# COMPACTNESS OF WEAK SOLUTIONS TO THE THREE-DIMENSIONAL COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. The compactness of weak solutions to the magnetohydrodynamic equations for the viscous, compressible, heat conducting fluids is considered in both the three-dimensional space $\mathbb{R}^3$ and the three-dimensional periodic domains. The viscosities, the heat conductivity as well as the magnetic coefficient are allowed to depend on the density, and may vanish on the vacuum. This paper provides a different idea from [15] to show the compactness of solutions of viscous, compressible, heat conducting magnetohydrodynamic flows, derives a new entropy identity, and shows that the limit of a sequence of weak solutions is still a weak solution to the compressible magnetohydrodynamic equations.

## 1. Introduction

Magnetohydrodynamics (MHD) is the theory of the macroscopic interaction of electrically conducting fluids with magnetic fields. It has a very broad range of applications. It is of importance in connection with many engineering problems, such as sustained plasma confinement for controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, and electromagnetic casting of metals. It also finds applications in geophysics and astronomy, where one prominent example is the so-called dynamo problem, that is, the question of the origin of the Earth's magnetic field in its liquid metal core.

Due to their practical relevance, MHD problems have long been the subject of intense cross-disciplinary research, but except for relatively simplified special cases, the rigorous mathematical and numerical analysis of such problems remains open. In the viscous incompressible case, MHD flow is governed by the Navier-Stokes equations and the Maxwell equations of the magnetic field. For the mathematical analysis in incompressible MHD equations, see [8, 12, 22] and the references therein. In the compressible case, the mathematical analysis is much more complicated, due to the oscillation of the density, the concentration of the temperature, and the coupling interaction of hydrodynamics with the magnetic field. The full system of the three-dimensional magnetohydrodynamic equations in the Eulerian coordinates can be read as follows([17, 18]):

$$\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \tag{1.1a}$$

$$(\rho\mathbf{u})_t + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = (\nabla\times\mathbf{H})\times\mathbf{H} + \mathrm{div}\Psi, \tag{1.1b}$$

$$\mathcal{E}_t + \mathrm{div}(\mathbf{u}(\mathcal{E}' + p)) = \mathrm{div}((\mathbf{u}\times\mathbf{H})\times\mathbf{H} + \nu\mathbf{H}\times(\nabla\times\mathbf{H}) + \mathbf{u}\Psi + \kappa\nabla\theta), \tag{1.1c}$$

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \qquad \mathrm{div}\mathbf{H} = 0, \tag{1.1d}$$

where $\Psi = 2\mu D(\mathbf{u}) + \lambda\,\mathrm{div}\mathbf{u}\,\mathbf{I}$ with $3\lambda + 2\mu \geq 0$ and $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)$ denotes the strain rate tensor; $\rho$ denotes the density, $\mathbf{u}\in\mathbb{R}^3$ the velocity, $\mathbf{H}\in\mathbb{R}^3$ the magnetic field,

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and $\theta$ the temperature; $\mathcal{E}$ is the total energy given by

$$\mathcal{E} = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) + \frac{1}{2}|\mathbf{H}|^2 \text{ and } \mathcal{E}' = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right),$$

with $e$ the internal energy, $\frac{1}{2}\rho|\mathbf{u}|^2$ the kinetic energy, and $\frac{1}{2}|\mathbf{H}|^2$ the magnetic energy. The equations of state $p = p(\rho,\theta)$, $e = e(\rho,\theta)$ relate the pressure $p$ and the internal energy $e$ to the density and the temperature of the flow; $\mathbf{I}$ is the $3\times 3$ identity matrix, and $(\nabla\mathbf{u})^\top$ is the transpose of the matrix $\nabla\mathbf{u}$. $\nu(\rho,\theta)$ is the magnetic field coefficient, $\kappa = \kappa(\rho,\theta)$ is the heat conductivity. In general, equations (1.1a), (1.1b), (1.1c) denote the conservations of mass, momentum, and energy, respectively. The equation (1.1d) is called the induction equation, and the electric field can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$,

$$\mathbf{E} = \nu\nabla\times\mathbf{H} - \mathbf{u}\times\mathbf{H}.$$

Although the electric field $\mathbf{E}$ does not appear in the MHD system (1.1a)-(1.1d), it is indeed induced according to the above relation by the moving conductive flow in the magnetic field.

In this paper, we are interested in the compactness of weak solutions to the compressible MHD equations (1.1) both in the three-dimensional space $\mathbb{R}^3$ and in the three-dimensional periodic domains. As it is well-known, the motivation of considering the compactness of weak solutions is to show the existence of weak solutions and the stability of weak solutions of nonlinear problems. In the literature, there have been a lot of studies on MHD by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [4, 5, 7, 9, 11, 13, 14, 15, 16, 18, 24] and the references cited therein. For instance, the smooth global solution near the constant state in one-dimensional case is investigated in [16]. However, many fundamental problems for compressible MHD with large, discontinuous initial data are still open.

Positive results on the existence of weak solutions with large, discontinuous data for compressible MHD equations have been obtained recently in [14, 15], specially in [15] for full compressible MHD equations with temperature-dependent viscosities. More precisely, it was shown in [15], under certain structural hypotheses imposed on the pressure $p$ and the heat conductivity coefficient $\kappa$, that the full compressible MHD system admits at least a global-in-time variational solution for large initial data. Those solutions satisfy the equations (1.1a), (1.1b), (1.1d) in the sense of distributions while the thermal energy equation (1.1c) is being replaced by two inequalities to be accordance with the second law of thermodynamics. This approach is in the spirit of the concept of *weak solutions with a defect measure* introduced by several authors in different contexts, see [6]. However, in order to obtain the estimates on the gradient of the velocity, the works in [14, 15] do rely strongly on the assumption that the shear viscosity $\mu$ is bounded below by a positive constant.

Our aim, in this paper, is to show the compactness of weak solutions to the full compressible MHD equations with viscosity coefficients vanishing on the vacuum both in the three-dimensional space $\mathbb{R}^3$ and in the three-dimensional periodic domains. Although the periodic case does not correspond to a physical configuration, its mathematical treatment is technically easier, while it retains the main mathematical difficulties of the problem of the flow. More importantly, in our context, the viscosities $\mu, \lambda$, the heat conductivity $\kappa$, and the magnetic coefficient $\nu$ can be allowed to depend on the density $\rho$ and the temperature $\theta$ of the flow. We remark that the similar problems for the compressible Navier-Stokes equations have been studied in [1, 2, 3, 21]. Comparing with those works on the compressible Navier-Stokes equations, we will encounter extra difficulties in studying the compressible MHD equations. More precisely, besides the possible oscillation of the density and the concentration of the temperature, the appearance of the magnetic field and the coupling effect between the hydrodynamic flow and the magnetic field should also been taken into consideration.

