. The NS-omega deconvolution family, including (1.1) as the zeroth order case, is given by

$$u_t - u \times D(\overline{\omega}) - \nu \triangle u + \nabla P = f, \omega = \nabla \times u, \nabla \cdot u = 0. \tag{1.2}$$

where $D : L^2(\Omega) \to L^2(\Omega)$ is a deconvolution operator.

In this report we prove existence of an attractor for (1.1), (1.2). Let $\mathbf{H}$ denote the closure of the smooth, periodic, zero mean, divergence free, vector functions in $L^2(\Omega)$ and $\mathbf{H}^s_\#(\mathbf{\Omega})$ their closure in the H^s norm; see Section 2.

Theorem 1.1 (Existence of an Attractor). *Let $\overline{u} = (-\alpha^2 \triangle + 1)^{-1} u$. Suppose D is a bounded, linear deconvolution operator that is smoothing in the sense that*

$$||D(\overline{v})||_{\mathbf{H}^2_\#} \leq C||v||. \tag{1.3}$$

Suppose

$$u_0, f \in \mathbf{H}^1_\#(\mathbf{\Omega}), \nabla \cdot u_0 = \nabla \cdot f = 0, \int_\Omega f dx = \int_\Omega u_0 dx = 0.$$

Then the NS-omega deconvolution model (1.2), including the NS-omega model (1.1) when $D = I$) has a maximal global attractor in $\mathbf{H}$.

In Section 4 we show that the maximal attractor is also an attractor in each $\mathbf{H}^s_\#(\boldsymbol{\Omega})$ and thus consists of $\mathbf{C}^\infty_\#(\boldsymbol{\Omega})$ functions. This parallels known results for the 2d NSE, Temam Chapter IV, Section 6.3 in [T88]; in the latter case, it is proven through establishing regularity of time derivatives and herein through space derivatives. Some preliminaries and the (standard) notation used are collected in Section 2. The proof of the above theorem is given in Section 3. The theory of attractors is highly developed, e.g., Temam [T88], Robinson [R01] and applied to the Leray and Leray deconvolution model in [CHOT05], and by Lewandowski and Preaux [LP08] (the report which inspired this effort). Under this theory, the proof of the above theorem reduces to verifying (i) existence of a bounded, absorbing set in $\mathbf{H}$ and (ii) compactness of the semigroup generated by (1.1), (1.2). The question of the dimension of the attractor and its dependence on the averaging radius α and deconvolution operator D (and how it compares with estimates of length scales of persistent eddies from turbulence phenomenology as well as estimates of attractor dimension of the Leray and NS-alpha regularizations) is a very important open problem.

2. Notation and preliminaries. $(\cdot,\cdot), ||\cdot||$ denotes the usual $L^2(\Omega)$ inner product and norm. The subscript # denotes the 2π periodic so the $C^\infty_\#(\Omega)$ denotes the C^∞ functions v with v and all derivatives 2π periodic. Let

$$\mathbf{H}_{\text{div}}(\Omega) : = \text{closure in } ||\cdot||_{\text{div}} := \sqrt{||\cdot||^2 + ||\operatorname{div}\cdot||^2} \text{ of } \{v \in C^\infty_\#(\Omega)^3 : \int_\Omega v dx = 0\},$$

$$\mathbf{H}^s_\#(\Omega) : = \text{closure in } ||\nabla^s \cdot|| \text{ of } \{v \in C^\infty_\#(\Omega)^3 : \int_\Omega v dx = 0 \text{ and } \nabla \cdot v = 0\}.$$

The norm on $\mathbf{H}^s_\#(\Omega)$ can be defined succinctly via Fourier series as

$$||v||_s^2 = \sum_{\mathbf{k}} |\mathbf{k}|^{2s} |\widehat{v}(\mathbf{k})|^2.$$

Define

$$\mathbf{H} = \{v \in \mathbf{H}_{\text{div}} : \nabla \cdot v = 0\}\ , \ \text{ and } \ \mathbf{V} := \mathbf{H}^1_\#(\Omega).$$

Let P_L denote the Helmholtz-Leray orthogonal projection of $L^2(\Omega)$ onto $\mathbf{H}$. The Stokes operator A is then $A = -P_L\triangle$ with $Domain(A) = \mathbf{H} \cap \mathbf{H}^2_\#(\Omega)$, e.g., Galdi [Gal94]. In the periodic case the vector Laplacian and Stokes operator coincide apart from their domains of definition. Let

$$\lambda_1 = \lambda_{\min}(A) = \min_{v \in \mathbf{V}} \frac{(\nabla v, \nabla v)}{||v||^2} \text{ so that } (f, v) \leq \frac{1}{\lambda_1} ||f|| ||\nabla v||.$$

2.1. Filtering and deconvolution. Koenderink [Koe84], see also [Lin94], has proven that the only filter satisfying the basic requirements of linearity, frame invariance and scale invariance is the Gaussian filter. Calculations with Gaussian filters can be expensive so *approximations* to the Gaussian are often used, such as the Pao filter, Pope [P00], Sagaut [S01],

$$\overline{v}(x) := \sum_{\mathbf{k}} (\alpha^2 |\mathbf{k}|^2 + 1)^{-1} \widehat{v}(\mathbf{k}) e^{i\mathbf{k}\cdot x}\ , \ \alpha := \text{ filter radius.}$$

The Pao filter is equivalent to a simple differential filter, Germano [Ger86], used in theoretical studies of deconvolution models, e.g., [Dunca04], [DE06], [BIL06], [LL03], and the NS-alpha model [FHT01]. The equivalent differential filter to the Pao filter is

$$\overline{v} = (-\alpha^2\triangle + 1)^{-1} v, \tag{2.1}$$

subject to periodic boundary conditions. In the periodic case, this differential filter preserves zero mean and incompressibility.

A deconvolution operator D is an *approximate* filter inverse. Typically D is a high order approximate inverse on the low Fourier modes and includes some sort of truncation or regularization to suppress the growth of noise in higher modes. Since many examples exist (due to approximate deconvolution's centrality in image processing, Bertero and Boccacci [BB98]) we assume some basic properties of D common to many used in practice.

- stability: $D : \mathbf{H} \to \mathbf{H}$ is a *bounded* linear operator,
- accuracy: $||v - D(\overline{v})|| \le C(v)\alpha^{\beta}$, for smooth v and some $\beta > 2$.
- smoothing: $||D(\overline{v})||_2 + ||D(\overline{v})||_1 + ||D(\overline{v})|| \le C(\alpha)||v||$, for all $v \in \mathbf{H}$.

Examples of filters satisfying these minimal conditions include van Cittert deconvolution [BB98], $D = \sum_{j=0}^{J}(-\alpha^2\triangle + 1)^{-j}$, (and its optimized variants, [LS07], [LS08b]), Tikhonov regularized deconvolution, $D = [(-\alpha^2\triangle + 1)^{-1} + \mu I]^{-1}$, and truncated SVD methods (which simplify in the periodic case to spectral methods) such as

$$Dv := \sum_{0<|\mathbf{k}|<\pi/\alpha} (\alpha^2|\mathbf{k}|^2 + 1)\widehat{v}(\mathbf{k})e^{i\mathbf{k}\cdot x} + \sum_{|\mathbf{k}|\ge\pi/\alpha} \widehat{v}(\mathbf{k})e^{i\mathbf{k}\cdot x}.$$

2.2. Á priori bounds and two Gronwall type lemmas. The uniform Gronwall lemma of Foias and Prodi [FP67] is a fundamental tool in the study of attractors and used to complete the proof of existence of an attractor from (3.11) below..