The novelty of this paper is to provide a new method to deal with the vanishing viscosities for compressible MHD flows. The loss of positivity of the viscosity coefficients implies that there is no hope to obtain directly the uniform bound on the gradient of the velocity. It is well known that the main difficulty, in proving the compactness of weak solutions of compressible MHD equations, is to pass to the limit for the nonlinear terms. In our context, the new kind of entropy equality will provide the estimates on the gradient of the density, which makes the nonlinear terms much easier to be dealt with and also give rise to a new estimate of $\rho\mathbf{u}^2$ in a functional space better than $L^\infty([0,T];L^1(\Omega))$. In other words, although the regularity on the velocity that we can get directly is much lower, the regularity on the density in our context is much higher. To achieve this aim, the entropy equation (3.9) and the thermal equation (4.2) need to be taken into consideration. But, unfortunately, the case with constant viscosity coefficients is excluded from our setting and the extension to the general case in which the viscosity coefficients depend on both the density and the temperature seems also out of the reach of our present work.

We organize the rest of this paper as follows. In Section 2, we will give the hypotheses in detail, introduce the definition of weak solutions, and state our compactness result (Theorem 2.1). In Section 3, we will derive the *a priori* estimates and a new kind of the entropy identity. In Section 4, some auxiliary integrability lemmas are showed. Finally, we will finish the proof of Theorem 2.1 in Section 5 using Aubin-Lions Lemma.

## 2. Assumptions and the Main Result

To our best knowledge, the rigorous mathematical analysis for compressible flows is beyond the available mathematical framework. Hence, we need add some restrictions to viscosity coefficients $\mu$, $\lambda$, the heat conductivity $\kappa$ and the magnetic coefficient $\nu$.

2.1. **Assumptions.** To begin with, we assume that $\mu(\rho)$ and $\lambda(\rho)$ are two $C^1(0,\infty)$ functions satisfying

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \tag{2.1}$$

As seen later on, this relation is fundamental to get higher regularity on the density. More precisely, with the help of this relation, we can show a new kind of entropy equality, which then gives the uniform bound on the gradient of the density. Next, due to our technical restrictions, we will need the following constraints: $\mu(0) = 0$ and there exist positive constants $c_0$, $c_1$, $A$ and $m > 1$, $2/3 < \beta < 1$ such that

$$\begin{cases} \text{for all } s < A, \quad c_0 s^{\beta-1} \le \mu'(s) \le \frac{s^{\beta-1}}{c_0} \text{ and } 3\lambda(s) + 2\mu(s) \ge c_0 s^\beta, \\ \text{for all } s \ge A, \quad c_1 s^{m-1} \le \mu'(s) \le \frac{s^{m-1}}{c_1} \text{ and } c_1 s^m \le 3\lambda(s) + 2\mu(s) \le \frac{s^m}{c_1}. \end{cases} \tag{2.2}$$

Observing that the assumption (2.2) implies that $\mu'(\rho) > 0$ for $\rho > 0$.

The heat conductivity coefficient $\kappa$ is assumed to satisfy

$$\kappa(\rho,\theta) = \kappa_0(\rho,\theta)(\rho+1)(\theta^a+1), \tag{2.3}$$

where $a \ge 2$, and $\kappa_0$ is a $C^0(R^2_+)$ function satisfying for all positive $\rho$ and $\theta$,

$$c_2 \le \kappa_0(\rho,\theta) \le \frac{1}{c_2},$$

for some positive constant $c_2$.

For the pressure, we assume that the equations of state are of ideal polytropic gas type

$$p = \rho\theta + p_e(\rho), \quad e = c_\nu\theta + P_e(\rho), \tag{2.4}$$

with $P_e(\rho) = \int_1^\rho p_e(\xi)/\xi^2 \mathrm{d}\xi$. We also require that $p_e(\rho)$ satisfies

$$\begin{cases} c_3\rho^{-l-1} \le p_e'(\rho) \le \frac{1}{c_3}\rho^{-l-1}, \text{ if } 0 \le \rho < A_0, \\ p_e'(\rho) \le c_4\rho^{k-1}, \text{ if } \rho > A_0, \end{cases} \tag{2.5}$$

for some $A_0 > 0$, $c_3 > 0$, $c_4 > 0$, $l > \frac{2\beta(3m-2)}{m-1} - 1$, and $k \le \left(m - \frac{1}{2}\right)\frac{5(l+1)-6\beta}{l+1-\beta}$.

For the magnetic coefficient, we need the following assumption:

$$\nu(\rho,\theta) \ge c_5\frac{\theta}{\rho} \text{ on } \{\rho > 0\}, \quad \frac{1}{c_6} \ge \nu(\rho,\theta) \ge c_6, \tag{2.6}$$

for some $c_5 > 0$, $c_6 > 0$.

2.2. **Main Result.** Before we state the compactness result, we need to specify the definition of weak solutions which we will address. It is necessary to require that the weak solutions should satisfy the natural energy estimates and from the viewpoint of physics, the conservation laws on mass, momentum and energy also should be satisfied at least in the sense of distributions. Based on those considerations, the definition of reasonable global-in-time weak solutions goes as follows.

**Definition 2.1.** A vector $(\rho, \mathbf{u}, \theta, \mathbf{H})$ is said to be a global-in-time weak solution to the full compressible MHD system (1.1a)-(1.1d), if and only if for any positive number $T$, the following conditions are satisfied:

- $$\rho\mathcal{E} \in L^\infty([0,T]; L^1(\Omega)), \quad \rho|\mathbf{u}|^2 \in L^\infty([0,T]; L^1(\Omega)),$$
$$\frac{\nabla\mu(\rho)}{\sqrt{\rho}} \in L^\infty([0,T]; L^2(\Omega)), \quad (\rho^{\beta/2} + \rho^{m/2})\nabla\mathbf{u} \in L^2([0,T]; L^2(\Omega)),$$
$$(1+\sqrt{\rho})\nabla\theta^{a/2} \in L^2([0,T]; L^2(\Omega)), \qquad (1+\sqrt{\rho})\frac{\nabla\theta}{\theta} \in L^2([0,T]; L^2(\Omega)),$$
for $a \ge 2$. Moreover, for large enough $s > 0$, we have
$$\rho, \quad \rho\mathbf{u}, \quad \rho\mathcal{E}, \quad \mathbf{H} \in C([0,T]; H^{-s}(\Omega)),$$
and hence, the initial data are satisfied in the sense of distributions (denoted by $\mathcal{D}'(\Omega \times (0,T))$.
- The equations (1.1a)-(1.1d) are satisfied in the sense of distributions.

Now our compactness result can be read as follows:

**Theorem 2.1.** *Let $\Omega$ be either the three-dimensional periodic domains or the three-dimensional space $\mathbb{R}^3$. Assume that $\mu$, $\lambda$, $\nu$, $\kappa$ are $C^1[0,\infty)$ functions satisfying the hypotheses* (2.1)-(2.6). *Let $\{(\rho_n, \mathbf{u}_n, \theta_n, \mathbf{H}_n)\}_{n=1}^\infty$ be a sequence of weak solutions of* (1.1a)-(1.1d) *satisfying the entropy equation* (3.9) *and the thermal energy equation* (4.2) *with the initial data*

$$\rho_n(x,0) = \rho_{0,n}(x), \quad \mathbf{u}_n(x,0) = \mathbf{u}_{0,n}(x),$$
$$\theta_n(x,0) = \theta_{0,n}(x), \quad \mathbf{H}_n(x,0) = \mathbf{H}_{0,n}(x),$$

*where $\rho_{0,n}$, $\mathbf{u}_{0,n}$, $\theta_{0,n}$, $\mathbf{H}_{0,n}$ satisfy*

$$\begin{cases} \rho_{0,n} \ge 0, \quad \rho_{0,n} \to \rho_0 \text{ in } L^1(\Omega), \\ \rho_{0,n}\ln\rho_{0,n} \to \rho_0\ln\rho_0 \in L^1(\Omega), \quad \rho_{0,n}\ln\theta_{0,n} \to \rho_0\ln\theta_0 \in L^1(\Omega), \\ \rho_{0,n}|\mathbf{u}_{0,n}|^2 \to \rho_0|\mathbf{u}_0|^2 \text{ in } L^1(\Omega), \\ \theta_{0,n} > 0, \quad \mathbf{H}_{0,n} \to \mathbf{H}_0 \text{ in } L^2(\Omega), \\ \rho_{0,n}e_{0,n} \to \rho_0 e_0 \in L^1(\Omega), \quad \frac{\nabla\mu(\rho_0)}{\sqrt{\rho_0}} \to \frac{\nabla\mu(\rho_0)}{\sqrt{\rho_0}} \in L^2(\Omega). \end{cases} \tag{2.7}$$

*Assume also that the sequence of the magnetic field $\{\mathbf{H}_n\}_{n=1}^{\infty}$ is uniformly bounded in $L^\infty(\Omega \times (0,T))$. Then, up to a subsequence, $\{(\rho_n, \mathbf{u}_n, \theta_n, \mathbf{H}_n)\}_{n=1}^{\infty}$ converges to a global-in-time weak solution $(\rho, \mathbf{u}, \theta, \mathbf{H})$ with initial data $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{H}_0)$. More precisely,*

$$
\begin{cases}
\{\rho_n\}_{n=1}^{\infty} \text{ converges strongly to } \rho \text{ in } C([0,T]; L^p(\Omega')) \text{ for all } 1 \le p < 6m-3; \\
\{\mathbf{u}_n\}_{n=1}^{\infty} \text{ converges weakly to } \mathbf{u} \text{ in } L^{q_1}([0,T]; W^{1,q_3}(\Omega')) \text{ for } q_1 > 5/3, q_3 > 15/8; \\
\{\theta_n\}_{n=1}^{\infty} \text{ converges strongly to } \theta \text{ in } L^p([0,T]; L^q(\Omega')) \text{ for all } p < a \text{ and } q < 3a \text{ with} \\
\qquad \frac{1}{q} = \frac{a-p}{p(a-1)r} + \frac{p-1}{3p(a-1)}, \text{ for all } r < 3/2; \\
\{\mathbf{H}_n\}_{n=1}^{\infty} \text{ converges weakly to } \mathbf{H} \text{ in } L^2([0,T]; H^1(\Omega)) \cap C([0,T]; L^2_{weak}(\Omega)),
\end{cases}
$$

*where $\Omega'$ is any sufficiently smooth and compact subset of $\Omega$.*

## 3. Energy Estimates and the Entropy Inequality

In this section, we dedicate to the well-known *a priori* estimates and a new kind of the entropy equality on weak solutions of the compressible MHD system (1.1a)-(1.1d). To begin with, from the total energy equation (1.1c), the physical energy inequality holds

$$
\begin{aligned}
E(t) &:= \int_\Omega \rho \left( e + \frac{1}{2}|\mathbf{u}|^2 \right)(t,x) + \frac{1}{2}|\mathbf{H}|^2(t,x)\, dx \\
&\le \int_\Omega \rho_0 \left( e_0 + \frac{1}{2}|\mathbf{u}_0|^2 \right) + \frac{1}{2}|\mathbf{H}_0|^2\, dx := E(0).
\end{aligned} \tag{3.1}
$$

As shown in [14, 15], the energy estimate (3.1) alone is not sufficient to build up a reasonable compactness theory of weak solutions to compressible MHD equations in the sense of distributions since we can not obtain any *a priori* estimate on the dissipation about the viscous stress and the gradient of the magnetic field. Comparing with *a priori* estimates for isentropic cases (see [14]), this is a major difference, because in the isentropic case, the viscous dissipation naturally provides a $H^1$ bound in spatial variables on the velocity $\mathbf{u}$. To establish the compactness theory in our *new* framework, the following calculation is crucial:

**Lemma 3.1.**

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left( \rho|\mathbf{u}|^2 + |\mathbf{H}|^2 \right) dx + \int_\Omega 2\mu(\rho) D(\mathbf{u}) : D(\mathbf{u})\, dx \\
&\quad + \int_\Omega \lambda(\rho)|\mathrm{div}\mathbf{u}|^2\, dx + \int_\Omega \nu|\nabla \times \mathbf{H}|^2\, dx = \int_\Omega p(\rho,\theta)\mathrm{div}\mathbf{u}\, dx,
\end{aligned} \tag{3.2}
$$