Lemma 2.1 (Uniform Gronwall Lemma). *Assume that y, g, h are positive, locally integrable functions on (t_0, ∞), and that for $t \ge t_0$*

$$y' \le gy + h, \text{ with}$$
$$\int_t^{t+r} y(s)ds \le k_1, \int_t^{t+r} g(s)ds \le k_2, \int_t^{t+r} h(s)ds \le k_3,$$

where k_1, k_2, k_3, and r are four positive constants. Then,

$$y(t + r) \le (\frac{k_1}{r} + k_3)e^{k_2}, \text{ for all } t \ge t_0. \tag{2.2}$$

Because the differential inequality (3.10) has sublinear growth in the nonlinear term, other related bounds can be obtained using an alternate Gronwall-type inequality.

Lemma 2.2. *Let $y(t)$ be a positive smooth function satisfying*

$$y' + \nu\lambda_1 y \le A + By^{\frac{3}{4}} \text{ , for } t > 0, y(0) = y_0,$$

where $A, B, y_0, \nu, \lambda_1$ are positive constants. Let $t_0 := \max\{\frac{2}{\nu\lambda_1}\ln(\frac{\nu\lambda_1 y_0}{2A}), 0\}$. Then,

$$y(t) \le \max\{y_0, (\frac{2B}{\nu\lambda_1})^4, \frac{4A}{\nu\lambda_1}\}, \text{ for } t \ge 0,$$
$$y(t) \le \max\{(\frac{2B}{\nu\lambda_1})^4, \frac{4A}{\nu\lambda_1}\}, \text{ for } t \ge t_0.$$

Proof. Define $Y := \max\{(\frac{2B}{\nu\lambda_1})^4, \frac{4A}{\nu\lambda_1}\}$ and divide $(0,\infty)$ into two collections of subintervals by

$$I_S := \{t : y(t) < Y\} \text{ and } I_L := \{t : y(t) \geq Y\},$$

so that $y(t) < Y$ for all $t \in I_S$. Thus, consider $t \in I_L$. Let $[a,b]$ be a maximal interval in I_L so either $a = 0$ and $y(a) = y_0$ or $a > 0$ and $y(a) = Y$. By construction, on I_L

$$\frac{1}{2}\nu\lambda_1 y(t) \geq By(t)^{\frac{3}{4}},$$

so that for $t \in [a,b], a > 0$, $y(t)$ satisfies

$$y' + \frac{1}{2}\nu\lambda_1 y \leq A, a \leq t \leq b, y(a) = Y.$$

An integrating factor then gives for $t \in [a,b]$,

$$y(t) \leq e^{-\frac{1}{2}\nu\lambda_1(t-a)}Y + \frac{2A}{\nu\lambda_1}[1-e^{-\frac{1}{2}\nu\lambda_1(t-a)}] \leq e^{-\frac{1}{2}\nu\lambda_1(t-a)}Y + Y[1-e^{-\frac{1}{2}\nu\lambda_1(t-a)}] \leq Y$$

There remains the case $a = 0$, i.e., $[a,b] = [0,b]$. The same analysis as the last case gives the bound

$$\begin{aligned} y(t) &\leq e^{-\frac{1}{2}\nu\lambda_1(t-a)}y_0 + \frac{2A}{\nu\lambda_1}[1-e^{-\frac{1}{2}\nu\lambda_1(t-a)}] \qquad (2.3)\\ &\leq e^{-\frac{1}{2}\nu\lambda_1(t-a)}\max\{y_0, Y\} + \max\{y_0, Y\}[1-e^{-\frac{1}{2}\nu\lambda_1(t-a)}], \text{ or} \\ y(t) &\leq max\{y_0, Y\} \end{aligned}$$

completing the proof of the first bound. For the second, by a direct calculation we have

$$e^{-\frac{\nu\lambda_1}{2}t}y_0 \leq \frac{2A}{\nu\lambda_1} \text{ , for } t \geq t_0.$$

Thus, from (2.3) for $t \geq t_0$, the second follows since

$$y(t) \leq \frac{2A}{\nu\lambda_1} + \frac{2A}{\nu\lambda_1}[1-e^{-\frac{1}{2}\nu\lambda_1 t}] \leq \frac{4A}{\nu\lambda_1}.$$

□

3. Existence of an attractor. The following was proven about the NS-omega model in [LST08]. The same proof can be used to prove existence for the NS-omega deconvolution model with van Cittert or Tikhonov deconvolution.

THEOREM 3.1 (Existence, Uniqueness and Regularity). *Let $\alpha > 0$ and $T > 0$ be fixed. Let the filter be the differential filter (2.1). For and $u_0 \in \mathbf{V}$, $f \in \mathbf{H}$, there exists a unique strong solution u to the NS-omega model (1.1) with*

$$u \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;\mathbf{H}^2_\#(\Omega)) \text{ and } u_t \in L^2((0,T)\times\Omega).$$

Further, u satisfies the energy equality. If the data is more regular

$$u_0 \in \mathbf{V} \cap \mathbf{H}^{m+1}_\#(\Omega), \text{ and } f \in L^2(0,T;\mathbf{H}^m_\#(\Omega)),$$

then

$$u \in L^{\infty}(0,T;\mathbf{H}_{\#}^{m+1}(\Omega)) \cap L^2(0,T;\mathbf{H}_{\#}^{m+2}(\Omega)), P \in L^2(0,T;\mathbf{H}_{\#}^{m+2}(\Omega)).$$

The NS-omega model is thus a well defined dynamical system and determines a (nonlinear) semi-group defined by

$$u(t;\cdot) := S(t)u_0.$$

PROPOSITION 3.2. *Suppose* $||\nabla \overline{u}|| + ||\triangle \overline{u}|| \le C(\alpha)||u||$ *and let*

$$t_1 := \max\{\frac{2}{\nu\lambda_1}\ln(\frac{\nu^2\lambda_1||\nabla u_0||^2}{4||f||^2}), 0\}.$$

Then, solutions to the NS-omega model satisfy the á priori bounds

$$||u(t)||^2 \le ||u_0||^2 e^{-\nu\lambda_1 t} + (\frac{1}{\nu\lambda_1}||f||)^2(1-e^{-\nu\lambda_1 t}),\ \text{for } t \ge 0, \tag{3.1}$$

$$||\nabla u(t)||^2 \le \max\{||\nabla u_0||^2, \frac{16C(\alpha)^4(\sup_{0,\infty)}||u||)^{10}}{\nu^8\lambda_1^4}, \frac{8||f||^2}{\nu^2\lambda_1}\},\ \text{for } t \ge 0, \tag{3.2}$$

$$||\nabla u(t)||^2 \le \max\{\frac{16C(\alpha)^4(\sup_{[0,\infty)}||u||)^{10}}{\nu^8\lambda_1^4}, \frac{8||f||^2}{\nu^2\lambda_1}\},\ \text{for } t \ge t_1. \tag{3.3}$$