*and*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left( \rho|\mathbf{u} + 2\nabla\varphi(\rho)|^2 + |\mathbf{H}|^2 \right) dx + \int_\Omega 2\mu(\rho) A(\mathbf{u}) : A(\mathbf{u})\, dx + \int_\Omega \nu|\nabla \times \mathbf{H}|^2\, dx \\
&= \int_\Omega p(\rho,\theta)\mathrm{div}\mathbf{u}\, dx - 2\int_\Omega \nabla p(\rho,\theta) \cdot \nabla\varphi(\rho)\, dx + 2\int_\Omega (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \frac{\nabla\mu(\rho)}{\rho}\, dx,
\end{aligned} \tag{3.3}
$$

*where $A(\mathbf{u}) = (\nabla\mathbf{u} - \nabla\mathbf{u}^\top)/2$ denotes the skew symmetric part of $\nabla\mathbf{u}$, and $\varphi'(x) = \mu'(x)/x$ for all $x > 0$. The notation $A : B$ denotes the dot product between two $n \times n$ matrices $A$ and $B$.*

*Proof.* The energy equality (3.2) is classical, and can be shown by multiplying the momentum equation (1.1b) by $\mathbf{u}$, the mass conservation equation (1.1a) by $|\mathbf{u}|^2/2$, and the magnetic equation (1.1d) by $\mathbf{H}$, then summing them together. Here we used the following identity:

$$
\int_\Omega (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathbf{u}\, dx = -\int_\Omega (\nabla \times (\mathbf{u} \times \mathbf{H})) \cdot \mathbf{H}\, dx.
$$

Now, we turn to show the equality (3.3). The idea is taken from [1], and the argument goes as follows. From the mass conservation equation, we deduce that

$$\partial_t\varphi(\rho)+\mathbf{u}\cdot\nabla\varphi(\rho)+\varphi'(\rho)\rho\mathrm{div}\mathbf{u}=0.$$

This gives, differentiating this equation with respect to the space variable $x_i$, $i=1,2,3$, noting $x=(x_1,x_2,x_3)$,

$$\partial_t\partial_i\varphi(\rho)+(\mathbf{u}\cdot\nabla)\partial_i\varphi(\rho)+(\partial_i\mathbf{u}\cdot\nabla)\varphi(\rho)+\partial_i(\varphi'(\rho)\rho\mathrm{div}\mathbf{u})=0.$$

Let us multiply this equation by $\rho\partial_i\varphi(\rho)$ and sum over $i$, by using the mass equation, then one can deduce

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\rho|\nabla\varphi(\rho)|^2\,dx+\int_\Omega\rho\nabla\varphi(\rho)\otimes\nabla\varphi(\rho):\nabla\mathbf{u}\,dx+\int_\Omega\nabla(\varphi'(\rho)\rho\mathrm{div}\mathbf{u})\cdot\nabla\mu(\rho)\,dx=0. \tag{3.4}$$

Multiplying the momentum equation by $\nabla\mu(\rho)/\rho$, we get

$$\begin{aligned}&\int_\Omega(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx+2\int_\Omega\mu(\rho)D(\mathbf{u}):\left(\frac{\nabla\nabla\mu(\rho)}{\rho}-\frac{\nabla\mu(\rho)\otimes\nabla\rho}{\rho^2}\right)dx\\&\quad+\int_\Omega\nabla p(\rho,\theta)\cdot\frac{\nabla\rho}{\rho}\mu'(\rho)\,dx+2\int_\Omega\nabla((\mu(\rho)-\mu'(\rho)\rho)\mathrm{div}\mathbf{u})\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx\\&=\int_\Omega(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx.\end{aligned} \tag{3.5}$$

Integrating by parts, equation (3.4) can be rewritten under the form

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\rho|\nabla\varphi(\rho)|^2\,dx+\int_\Omega\frac{\mu(\rho)\nabla\mu(\rho)\cdot\nabla\mathbf{u}\cdot\nabla\rho}{\rho^2}\mathrm{d}-\int_\Omega\frac{\mu(\rho)\nabla\mathbf{u}:\nabla\nabla\mu(\rho)}{\rho}\,dx\\&-\int_\Omega\frac{\mu(\rho)\nabla\mathrm{div}\mathbf{u}\cdot\nabla\mu(\rho)}{\rho}\,dx+\int_\Omega\nabla(\varphi'(\rho)\rho\mathrm{div}\mathbf{u})\cdot\nabla\mu(\rho)\,dx=0.\end{aligned} \tag{3.6}$$

Adding equation (3.5) to equation (3.6) multiplied by 2, we get

$$\begin{aligned}&\int_\Omega(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx-2\int_\Omega\frac{\mu(\rho)\nabla\mathrm{div}\mathbf{u}\cdot\nabla\mu(\rho)}{\rho}\,dx\\&\quad+\frac{1}{2}\frac{d}{dt}\int_\Omega2\rho|\nabla\varphi(\rho)|^2\,dx+2\int_\Omega\nabla((\mu(\rho)-\mu'(\rho)\rho)\mathrm{div}\mathbf{u})\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx\\&\quad+2\int_\Omega\nabla(\mu'(\rho)\mathrm{div}\mathbf{u})\cdot\nabla\mu(\rho)\,dx+\int_\Omega\nabla p(\rho,\theta)\cdot\frac{\nabla\rho}{\rho}\mu'(\rho)\,dx\\&=\int_\Omega(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx.\end{aligned} \tag{3.7}$$

By splitting the terms involving div$\mathbf{u}$ and by summing them, we get

$$\begin{aligned}&\int_\Omega(\partial_t\mathbf{u}+u\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega2\rho|\nabla\varphi(\rho)|^2\,dx+\int_\Omega\nabla p(\rho,\theta)\cdot\frac{\nabla\rho}{\rho}\mu'(\rho)\,dx\\&=\int_\Omega(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx.\end{aligned} \tag{3.8}$$

But as for the first term in (3.8), we can calculate

$$\int_\Omega(\partial_t\mathbf{u}+u\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx=\frac{d}{dt}\int_\Omega\mathbf{u}\cdot\nabla\mu(\rho)\,dx-\int_\Omega\mathbf{u}\cdot\nabla\partial_t\mu(\rho)\,dx+\int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx.$$

By using now the mass equation and integrating by parts the last two terms, this gives

$$\int_\Omega (\partial_t \mathbf{u} + u\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx = \frac{d}{dt}\int_\Omega \mathbf{u}\cdot\nabla\mu(\rho)\,dx - \int_\Omega \mu'(\rho)\mathrm{div}(\rho\mathbf{u})\mathrm{div}\mathbf{u}\,dx$$
$$- \int_\Omega \mu(\rho)\mathbf{u}\cdot\nabla\mathrm{div}\mathbf{u}\,dx - \int_\Omega \mu(\rho)\partial_i\mathbf{u}_j\partial_j\mathbf{u}_i\,dx.$$

Integrating by parts in the third term, we get

$$\int_\Omega (\partial_t \mathbf{u} + u\cdot\nabla\mathbf{u})\cdot\nabla\mu(\rho)\,dx = \frac{d}{dt}\int_\Omega \mathbf{u}\cdot\nabla\mu(\rho)\,dx - \int_\Omega (\rho\mu'(\rho)-\mu(\rho))|\mathrm{div}\mathbf{u}|^2\,dx$$
$$- \int_\Omega \mu(\rho)\partial_i\mathbf{u}_j\partial_j\mathbf{u}_i dx.$$

Adding the above identity with (3.8), we get the following equality

$$\frac{d}{dt}\int_\Omega \mathbf{u}\cdot\nabla\mu(\rho)dx - \int_\Omega (\rho\mu'(\rho)-\mu(\rho))|\mathrm{div}\mathbf{u}|^2 dx - \int_\Omega \mu(\rho)\partial_i\mathbf{u}_j\partial_j\mathbf{u}_i dx$$
$$+ \frac{1}{2}\frac{d}{dt}\int_\Omega 2\rho|\nabla\varphi(\rho)|^2 dx + \int_\Omega \nabla p(\rho,\theta)\cdot\frac{\nabla\rho}{\rho}\mu'(\rho)dx$$
$$= \int_\Omega (\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}dx.$$

Adding this last equation multiplied by 2 to the energy estimate (3.2) gives (3.3). □

Next, we introduce the concept of the entropy $s(\rho,\theta)$ which satisfies the entropy equation ([7, 15, 17, 18]):

$$\partial_t(\rho s) + \mathrm{div}(\rho s\mathbf{u}) + \mathrm{div}\left(\frac{\kappa(\theta)}{\theta}\right) = \frac{1}{\theta}(\nu|\nabla\times\mathbf{H}|^2 + \Psi:\nabla\mathbf{u}) - \frac{\kappa(\theta)|\nabla\theta|^2}{\theta^2}, \tag{3.9}$$

with $s(\rho,\theta) = c_v\ln\theta - \ln\rho$, where $c_v$ is a positive constant denoting the specific heat at constant volume. The entropy equation is useful in compressible flows, because it provides naturally the estimates in the gradient of the temperature. More precisely, integrating (3.9) over $\Omega\times(0,t)$, the following proposition is verified:

*Proposition* 3.1. Assume that $\rho_0 s_0 \in L^1(\Omega)$. Then, for all $t\geq 0$, one has:

$$\int_0^t\int_\Omega \frac{1}{\theta}(\nu|\nabla\times\mathbf{H}|^2 + \Psi:\nabla\mathbf{u}) + \frac{\kappa(\rho,\theta)|\nabla\theta|^2}{\theta^2}dx\mathrm{d}t \leq \int_\Omega \rho s + |\rho_0 s_0| dx. \tag{3.10}$$

Observing that

$$\rho s \leq c_v\rho\theta - \rho\ln\rho.$$

the first term on the right-hand side of (3.10) can be estimated by

$$\int_\Omega \rho s dx \leq \int_\Omega c_v\rho\theta dx - \int_\Omega \rho\ln\rho dx.$$

Multiplying (1.1a) by $1+\ln\rho$, we get

$$\partial_t(\rho\ln\rho) + \mathrm{div}(\rho\mathbf{u}\ln\rho) + \rho\mathrm{div}\mathbf{u} = 0.$$

Thus, we have

$$\int_\Omega \rho\ln\rho dx = \int_\Omega \rho_0\ln\rho_0 dx + \int_0^t\int_\Omega \rho\mathrm{div}\mathbf{u}\,dx\mathrm{d}s.$$

Therefore, the right-hand side of (3.10) can be estimated by

$$\begin{aligned}\int_\Omega \rho s\,dx &\le \int_\Omega c_v\rho\theta\,dx + \int_\Omega |\rho_0\ln\rho_0|\,dx + \int_0^t\!\!\int_\Omega \rho|\mathrm{div}\mathbf{u}|\,dx\mathrm{d}s\\ &\le \int_\Omega c_v\rho\theta\,dx + \int_\Omega |\rho_0\ln\rho_0|\,dx + \int_0^t\!\!\int_\Omega \frac{\sqrt{\rho}}{\sqrt{3\lambda+2\mu}}\frac{\sqrt{3\lambda+2\mu}}{\sqrt{\theta}}|\mathrm{div}\mathbf{u}|\sqrt{\rho\theta}\,dx\mathrm{d}t,\end{aligned} \tag{3.11}$$

and, then using the classical Young's inequality, the bound of $\rho\theta$ in $L^\infty([0,T];L^1(\Omega))$, and the assumption (2.2) that ensures that $s\mapsto s/3\lambda(s)+2\mu(s)$ belongs to $L^\infty(R_+)$, we conclude that the terms on the left-hand side of (3.10) is bounded in $L^1(\Omega\times(0,T))$.