Proof. Since the NS-omega model has a unique strong solution, we may multiply (1.1) by u and ∇u and integrate over Ω. The first choice yields

$$\frac{1}{2}\frac{d}{dt}||u||^2 + \nu||\nabla u||^2 = (f,u) \tag{3.4}$$

Since $||u||^2 \le \lambda_1||\nabla u||^2$, we have

$$\frac{d}{dt}||u||^2 + \nu\lambda_1||u||^2 \le \frac{1}{\nu\lambda_1}||f||^2$$

and thus the first bound follows:

$$||u(t)||^2 \le ||u_0||^2 e^{-\nu\lambda_1 t} + (\frac{1}{\nu\lambda_1}||f||)^2(1-e^{-\nu\lambda_1 t}). \tag{3.5}$$

For the second and third, take the inner product of (1.1) with $-\triangle u$. This gives

$$\frac{1}{2}\frac{d}{dt}||\nabla u||^2 + \int_{\Omega} u \times \nabla \times \overline{u} \cdot (-\triangle u)dx + \nu||\nabla u||^2 = (f, -\triangle u) \le ||f||||\triangle u||.$$

For the nonlinear term we use the following bound (which follows via Holder's inequality and the Sobolev embedding theorem as in the normal bounds on the NSE nonlinearity, e.g., Constantine and Foias [CF88], Temam [T88])

$$|\int_{\Omega} u \times \nabla \times \overline{u} \cdot (-\triangle u)dx| \le C||u||^{\frac{1}{4}}||\nabla u||^{\frac{3}{4}}||\nabla \overline{u}||^{\frac{1}{4}}||\triangle \overline{u}||^{\frac{3}{4}}||\triangle u||.$$

Since the filter is smoothing $||\nabla \overline{u}|| + ||\triangle \overline{u}|| \le C(\alpha)||u||$ so

$$|\int_\Omega u \times \nabla \times \overline{u} \cdot (-\triangle u) dx| \le C||u||^{\frac{5}{4}}||\nabla u||^{\frac{3}{4}}||\triangle u|| \le \frac{\nu}{2}||\triangle u||^2 + C(\alpha)\nu^{-1}||u||^{\frac{5}{2}}||\nabla u||^{\frac{3}{2}}.$$

Thus,

$$\frac{d}{dt}||\nabla u||^2 + \nu||\triangle u||^2 \le \frac{2}{\nu}||f||^2 + C(\alpha)\nu^{-1}||u||^{\frac{5}{2}}||\nabla u||^{\frac{3}{2}}, or \tag{3.6}$$

$$\frac{d}{dt}||\nabla u||^2 + \nu\lambda_1||\nabla u||^2 \le \frac{2}{\nu}||f||^2 + C(\alpha)\nu^{-1}||u||^{\frac{5}{2}}||\nabla u||^{\frac{3}{2}}. \tag{3.7}$$

Lemma 2.2 with $y_0 = ||\nabla u_0||^2$, $A = \frac{2}{\nu}||f||^2$ and $B = C(\alpha)\nu^{-1}\sup_{[0,\infty)}||u||^{\frac{5}{2}}$ gives

$$y(t) \le \max\{||\nabla u_0||^2, \frac{16C(\alpha)^4(\sup_{[0,\infty)}||u||)^{10}}{\nu^8\lambda_1^4}, \frac{8||f||^2}{\nu^2\lambda_1}\}, t \ge 0,$$

and for $t \ge t_1 := \max\{\frac{2}{\nu\lambda_1}\ln(\frac{\nu^2\lambda_1||\nabla u_0||^2}{4||f||^2}), 0\}$,

$$y(t) \le \max\{\frac{16C(\alpha)^4(\sup_{[0,\infty)}||u||)^{10}}{\nu^8\lambda_1^4}, \frac{8||f||^2}{\nu^2\lambda_1}\}.$$

□

3.1. Basic properties of attractors. To prepare for the proof of Theorem 1.1 we collect some information about attractors from, for example, Temam [T88] , Robinson [R01], Doering and Gibbon [DG95].

DEFINITION 3.3. *We say $\mathcal{A} \subset \mathbf{H}$ is a global or maximal attractor in $\mathbf{H}$ for the dynamical system (1.1) if and only if*

(i) $\mathcal{A}$ is compact in $\mathbf{H}$.

(ii) For all $t > 0, S(t)\mathcal{A} \subset \mathcal{A}$.

(iii) For every bounded set $B \subset \mathbf{H}$, $\rho(S(t)B, \mathcal{A}) := \sup_{v\in B}\inf_{u\in\mathcal{A}}||u-v||$ goes to zero as $t \to \infty$.

DEFINITION 3.4. *The set $A \subset \mathbf{H}$ is an absorbing set if and only if, for every bounded subset $B \subset \mathbf{H}$ there exists $t_1 > 0$ such that for all $t \ge t_1$ one has $S(t)(B) \subset A$.*

The semigroup $S(t)$ is uniformly compact if and only if for every bounded subset $B \subset \mathbf{H}$, there exists $t_2 = t_2(B)$ such that $\overline{\cup_{t\ge t_2}S(t)(B)}$ is compact.

Let $\varpi(A)$ denote the set $\varpi(A) := \cap_{s\ge 0}\overline{\cup_{t\ge s}S(t)(B)}$.

The proof of the following can be found in Temam [T88] , Robinson [R01].

THEOREM 3.5. *Suppose that there exists an absorbing bounded set A and that the semigroup $S(t)$ is uniformly compact. Then, $\mathcal{A}_0 = \varpi(A)$ is the global attractor for the dynamical system defined by $S(t)$.*

3.2. Proof of Theorem 1.1. We shall show that $S(t)$ has an absorbing set and is compact. First we establish existence of an absorbing set. (This step follows the NSE case closely.) From Proposition 3.2, the estimate (3.1) gives

$$||u(t)||^2 \le ||u_0||^2 e^{-\nu\lambda_1 t} + (\frac{1}{\nu\lambda_1}||f||)^2(1 - e^{-\nu\lambda_1 t}). \tag{3.8}$$

As for the NSE (e.g., Temam [T88] , Robinson [R01]), let $\rho_0 = \frac{1}{\nu\lambda_1}||f||$ and $\rho' > \rho_0$. Then, from (3.1), given u_0 in $\mathbf{H}$ for $t > T := \max\{\frac{1}{\nu\lambda_1}\ln(\frac{||u_0||^2}{\rho'^2-\rho_0^2}), 0\}$, it follows that $||u(t)|| \le \rho'$. In other words, $B_{\rho'}(0) = \{v \in \mathbf{H} : ||v|| < \rho'\}$ is an absorbing set in $\mathbf{H}$.

For the second step we show $S(t)$ is uniformly compact by obtaining a uniform estimate on the $\mathbf{V}$ norm of solutions. To prove compactness the second and third bounds in Proposition 3.2 do not suffice because they require $u_0 \in \mathbf{V}$ instead of $\mathbf{H}$. Thus we shall apply the uniform Gronwall lemma. Integrating (3.4) over $(t, t+r)$ and using standard inequalities gives

$$\frac{1}{2}(||u(t+r)||^2 - ||u(t)||^2) + \frac{1}{2}\int_t^{t+r} \nu||\nabla u(t')||^2 dt' \leq \frac{r}{2\nu\lambda_1}||f||^2.$$

For $t > T$, $u(t) \in B_{\rho'}(0)$ so $||u(t)|| < \rho'$ and $||u(t+r)|| < \rho'$. Thus for $t > T$

$$\int_t^{t+r} ||\nabla u(t')||^2 dt' \leq \frac{r}{\nu^2\lambda_1}||f||^2 + \frac{2\rho'^2}{\nu}. \tag{3.9}$$

We begin with (3.6), (3.7) from the proof of Proposition 3.2:

$$\frac{d}{dt}||\nabla u||^2 + \nu||\triangle u||^2 \leq \frac{2}{\nu}||f||^2 + C(\alpha)\nu^{-1}||u||^{\frac{5}{2}}||\nabla u||^{\frac{3}{2}}. \tag{3.10}$$

The Poincaré-Friedrichs inequality can be used on the last term on the RHS to give

$$\frac{d}{dt}||\nabla u||^2 + \nu||\nabla u||^2 \leq \frac{2}{\nu}||f||^2 + C(\alpha)\nu^{-1}||u||^2||\nabla u||^2. \tag{3.11}$$

The proof can now be completed by applying the uniform Gronwall lemma. With the following identifications of y, g and h we calculate

$$y(t) = ||\nabla u||^2, g(t) = C(\alpha)\nu^{-1}||u||^2, h = \frac{2}{\nu}||f||^2,$$

$$\begin{aligned}
\int_t^{t+r} y(t')dt' &= \int_t^{t+r} ||\nabla u(t')||^2 dt' \leq \frac{r}{\nu^2\lambda_1}||f||^2 + \frac{2\rho'^2}{\nu} =: k_1, \\
\int_t^{t+r} g(t')dt' &= \int_t^{t+r} \frac{C(\alpha)}{\nu}||u(t')||^2 dt' \leq \frac{C(\alpha)}{\nu}\rho'^2 r =: k \\
\int_t^{t+r} h(t')dt' &= \int_t^{t+r} ||\nabla u(t')||^2 dt' \leq \frac{2}{\nu}||f||^2 r =: k_3,
\end{aligned}$$

and thus

$$||\nabla u(t)||^2 \leq \{\frac{1}{\nu^2\lambda_1}||f||^2 + \frac{2\rho'^2}{\nu r} + \frac{C(\alpha)}{\nu}\rho'^2 r\}e^{\frac{2}{\nu}||f||^2 r} =: R_1 \tag{3.12}$$

Thus for $t \geq T + r$, $u(t)$ lies in a ball of radius R_1 in $\mathbf{V}$. Since R_1 is independent of u_0, (3.12) implies that for any bounded set $B \subset \mathbf{V}$,

$$\cup_{t \geq T+r} S(t)B$$

is a bounded set in $\mathbf{V}$. By the Reillich Lemma (e.g., [Gal94]), this set is compact in $\mathbf{H}$ and so $S(t)$ is uniformly compact. This completes the proof of existence of an absorbing set and compactness of $S(t)$ and thus of a global attractor.

4. Smoothness of the Attractor. This section proves regularity of the maximal attractor by showing that an attractor of the NS-omega model exists in each space $\mathbf{H}^s_\#$. As in [T88], this implies that $\mathcal{A}$ consists of $\mathbf{C}^\infty_\#$ functions. So as to not overburden the presentation with further assumptions on the deconvolution operator, we consider the base model (1.1). The results in this section extend to the examples of deconvolution operators given above and to general deconvolution operators under mild assumptions on smoothing in $\mathbf{H}^s_\#(\mathbf{\Omega})$, such as

$$||D(\overline{v})||_{\mathbf{H}^{s+2}_\#} \leq C||v||_{\mathbf{H}^s_\#}. \tag{4.1}$$

To prove regularity of the attractor we use a bootstrap argument to prove existence for each s, sharpening the regularity proof in Theorem 3.4 in [LST08]. We first prove uniform boundedness of spacial derivatives.

LEMMA 4.1. *Consider the NS-omega model with $u_0 \in \mathbf{H}^s_\#$, $f \in \mathbf{H}^s_\#, s \geq 0$. Then there is a finite constant C such that*

$$\sup_{[0,\infty)} ||u(t)||_s \leq \rho_s < \infty, .$$

Further, for every s there is a finite constant $C(||\nabla^s f||, \nu, \alpha, s)$ such that

$$\int_t^{t+r} ||\nabla \partial^s u(t')||^2 dt' \leq 2rC(||\nabla^s f||, \nu, \alpha, s) + \frac{2\rho_s^2}{\nu} < \infty. \tag{4.2}$$

Proof. We have proven this for $m = 0, 1$ in the previous section. Thus we consider $m \geq 2$. Letting ∂^m denote any partial derivative of order m, take an $m+1^{st}$ derivative ∂^{m+1} of the NS-omega model. This gives

$$(\partial^{m+1}u)_t - \partial^{m+1}(u \times \overline{\omega}) - \nu \triangle \partial^{m+1} u + \nabla \partial^{m+1} P = \partial^{m+1} f, \nabla \cdot \partial^{m+1} u = 0 \tag{4.3}$$

with periodic and zero mean boundary conditions. Multiplying by $\partial^{m+1}u$, integrating and using basic inequalities gives

$$\frac{1}{2}\frac{d}{dt}||\partial^{m+1}u||^2 + \frac{\nu}{2}||\nabla \partial^{m+1}u||^2 \leq C||\nabla^m f||^2 + (\partial^{m+1}(u \times \overline{\omega}), \partial^{m+1}u), \tag{4.4}$$

where the last term is the critical one. Expanding gives

$$\begin{aligned}(\partial^{m+1}(u \times \overline{\omega}), \partial^{m+1}u) &= \sum_{|\beta| \leq m+1} \binom{m+1}{\beta} \int_\Omega \partial^\beta \overline{\omega} \times \partial^{m+1-\beta} u \cdot \partial^{m+1} u dx = \\ &= \int_\Omega \{\partial^{m+1}\overline{\omega} \times u \cdot \partial^{m+1}u + (m+1)\partial^m \overline{\omega} \times \partial^1 u \cdot \partial^{m+1} u + \\ &\quad + \cdots + \\ &\quad + (m+1)\partial^1 \overline{\omega} \times \partial^m u \cdot \partial^{m+1} u + \overline{\omega} \times \partial^{m+1} u \cdot \partial^{m+1} u\} dx.\end{aligned}$$

By the smoothing property of the filter we have for $0 \leq \theta \leq \frac{1}{2}$ that

$$\begin{aligned}
(\partial^{m+1}(u\times\overline{\omega}), \partial^{m+1}u) &\leq C(m)\sum_{|\beta|\leq m+1}\int_\Omega |\partial^\beta\overline{\omega}\times\partial^{m+1-\beta}u\cdot\partial^{m+1}u|dx \\
&\leq C(m)\sum_{|\beta|\leq m+1}||\partial^\beta\overline{\omega}||_\theta||\partial^{m+1-\beta}u||_{\frac{1}{2}-\theta}||u||_{m+2}, \\
&\leq \frac{\nu}{8}||u||^2_{m+2} + C(m,\nu)\sum_{|\beta|\leq m+1}||\partial^\beta\overline{\omega}||^2_\theta||\partial^{m+1-\beta}u||^2_{\frac{1}{2}-\theta} \\
&\leq \frac{\nu}{8}||u||^2_{m+2} + C(m,\nu,\alpha)\sum_{|\beta|\leq m+1}||u||^2_{\theta+|\beta|+1-2}||u||^2_{m+\frac{3}{2}-\beta-\theta}.
\end{aligned}$$