Hence, if $\rho_0 s_0$ and $\rho_0\ln\rho_0$ belong to $L^1(\Omega)$, then the components of the following four quantities $\sqrt{3\lambda+2\mu}|\mathrm{div}\mathbf{u}|/\sqrt{\theta}$, $\sqrt{\mu}D(\mathbf{u})/\sqrt{\theta}$, $\sqrt{\nu}\nabla\times\mathbf{H}/\sqrt{\theta}$, $(\sqrt{\rho}+1)\nabla\theta^{\frac{a}{2}}$ and $(\sqrt{\rho}+1)\nabla\ln\theta$ are bounded in $L^2(\Omega\times(0,T))$. We note that the last two bounds involving the temperature gradient provide the following useful estimates:

$$(\sqrt{\rho}+1)\nabla\theta^\alpha\in L^2(\Omega\times(0,T)),\ \text{for all } \alpha \text{ such that } 0\le\alpha\le a/2. \tag{3.12}$$

In order to get enough *a priori* estimates from Lemma 3.1 and the initial condition (2.7), we have to control the following terms:

$$\int_\Omega p\mathrm{div}\mathbf{u}\,dx,\quad \int_\Omega \nabla p\cdot\nabla\varphi(\rho)\,dx,\quad \int_\Omega (\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx.$$

To this end, the following estimates are useful:

**Lemma 3.2.** *Let $\Omega$ be the three-dimensional periodic box or the whole space $\mathbb{R}^3$. For all $\rho$ satisfying $\rho^{-1/2}\nabla\mu(\rho)\in L^2(\Omega)$, one has*

$$\begin{cases}\|\rho^{m-1/2}\chi_{\{\rho>2A\}}\|_{L^6(\Omega)}\le c\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\|_{L^2(\Omega)},\\ \|\rho^{\beta-1/2}\chi_{\{\rho\le A/2\}}\|_{L^6(\Omega)}\le c\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\|_{L^2(\Omega)},\end{cases}$$

*for some positive constant c, where A is from* (2.2)*, and $\chi$ is the characteristic function.*

*Proof.* Let us consider the function $\eta=\alpha\xi^{m-1/2}$, where $\xi:(0,\infty)\to(0,\infty)$ is a smooth increasing function such that $\xi(s)=s$ for $s>2A$ and $\xi(s)=0$ for $s<A$ and $\alpha$ is a positive constant. By hypothesis (2.2), we can choose $c$ such that $\eta'(s)\le c\mu'(s)/\sqrt{s}$ for all $s>0$. Using Sobolev's inequality, we have

$$\|\eta(\rho)\|_{L^6(\Omega)}\le c\|\nabla\eta(\rho)\|_{L^2(\Omega)}\le c\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2(\Omega)}.$$

The left-hand side of above inequality is bigger than $\|\rho^{m-1/2}\chi_{\{\rho>2A\}}\|_{L^6(\Omega)}$. This implies

$$\|\rho^{m-1/2}\chi_{\{\rho>2A\}}\|_{L^6(\Omega)}\le c\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2(\Omega)}.$$

To prove the second part, a similar approach can be applied. Indeed, choosing the function $\eta=\alpha\xi^{\beta-1/2}$ such that $|\eta'(s)|\le c\mu'(s)/\sqrt{s}$ for all $s>0$, where $\xi:(0,\infty)\to(0,\infty)$ is a smooth positive function such that $\xi(s)=0$ for $s>A$ and $\xi(s)=s$ for $s<A/2$. By Sobolev's inequality, we have

$$\|\eta(\rho)\|_{L^6(\Omega)}\le c\|\nabla\eta(\rho)\|_{L^2(\Omega)}\le c\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2(\Omega)}.$$

The left-hand side of above inequality is bigger than $\|\rho^{\beta-1/2}\chi_{\{\rho<A/2\}}\|_{L^6(\Omega)}$. This implies

$$\|\rho^{\beta-1/2}\chi_{\{\rho<A/2\}}\|_{L^6(\Omega)}\le c\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2(\Omega)}.$$

□

From Lemma 3.2, we know $\rho \in L^\infty([0,T]; L^{6m-3}(\Omega'))$ for any bounded subset $\Omega'$ of $\Omega$.

**Lemma 3.3** (**The control of** $\int_\Omega p\,\mathrm{div}\mathbf{u}\,dx$**).**

$$\begin{aligned}\int_\Omega p\,\mathrm{div}\mathbf{u}\,dx \le& -\frac{d}{dt}\int_\Omega \rho P_e(\rho)\,dx + \varepsilon\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}^2\\ &+ c_\varepsilon\left(\|\rho\theta\|_{L^1}^2 + \|\theta\|_{L^6}^2 + \|\theta\|_{L^3}^2\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2}^2\right),\end{aligned}$$

*for all positive* $\varepsilon$.

*Proof.* By the continuity equation in the renormalized sense, we have

$$\begin{aligned}\int_\Omega p\,\mathrm{div}\mathbf{u}\,dx &= \int_\Omega p_e(\rho)\mathrm{div}\mathbf{u}\,dx + \int_\Omega \rho\theta\mathrm{div}\mathbf{u}\,dx\\ &= -\frac{d}{dt}\int_\Omega \rho P_e(\rho)\,dx + \int_\Omega \rho\theta\mathrm{div}\mathbf{u}\,dx.\end{aligned} \tag{3.13}$$

For the second term on the right-hand side of (3.13), we can estimate

$$\begin{aligned}\left|\int_\Omega \rho\theta\mathrm{div}\mathbf{u}\,dx\right| \le& \|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}\\ &\times(\|\rho\theta\chi_{\{\rho<A\}}/\sqrt{3\lambda+2\mu}\|_{L^2} + \|\rho\theta\chi_{\{\rho\ge A\}}/\sqrt{3\lambda+2\mu}\|_{L^2})\\ \le& c\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}\\ &\times(\|\rho^{2/5}\theta\|_{L^2}\|\rho^{(6-5\beta)/10}\chi_{\{\rho<A\}}\|_{L^\infty} + A^{-m/2}\|\rho\chi_{\{\rho\ge A\}}\|_{L^6}\|\theta\|_{L^3})\\ \le& c\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}\\ &\times(\|\rho\theta\|_{L^1}^{2/5}\|\theta\|_{L^6}^{3/5}A^{(6-5\beta)/10} + A^{(3-3m)/2}\|\rho\chi_{\{\rho\ge A\}}\|_{L^{6m-3}}^{m-1/2}\|\theta\|_{L^3}).\end{aligned}$$

Thus, in view of Lemma 3.2 and Young's inequality, we have

$$\begin{aligned}\left|\int_\Omega \rho\theta\mathrm{div}\mathbf{u}\,dx\right| \le& c\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}\\ &\times\left(\|\rho\theta\|_{L^1} + \|\theta\|_{L^6} + \|\theta\|_{L^3}\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2}\right)\\ \le& \varepsilon\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}^2\\ &+ c_\varepsilon\left(\|\rho\theta\|_{L^1}^2 + \|\theta\|_{L^6}^2 + \|\theta\|_{L^3}^2\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2}^2\right).\end{aligned}$$

□

By(3.12) and Sobolev's inequality, $\theta \in L^2([0,T]; L^6(\Omega)) \cap L^2([0,T]; L^3(\Omega))$, since $a \ge 2$. Also, in view of the total energy conservation inequality (3.1) and our assumptions, we know that $\rho P_e(\rho) \in L^\infty([0,T]; L^1(\Omega))$ and $\rho\theta \in L^\infty([0,T]; L^1(\Omega))$. Thus, it is possible, by taking $\varepsilon$ small enough, to get some *a priori* estimates from (3.2) and (3.3) via Gronwall's inequality.