We thus have for t large enough and $0 \leq \theta \leq \frac{1}{2}$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\partial^{m+1}u||^2 + \frac{7\nu}{8}||\nabla\partial^{m+1}u||^2 &\leq C||\nabla^m f||^2 \\
&\quad + C(m,\nu,\alpha)\sum_{|\beta|\leq m+1}||u||^2_{|\beta|+\theta-1}||u||^2_{(m+1-|\beta|)+(\frac{1}{2}-\theta)}.
\end{aligned}$$

The result will be proved using the induction hypothesis provided for each $\beta, 0 \leq |\beta| \leq m+1$, we can pick $\theta, 0 \leq \theta \leq \frac{1}{2}$, with

$$|\beta| + \theta - 1 \leq m, \text{ and } (m+1-|\beta|) + (\frac{1}{2}-\theta) \leq m.$$

(As then each term in the above sum is uniformly bounded in t for t large enough.) From the first inequality we pick

$$\theta = 0 \text{ if } |\beta| = m+1 \text{ and } \theta = \frac{1}{2} \text{ if } |\beta| \leq m.$$

If $\theta = 0$, $|\beta| = m+1$ then the second inequality becomes

$$1 + \frac{1}{2} \leq m+1$$

which holds since $m \geq 1$. If $\theta = \frac{1}{2}, |\beta| \leq m$ the second constraint becomes

$$(m+1-|\beta|) \leq m$$

which also holds because $m \geq 1$. This shows that

$$\frac{d}{dt}||\partial^{m+1}u||^2 + \nu||\nabla\partial^{m+1}u||^2 \leq C(||\nabla^m f||, \nu, \alpha, m)(< \infty), \tag{4.5}$$

from which the result follows using the Poincaré-Friedrichs inequality and an integrating factor. The second claim follows by integrating (4.5) over $(t, t+r)$ and using standard inequalities gives

$$||\partial^{m+1}u(t+r)||^2 - ||\partial^{m+1}u(t)||^2 + \int_t^{t+r}\nu||\nabla\partial^{m+1}u(t')||^2dt' \leq 2rC(||\nabla^m f||, \nu, \alpha, m).$$

For $||\partial^{m+1}u(t)||$ and $|||\partial^{m+1}u(t+r)|| \leq \rho_{m+1}$ we have

$$\int_t^{t+r} ||\nabla\partial^{m+1}u(t')||^2 dt' \leq 2rC(||\nabla^m f||, \nu, \alpha, m) + \frac{2\rho_{m+1}^2}{\nu} < \infty. \qquad (4.6)$$

completing the proof. □

The above lemma gives the necessary á priori bounds to apply the uniform Gronwall lemma as in the $s = 0$ case. Thus we can conclude existence of an attractor.

THEOREM 4.2 (Smooth Attractors). *Suppose*

$$u_0, f \in \mathbf{H}^s_{\#}(\mathbf{\Omega}), \nabla \cdot u_0 = \nabla \cdot f = 0, \int_\Omega f dx = \int_\Omega u_0 dx = 0.$$

Then $\mathcal{A}$ is an attractor of the NS-omega model (1.1) has a global attractor in each $\mathbf{H}^s_{\#}(\mathbf{\Omega})$ and thus consists of $\mathbf{C}^\infty_{\#}(\mathbf{\Omega})$ functions.

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5. Motivation for the NS-omega model. In 1934, J. Leray introduced the following NSE regularization (now known as the Leray model) as a theoretical tool:

$$u_t + \overline{u}\cdot\nabla u - \nu\Delta u + \nabla p = f \text{ and } \nabla\cdot u = 0\,, \quad \text{in } \Omega\times(0,T).$$

He chose $\overline{u} = g_\alpha \star u$, where g_α is a Gaussian associated with a length scale α, and proved existence and uniqueness of strong solutions to it and that a subsequence u_{α_j} converges to a weak solution of the NSE as $\alpha_j \to 0$. If that weak solution is a smooth, strong solution it is not difficult to prove additionally that $||u_{NSE} - u_{LerayModel}|| = O(\alpha^2)$ using only $||u-\overline{u}|| = O(\alpha^2)$.

The Camassa-Holm / Navier-Stokes-alpha NSE regularization, proposed as a possible basis for simulation of under-resolved flows, is analogous to the above Leray regularization when the Navier-Stokes equations (NSE) are written in rotational form. Indeed, the NSE, with nonlinear term in rotational form, are given by

$$u_t + \omega\times u - \nu\Delta u + \nabla P = f, \tag{5.1}$$

$$\omega = \nabla\times u, P = p + \frac{1}{2}|u|^2, \text{ and} \tag{5.2}$$

$$\nabla\cdot u = 0\,, \quad \text{in } \Omega\times(0,T). \tag{5.3}$$

The *NS-alpha model* is given by

$$u_t + \omega\times\overline{u} - \nu\Delta u + \nabla P = f, \tag{5.4}$$

$$\omega = \nabla\times u, \text{ and} \tag{5.5}$$

$$\nabla\cdot\overline{u} = 0\,, \quad \text{in } \Omega\times(0,T). \tag{5.6}$$

Here $\overline{u}$ is the differentially filtered / smoothed / mollified model velocity, given by

$$-\alpha^2\Delta\overline{u} + \overline{u} = u \text{ , in } \Omega \text{ and} \tag{5.7}$$

$$+ \text{ boundary conditions on } \partial\Omega. \tag{5.8}$$

Motivated by the requirements of practical computing with NSE regularizations we have considered a related (and to our limited knowledge new) regularization, the NS-omega model[1]. The related NS-omega model is (to my knowledge) not previously studied but emerges naturally from the above similarities between the NS-alpha model and rotational Leray regularizations. The first averages the vorticity term instead of the velocity (hence calling it the *NS-omega model* seemed natural). It is given by

$$u_t + \overline{\omega} \times u - \nu \Delta u + \nabla P = f, \tag{5.9}$$

$$\overline{\omega} \ := \nabla \times \overline{u}, \text{ and} \tag{5.10}$$

$$\nabla \cdot u = 0 \,, \quad \text{in } \ \Omega \times (0, T). \tag{5.11}$$

With that said, one can also study combinations such as the following; the *NS-alpha and omega model* is then naturally

$$u_t + \overline{\omega} \times \overline{u} - \nu \Delta u + \nabla P = f, \tag{5.12}$$

$$\overline{\omega} = \nabla \times \overline{u}, \text{ and} \tag{5.13}$$

$$\nabla \cdot \overline{u} = 0 \,, \quad \text{in } \ \Omega \times (0, T). \tag{5.14}$$

Concerning the precise formulation of the above nonlinear terms: here is no difference between $\overline{\nabla \times u}$ and in the periodic case. However, in the non-periodic case there can be a very significant difference. On the simplest level, computing $\nabla \times \overline{u}$ requires boundary conditions for the velocity (which are known) whereas $\overline{\nabla \times u}$ requires boundary conditions for the vorticity, which are unknown. There are both similarities and differences among the above five (Leray, NSE, NS-alpha, NS-omega and NS-alpha and omega). One is the above difference in the treatment of the incompressibility condition in the non-periodic case.

One main motivation for the above variations is that the perfect model of fluid motion is already known: the Navier-Stokes equations. Thus "models" like those discussed above exist only as an intermediate step to an under resolved numerical simulation of turbulent flow. Thus, two critical features of any such model / regularization are

- its solutions must faithfully represent the qualitative properties of solutions of the NSE, and
- it must be amenable to efficient numerical simulation with robust methods.