**Lemma 3.4** (**The control of** $\int_\Omega \nabla p\cdot\nabla\varphi(\rho)\,dx$**).**

$$\begin{aligned}-\int_\Omega \nabla p\cdot\nabla\varphi(\rho)\,dx \le& -\int_\Omega \varphi'(\rho)\theta|\nabla\rho|^2\,dx + \int_\Omega\left(c_\varepsilon\kappa(\rho,\theta)\frac{|\nabla\theta|^2}{\theta^2} + \varepsilon\frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx\\ &-\int_\Omega |\nabla\rho^{-\frac{l+1-\beta}{2}}|^2\chi_{\{\rho<A_1\}}\,dx,\end{aligned}$$

*with* $A_1 = \min\{A, A_0\}$.

*Proof.* By our assumption (2.4), we have

$$\nabla p = \theta\nabla\rho + \rho\nabla\theta + p_e'(\rho)\nabla\rho.$$

Hence, we have

$$\begin{aligned}
-\int_\Omega \nabla p\cdot\nabla\varphi(\rho)\,dx &= -\int_\Omega \varphi'(\rho)\theta|\nabla\rho|^2\,dx - \int_\Omega \varphi'(\rho)\rho\nabla\theta\cdot\nabla\rho\,dx\\
&\quad - \int_\Omega p_e'(\rho)\varphi'(\rho)|\nabla\rho|^2\,dx\\
&\le -\int_\Omega \varphi'(\rho)\theta|\nabla\rho|^2\,dx - \int_\Omega \varphi'(\rho)\rho\nabla\theta\cdot\nabla\rho\,dx\\
&\quad - c\int_\Omega |\nabla\rho^{-\frac{l+1-\beta}{2}}|^2\chi_{\{\rho<A_1\}}\,dx.
\end{aligned} \tag{3.14}$$

because $\varphi'(\rho) > 0$ and $p_e'(\rho)\ge 0$.

As for the second term on the right-hand side of (3.14), we have, by our assumption (2.3),

$$\begin{aligned}
\left|\int_\Omega \varphi'(\rho)\rho\nabla\theta\cdot\nabla\rho\,dx\right| &\le \int_\Omega |\varphi'(\rho)\rho\nabla\theta\cdot\nabla\rho|\,dx\\
&\le \int_\Omega \left(c_\varepsilon\kappa(\rho,\theta)\frac{|\nabla\theta|^2}{\theta^2} + \varepsilon\frac{\rho\theta^2}{\kappa(\rho,\theta)}\frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx\\
&\le \int_\Omega \left(c_\varepsilon\kappa(\rho,\theta)\frac{|\nabla\theta|^2}{\theta^2} + \varepsilon\frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
-\int_\Omega \nabla p\cdot\nabla\varphi(\rho)\,dx \le &-\int_\Omega \varphi'(\rho)\theta|\nabla\rho|^2\,dx + \int_\Omega \left(c_\varepsilon\kappa(\rho,\theta)\frac{|\nabla\theta|^2}{\theta^2} + \varepsilon\frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx\\
&- \int_\Omega |\nabla\rho^{-\frac{l+1-\beta}{2}}|^2\chi_{\{\rho<A_1\}}\,dx.
\end{aligned}$$

□

Noting that Proposition 3.1 implies $\kappa(\rho,\theta)\frac{|\nabla\theta|^2}{\theta^2} \in L^1(\Omega\times(0,T))$. Therefore, it is also possible, by incorporating the estimate into (3.2) and (3.3), to get some *a priori* estimates via Gronwall's inequality.

**Lemma 3.5** (**The control of** $\int_\Omega(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx$**).**

$$\left|\int_\Omega(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx\right| \le c\int_\Omega\left(\frac{|\nabla\times\mathbf{H}|^2\nu(\rho,\theta,\mathbf{H})}{\theta} + \frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx.$$

*Proof.* Indeed, we can estimate, by our assumption (2.6) and the uniform bound of $\mathbf{H}_n$ in $L^\infty(\Omega\times(0,T))$,

$$\begin{aligned}
\left|\int_\Omega(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}\,dx\right| &\le \int_\Omega |(\nabla\times\mathbf{H})\times\mathbf{H}\cdot\frac{\nabla\mu(\rho)}{\rho}|\,dx\\
&\le c\int_\Omega\left(\frac{|\nabla\times\mathbf{H}|^2|\mathbf{H}|^2}{\rho} + \frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx\\
&\le c\int_\Omega\left(\frac{|\nabla\times\mathbf{H}|^2\nu(\rho,\theta)}{\theta} + \frac{|\nabla\mu(\rho)|^2}{\rho}\right)dx.
\end{aligned}$$