As an example of the motivation for the NS-omega model consider two standard, robust, fully discrete unconditionally stable algorithms, the CN (Crank-Nicolson) and CNLE (Crank-Nicolson with linear extrapolation) methods[2]. Suppressing the spacial discretization, the CN method for $u_n \simeq u(t_n)$ is for any of the three models

$$\frac{1}{\Delta t}(u_{n+1} - u_n) + b^h(\omega_{n+1/2}, u_{n+1/2}) + \nabla P_{n+1/2} - \nu \triangle^h u_{n+1/2} = f_{n+1/2}, \text{ and} \tag{5.15}$$

[1]In joint work with C Trenchea and I Stanculescu, W. Layton, I. Stanculescu and C. Trenchea, *Theory of the NS-omega model of turbulence*, (submitted, http://www.math.pitt.edu/techreports.html), 2008., a Leray-type theory has been established for for the NS-omega model and estimates of compolexity made based on turbulence phenomenology.

[2]The numerical analysis of these two is developed in W. Layton, C. Manica, M. Neda and L. Rebholz, *Numerical analysis and computational comparisons of the NS-alpha and NS-omega regularizations*, to appear: CMAME, 2008.

$$\text{for the alpha model: } \nabla \cdot \overline{u_{n+1}} = 0 \ , \tag{5.16}$$

$$\text{for the omega model: } \nabla \cdot u_{n+1} = 0 \tag{5.17}$$

$$\text{for the alpha-and-omega model: } \nabla \cdot \overline{u_{n+1}} = 0 \ . \tag{5.18}$$

Briefly, this method requires the solution of a large ($2N_{velocity} + N_{pressure}$ unknowns) coupled *nonlinear* system at every time step. The NS-omega model can be solved much more economically by Baker's CNLE method (again, simplifying and suppressing spatial discretizations), given by

$$\frac{1}{\Delta t}(u_{n+1} - u_n) + \overline{(\frac{3}{2}\omega_n - \frac{1}{2}\omega_{n-1})} \times u_{n+1/2} + \nabla P_{n+1/2} - \nu \triangle^h u_{n+1/2} = f_{n+1/2}, \tag{5.19}$$

$$\text{where } \omega_n = \nabla \times u_n \tag{5.20}$$

$$\text{and } (\nabla \cdot u^h_{n+1}, \chi^h) = 0 \ , \forall \chi^h \in Q^h. \tag{5.21}$$

This method filters an explicitly known (from previous time levels) variable. This simple method is second order accurate, unconditionally and nonlinearly stable and requires only the solution of one smaller ($N_{velocity}+N_{pressure}$ unknowns) *linear* system at every time step. To our knowledge, no comparable simplification exists for the NS-alpha model which shares all these favorable properties.

The remaining critical question concerns physical fidelity of the NS-omega model.

The ideas we study are outgrowths of the seminal work of J. Leray, the recent work on the NS-α model, the early work of G. Baker from 1976 on extrapolated Crank-Nicolson methods and our previous work on the numerical analysis of approximate deconvolution models of turbulence.

The "models" discussed above are, strictly speaking, not models as such. Their solutions have no intrinsic meaning other than as approximations to the solutions of the NSE. Thus, they are important as are regularizations of the NSE and proposed as a basis for reliable simulation of turbulent flows. Thus, we stress that the ultimate question is to study convergence of discretizations the continuum regularizations to solutions of the NSE as the spatial mesh widths (typically h and $\triangle t$), time steps and filter width α all $\rightarrow 0$.

6. More about Averaging or Filtering. Averaging is a very common procedure in turbulence modeling. The classic example is time averaging or Reynolds averaging (after Osbourne Reynolds) over a finite time window :

$$\langle u \rangle_{[0,T]} (x,T) := \frac{1}{T}\int_0^T u(x,t)dt, \text{ or } \langle u \rangle_{[t-T,t]} (x,t) := \frac{1}{T}\int_{t-T}^t u(x,t')dt'.$$

Motivated by practical computations, the key idea is that in an under resolved simulation a computed velocity should properly represent an average over one (or a few) mesh cells. This leads to a space filtered $\overline{u}(x,t)$, such as

$$\overline{u}(x,t) := \frac{1}{\alpha^3}\int_{[-\frac{\alpha}{2},+\frac{\alpha}{2}]^3} u(x-x',t)dx'. \tag{6.1}$$

Alternately, equivalently and mathematically clearer, weighed local averages are defined by convolution or filtering with the chosen filter kernel, such as a top hat kernel for the above average or a Gaussian:

$$\overline{u}(x,t) := g_\alpha \star u(x,t), \text{ where } g_\alpha \star u(x,t) := \int_{\mathbb{R}^3} g_\alpha(x')u(x-x',t)dx',$$
$$g_\alpha(x) := \alpha^{-3} g(x/\alpha), \text{ and } g(x) := \text{Gaussian}.$$

Viewing averaging from the point of view of scale space, one thinks of a fluid velocity as naturally composed of a scale of velocities (scaled by the filter length scale α)

$$u = u(x,t;\alpha)(= (g_\alpha \star u)(x,t)).$$

In practical computing, the averaging radius α is related to the (possibly locally varying) mesh width and filters are often chosen based on computational convenience.

6.1. What is the right filter?. Ignoring computational convenience for the moment, it is interesting also to consider filtering from the ideal continuum point of view. If the averages are viewed as containing information of physical meaning, there already results on acceptable filters beginning with the famous paper of Koenderink[3]. Scale space analysis[4] begins with some basic postulates that any physically reasonable filter should satisfy. There are various ways to develop the essential result. One begins by assuming the filter is

CONDITION 6.1. *linearity:*

$$\overline{\alpha u + \beta v} = \alpha \overline{u} + \beta \overline{v}$$

CONDITION 6.2. *spacial invariance (no preferred location):*

$$\overline{u(x-a)} = \overline{u}(x-a)$$

CONDITION 6.3. *isotropy (no preferred orientation): for all rotations R:*

$$\overline{u}(x) = R^* \overline{u(Rx)}$$

CONDITION 6.4. *scale invariance, or the semigroup property (no preferred size): if $\overline{u}(x,t) := g_\alpha \star u(x,t)$ (so $\overline{\overline{u}} := g_\alpha \star g_\alpha \star u$) then*

$$\overline{\overline{u}} = g_{\sqrt{2}\alpha} \star u.$$

For example, in 1984 Koenderink proved the following.

THEOREM 6.5 (Koenderink 1984). *The only filter satisfying C1, C2, C3 and C4 is the Gaussian filter.*

Thus, there is really only one mathematical correct filter: the Gaussian. The same conclusion can be arrived at by other plausible conditions on the filter such as filtering not creating new structures such as local extrema and causality in the scale variable α. Convolution with the Gaussian is quite expensive computational so that in spite of this strong uniqueness result usually other filters are used. Koenderink's result indicate

[3] J.J. KOENDERINK, *The structure of images*, *Biol. Cybernetics*, 50 (1984) 363-370.

[4] See, e.g., T. LINDEBERG, *Scale-space theory in computer vision, Kluwer, Dordrecht, 1994.*

that other filters for large eddy simulation must be assessed as approximations to the Gaussian. One filtering method which is very convenient mathematical and not too unreasonable computational is a special differential filter. Define the velocity averages by:

$$\overline{u}(x,t) := (-\alpha^2\triangle + 1)^{-1}u(x,t). \tag{6.2}$$

Differential filters were proposed for large eddy simulation by Germano[5] and the above differential filter is a well known regularization of evolution equations. The close connection of the above differential filter to the Gaussian filter can be seen two ways. In Galdi and Layton[6], it was derived as an approximation of the Gaussian filter by Padé approximations of exponentials as follows. Fourier transformation of

$$\overline{u}(x,t) := g_\alpha \star u(x,t)$$

gives

$$\widehat{\overline{u}}(k,t) := \widehat{g_\alpha}\widehat{u}(k,t) \text{ , where } \widehat{g_\alpha} = e^{-|\alpha k|^2}.$$

Since $e^{-|\alpha k|^2} \to 0$ as $|k| \to \infty$, the filter suppresses fluctuations; in other words it is smoothing. This is a fundamental property that must be preserved under approximation. The simplest rational approximation preserving this is the $(0,1)$ Padé approximation given by

$$\begin{aligned} e^{-\theta} &= \frac{1}{1+\theta} + O(\theta^2) \text{ , as } \theta \to 0, \text{ so} \\ e^{-|\alpha k|^2} &= \frac{1}{1+|\alpha k|^2} + O(|\alpha k|^4). \end{aligned}$$

Using this approximation in the above for $\widehat{g_\alpha}$ and inverting the Fourier transform recovers the differential filter (#).

Alternately, the Gaussian is the heat kernel. Thus, one way to compute the average velocity $\overline{u}(x,t) = g_\alpha \star u(x,t)$ is to solve the following evolution equation:

$$\begin{aligned} v_s(x,s) &= \triangle v(x,s) \text{ for } s > 0, \\ v(x,0) &= u(x), \end{aligned}$$

then set

$$\overline{u}(x,t) := v(x,s)|_{s=\alpha^2}.$$

This gives the exact Gaussian filtered velocity $\overline{u}(x,t) := g_\alpha \star u(x,t)$. Since the averaging radius α is small, α^2 is smaller still and we can reasonably approximate $v(x,\alpha^2)$ by one step of backward Euler, leading back to the differential filter

$$\overline{u}(x,t) := (-\alpha^2\triangle + 1)^{-1}u(x,t).$$

In comparison with the Gaussian, the differential filter is only approximately scale invariant. Indeed, if we compute $\overline{\overline{u}}$ we find that it fails the semigroup property by

[5] M. Germano, *Differential filters of elliptic type*, Phys. Fluids, 29(1986), 1757-1758.

[6] G. P. Galdi and W. J. Layton, *Approximation of the large eddies in fluid motion II: A model for space-filtered flow*, Math. Models and Methods in the Appl. Sciences, 10(2000), 343-350.

$O(\alpha^4)$

$$\begin{aligned}\overline{\overline{u}} &= (-\alpha^2\triangle + 1)^{-1}(-\alpha^2\triangle + 1)^{-1}u \\ &\neq (-(\sqrt{2}\alpha)^2\triangle + 1)^{-1}u, \\ &\quad \text{but rather for smooth } u, \\ \overline{\overline{u}} &= (-(\sqrt{2}\alpha)^2\triangle + 1)^{-1}u + O(\alpha^4).\end{aligned}$$

Many important theoretical and practical questions remain at this very first step.

7. More on Deconvolution. If we can invert the filter exactly (so called, *exact deconvolution*), then the closure problem is solved in principle. For example, the differential filter has an exact filter inverse[7]

$$A := -\alpha^2\triangle + 1$$

which exists as an *unbounded* operator on $L^2(\Omega)$ with dense domain and closed range. To obtain a useful regularization, however, information must be lost by *approximate* deconvolution. This means that the extra terms should be smoothing in some sense. Our intuition is that this is accomplished when the approximate deconvolution operator is a *bounded operator.*

Alternately, an unbounded deconvolution operator will suffer from small divisor problems: *the (inevitable) noise from data and discretization will be magnified by the model rather than damped.* Unfortunately, it is well-known to be impossible for an interesting filter to have a bounded (exact) inverse[8]. Consider the deconvolution problem

$$\text{given } \overline{u} \text{ (+ } noise\text{) solve } \overline{u} = Gu \text{ for } u.$$

THEOREM 7.1. *Let X be a Hilbert space and $G : X \to X$ be a compact linear operator. Then, if G^{-1} is bounded then $dim(X)$ is finite.*

The following is an immediate consequence.

COROLLARY 7.2. *Let $G\phi = \overline{\phi}$ denote the filtering operator. If G is smoothing so $G : L^2(\Omega) \to H^s(\Omega)$ is bounded for some $s > 0$ then G cannot have an exact inverse that is a bounded linear operator : $L^2(\Omega) \to L^2(\Omega)$.*

The problem of inverting the filter G is is ill-posed and thus the closure problem itself must be ill posed. This does not mean that accurate approximate closure is impossible. There are after all many good methods for approximate solution of ill posed problems!

7.1. Approximate deconvolution as an ill-posed a problem. The filtering or convolution problem is: given ϕ compute $\phi \to G\phi = \overline{\phi} := g \star \phi$. The de-filtering or deconvolution problem is: given $\overline{\phi}$ (possibly $+$*noise*) solve the following equation approximately for ϕ

$$\text{given } \overline{\phi} \text{ solve } \overline{\phi} = G\phi \text{ for } \phi.$$

[7] The same argument can be made for any filter with $\widehat{g}(k) \neq 0$. Filters which do not satisfy this are often called "lossy" filters.

[8] L. C. Berselli, T. Iliescu, and W. Layton, *Mathematics of Large Eddy Simulation of Turbulent Flows.* Springer, Berlin, 2006.

AND

M. Bertero and B. Boccacci, *Introduction to Inverse Problems in Imaging*, IOP Publishing Ltd., 1998.

Definition 7.3. *An approximate deconvolution operator D is a bounded linear operator $D : L^2(\Omega) \to L^2(\Omega)$ satisfying*

$$\text{for smooth functions } \phi :$$
$$\phi = D\, \overline{\phi} + O(\alpha^\alpha) \ \alpha \to 0 \text{ for some } \alpha \geq 2.$$

The deconvolution error is

$$e_{DCV}(\phi) = \phi - D_N \overline{\phi}.$$

The deconvolution problem is an important problem in image processing so there are many methods specifically directed at the deconvolution problem. We shall consider a few examples.

Example 7.4. *Tikhonov-Lavrentiev regularization.*

Let $X = L^2(\Omega)$ and $(\cdot, \cdot), ||\cdot||$ denote the inner product and induced norm on X. If G is a symmetric, positive definite operator, solving the deconvolution problem

$$\text{given } \overline{\phi} \text{ solve } \overline{\phi} = G\phi \text{ for } \phi.$$

in the Hilbert space X is formally equivalent to minimizing the quadratic functional

$$\phi = \arg\min_{v \in X} J(v)$$
$$J(v) \ : \ = \frac{1}{2}(Gv, v) - (\overline{\phi}, v).$$

The exact solution is obviously $v = \phi$. However, when noise $\varepsilon \in X$ is present it is easy to construct simple examples in which the minimization problem

$$\phi = \arg\min_{v \in X} \frac{1}{2}(Gv, v) - (\overline{\phi} + \varepsilon, v)$$

has no solution.

Tikhonov-Lavrentiev regularization picks a regularization parameter $\mu > 0$ and computes the approximation of the deconvolution problem by the approximate minimization problem:

$$\phi_\mu = \arg\min_{v \in X} \frac{1}{2}(Gv, v) - (\overline{\phi}, v) + \frac{\mu}{2}||v||^2.$$

The classical Tikhonov-Lavrentiev method has error even on the largest scales. C. Manica and I. Stanculescu recently gave an important refinement which reduces the associated error on the large scales significantly. The modification is

$$\phi_\mu = \arg\min_{v \in X} J_\mu(v)$$
$$J_\mu(v) =^{def} (1 - \mu)\{\frac{1}{2}(Gv, v) - (\overline{\phi}, v)\} + \frac{\mu}{2}||v||^2.$$

The Euler-Lagrange equations of the above is

$$\phi_\mu = ((1 - \mu)G + \mu I)^{-1}\overline{\phi}$$

so the approximate deconvolution operator induced by Tikhonov regularization is

$$D_\mu = ((1-\mu)G + \mu I)^{-1}.$$

When the filter operator is not a SPD, the deconvolution problem is converted into the SPD deconvolution problem

$$\text{given } \overline{\phi} \text{ solve } G^*\overline{\phi} = G^*G\phi \text{ for } \phi.$$

by least squares and the associated deconvolution operator of this is then

$$D_\mu = (G^*G + \mu I)^{-1}G^*.$$

This is the (full) Tikhonov regularization. For example, with the differential filter

$$\overline{u}(x,t) := (-\alpha^2\triangle + 1)^{-1}u(x,t),$$

we have formally

$$D_\mu = ((-\alpha^2\triangle + 1)^{-1} + \mu I)^{-1}.$$

Thus given a filtered variable $\overline{\phi}$, its deconvolved variable is calculated by solving:

$$\{\mu(-\alpha^2\triangle + 1) + 1\}\phi_\mu = (-\alpha^2\triangle + 1)\overline{\phi} \text{ , or, equivalently}$$

$$\phi_\mu = \frac{1}{\mu}\overline{\phi} - \frac{1}{\mu}\{\mu(-\alpha^2\triangle + 1) + 1\}^{-1}\overline{\phi}.$$

Example 7.5 (The van Cittert algorithm). *In 1931 (!) van Cittert studied a very simple approximate deconvolution algorithm. The van Cittert algorithm is equivalent to first order Richardson iteration for solving the ill posed operator equation* $G\phi = \overline{\phi}$ *or simple iteration in*

$$\textit{given } \overline{u} \textit{ solve } u = u + \{\overline{u} - A^{-1}u\} \textit{ for } u.$$

Algorithm 7.6 (van Cittert Approximate Deconvolution). *Set* $v_0 = \overline{u}$,
for $n = 0, 1, 2, ..., N-1$, *perform*
$v_{n+1} = v_n + \{\overline{u} - A^{-1}v_n\}$
Define $D_N\overline{u} := v_N$.

By eliminating the intermediate steps, the N^{th} de-convolution operator D_N is given explicitly by

$$D_N\phi := \sum_{n=0}^{N}(I - A^{-1})^n\phi. \tag{7.1}$$

For example, the approximate de-convolution operator corresponding to $N = 0, 1, 2$ are:

$$D_0\overline{u} = \overline{u},$$
$$D_1\overline{u} = 2\overline{u} - \overline{\overline{u}},$$
$$D_2\overline{u} = 3\overline{u} - 3\overline{\overline{u}} + \overline{\overline{\overline{u}}}.$$

It is known that D_N is bounded, SPD and an asymptotic filter inverse of accuracy $O(\alpha^{2N+2})$:

$$\phi = D_N\overline{\phi} + O(\alpha^{2N+2}) \text{ , for smooth } \phi.$$

PROPOSITION 7.7. *Let $G = A^{-1}$ be the differential filter 6.2. Then, both G and $I - G$ are SPD; further $0 \le \lambda(G) \le 1$ and $1 \le \lambda(D_N) \le N+1$. The operator D_N is bounded*

$$||D_N||_{L(L^2(\Omega)\rightarrow L^2(\Omega))} \le N+1.$$

Further,

$$\phi = D_N\overline{\phi} + O(\alpha^{2N+2}) \text{ , for smooth } \phi.$$

One immediate consequence of the above asymptotic result is convergence as $\alpha \rightarrow 0$ for fixed N. This is typical for filtering, (see the recent work of Stanculescu).

COROLLARY 7.8. *For $\phi \in L^2(\Omega), D_N\overline{\phi} \rightarrow \phi$ as $\alpha \rightarrow 0$ for fixed N.*

Proof. Let $\varepsilon > 0$ be given and let ψ be a smooth function with $||\phi - \psi|| < [1 + (N+1)||G||]^{-1}\frac{\varepsilon}{3}$. Write

$$||\phi - D_N\overline{\phi}|| \le ||\phi - \psi|| + ||\psi - D_N\overline{\psi}|| + ||D_N\overline{(\phi - \psi)}||,$$

so that

$$||\phi - D_N\overline{\phi}|| \le ||\phi - \psi|| + O(\alpha^{2N+2}) + ||D_N||||G||||\phi - \psi|| \le$$
$$\le \frac{\varepsilon}{3} + O(\alpha^{2N+2}) < \varepsilon, \text{ for } \alpha \text{ small enough.}$$

□

EXAMPLE 7.9 (Optimized van Cittert deconvolution). *Since the van Cittert algorithm is equivalent to first order Richardson for the operator equation $G\phi = \overline{\phi}$, relaxation parameters can be introduced and no extra cost. with proper choice of the optimization parameters significant improvement of accuracy is possible.*

ALGORITHM 7.10 (van Cittert deconvolution with relaxation). *Set $v_0 = \overline{u}$,*
For $n = 0, 1, 2, ..., N-1$,
select relaxation parameter ω_n and compute
$v_{n+1} = v_n + \omega_n\{\overline{u} - A^{-1}v_n\}$
Define $D_N^{\omega}\overline{u} := v_N$.

Optimization of the parameters ω_n depends on the objective and the exact choice of filters two cases have been studied by Stanculescu.

CASE 7.11 (K41 optimized deconvolution). *In the work of Stanculescu the optimal parameters were derived to minimize the norm of the deconvolution error $||\phi - D_N\overline{\phi}||$ over the resolved scales for velocity fields coming from turbulent flows with the inertial range energy spectrum typical of homogeneous isotropic turbulence:*

Find $(\omega_0, \omega_1, \cdots, \omega_N)$ minimizing
$$||u - D_N\overline{u}||$$
subject to $\widehat{E}(k) = \alpha_{Kolmogorov}\varepsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$.

CASE 7.12 (Optimization for general velocity fields). *Another possibility is to optimize the norm of the deconvolution error over general, square integrable velocity fields. This leads to the minimax problem*

$$\min_{\omega_j} \max_{\phi \in L^2} ||\phi - D_N \overline{\phi}||$$

which was solved by Stanculescu as well.

EXAMPLE 7.13 (Geurts' approximate filter inverse operators). *Exploiting special features of the top hat filter, Geurts*[9] *constructed efficient and ingenious approximate filter inverses of varying degrees of accuracy of the top hat filter.*

[9] B. J. GEURTS, *Inverse modeling for large eddy simulation*, Phys. Fluids, 9(1997), 3585.