□

The entropy inequality (3.10) implies that $\frac{|\nabla\times\mathbf{H}|^2\nu(\rho,\theta)}{\theta}$ belongs to $L^1(\Omega\times(0,T))$, provided $\rho_0 s_0$ and $\rho_0\ln\rho_0$ belong to $L^1(\Omega)$. Therefore, from Lemma 3.1-3.5, we can deduce the following *a priori* estimates via Gronwall's inequality:

$$\begin{cases} \|\sqrt{\rho}\mathbf{u}\|_{L^\infty([0,T];L^2(\Omega))}\le c, \quad \|\rho^{-1/2}\nabla\mu(\rho)\|_{L^\infty([0,T];L^2(\Omega))}\le c,\\ \|(\rho^{m/2}+\rho^{\beta/2})\nabla\mathbf{u}\|_{L^2(\Omega\times(0,T))}\le c, \quad \|\frac{\rho^{m/2}+\rho^{\beta/2}}{\theta^{1/2}}\nabla\mathbf{u}\|_{L^2(\Omega\times(0,T))}\le c,\\ \|(\theta)^{1/2}(\rho^{(\beta-1)/2}+\rho^{(m-1)/2})\frac{\nabla\rho}{\rho}\|_{L^2(\Omega\times(0,T))}\le c, \quad \|\rho P_e(\rho)\|_{L^\infty([0,T];L^1(\Omega))}\le c,\\ \|\sqrt{\frac{\rho\theta}{\mu'(\rho)}}\nabla\varphi(\rho)\|_{L^2(\Omega\times(0,T))}\le c, \quad \|\rho\theta\|_{L^\infty([0,T];L^1(\Omega))}\le c,\\ \|\mathbf{H}\|_{L^\infty([0,T];L^2(\Omega))}\le c, \quad \|\sqrt{\nu}\nabla\times\mathbf{H}\|_{L^2(\Omega\times(0,T))}\le c,\\ \|\sqrt{1+\rho}\nabla\theta^\alpha\|_{L^2(\Omega\times(0,T))}\le c, \quad \|\nabla\rho^{-\frac{l+1-\beta}{2}}\chi_{\{\rho<A_1\}}\|_{L^2(\Omega\times(0,T))}\le c, \end{cases} \tag{3.15}$$

for all $\alpha\in[0,a/2]$.

## 4. Some Integrability Lemmas

As mentioned in [15, 20], the lack of *a priori* estimates on approximation solutions to the compressible flow is the main difficulty to prove the existence and the compactness of global-in-time weak solutions. Indeed, the basic and natural *a priori* estimates are not sufficient, since the energy equation does not hold so far even in the distribution theory framework. For more details, we refer the readers to [15].

This difficulty has been circumvented in [10, 15] by restricting the generality of the equations of state (2.3) and (2.4), and defining variational solutions for which the energy equation (1.1c) becomes two inequalities in the sense of distributions. However, this approach requires significant restrictions on the equations of state, in particular the ideal gas case is excluded.

This section is devoted to the local integrability analysis of the various energy fluxes such as $\rho\mathbf{u}|\mathbf{u}|^2$, $\rho\mathbf{u}e$, $\mathbf{u}p$, $\kappa\nabla\theta$. One of the crucial steps is the additional integrability obtained on $\rho$.

4.1. **Integrability of the velocity.** Let us begin with some bounds on the velocity with density dependent weights.

**Lemma 4.1.** *Let $\Omega$ be either the whole space $\mathbb{R}^3$ or the three-dimensional periodic box, and $T>0$. Let $\mathbf{u}$ be a vector field over $\Omega\times(0,T)$ such that $\mathbf{u}\in L^{q_1}([0,T];L^{q_2}_{loc}(\Omega))$, $\sqrt{\rho}\mathbf{u}\in L^\infty([0,T];L^2(\Omega))$, and $\rho\in L^\infty([0,T];L^p_{loc}(\Omega))$ such that*

$$q_1\in(1,2), \quad \textit{and} \quad \frac{1}{p}+\frac{2q_1}{q_2(q_1-1)}<1.$$

*Then, there exists $\delta>3$ such that $\rho^{1/3}\mathbf{u}\in L^\delta(\Omega'\times(0,T))$ for all bounded subsets $\Omega'$ in $\Omega$.*

*Proof.* For the proof we refer the reader to Lemma 6.1 in [1]. □

In order to apply Lemma 4.1 to improve the integrability of $\rho^{1/3}\mathbf{u}$, we need first to deduce the integrability of the velocity $\mathbf{u}$. Indeed, following the computation in [1], one may write $\nabla\mathbf{u}=\rho^{-\beta/2}\rho^{\beta/2}\nabla\mathbf{u}$, and then deduce that

$$\begin{aligned} &\|\nabla\mathbf{u}\|_{L^{q_1}([0,T];L^{q_3}(\Omega'))}\\ &\le C(\Omega')\left(1+\|\rho^{-\beta/2}\chi_{\{\rho<A_1\}}\|_{L^{2j}([0,T];L^{6j}(\Omega'))}\right)\|\rho^{\beta/2}\nabla\mathbf{u}\|_{L^2(\Omega\times(0,T))}\\ &\le C(\Omega')\left(1+\|\nabla\rho^{-(l+1-\beta)/2}\chi_{\{\rho<A_1\}}\|_{L^2([0,T];L^2(\Omega'))}\right)\|\rho^{\beta/2}\nabla\mathbf{u}\|_{L^2(\Omega\times(0,T))}, \end{aligned} \tag{4.1}$$

with

$$j=\frac{l+1-\beta}{\beta}, \quad q_1=2\left(1-\frac{\beta}{l+1}\right), \quad \text{and } \frac{1}{q_3}=\frac{1}{6j}+\frac{1}{2}.$$

In Lemma 4.1, letting $q_2 = \frac{3q_3}{3-q_3} = 3q_1$, taking $p = 6m-3$, we deduce that $\rho^{1/3}\mathbf{u} \in L^\delta((0,T)\times\Omega')$ for some $\delta > 3$.

As a byproduct of previous analysis, we also can derive useful bounds on energy flux. More precisely, one has for all bounded subset $\Omega'$ of $\Omega$,

$$\|\rho^{-l}\mathbf{u}\|_{L^s([0,T];L^r(\Omega'))} \le C\|\rho^{-l}\|^{2/3}_{L^\infty([0,T];L^1(\Omega'))}\|\rho^{-l}\|^{1/3}_{L^{2j_1}([0,T];L^{6j_1}(\Omega'))}\|\mathbf{u}\|_{L^{q_1}([0,T];L^{q_2}(\Omega'))},$$

with

$$j_1 = \frac{l+1-\beta}{2l}, \quad \frac{1}{s} = \frac{5l+3}{6(l+1-\beta)}, \quad \frac{1}{r} = \frac{17l+15-12\beta}{18(l+1-\beta)}.$$

Noting that the hypothesis (2.5) implies that $s > 1$ and $r > 1$.

In order to bound the energy fluxes, it remains to control $\rho^k\mathbf{u}$ in $L^\delta((0,T)\times\Omega')$ for some $\delta > 1$. Since $\mathbf{u}$ is bounded in $L^{q_1}([0,T];L^{q_2}_{loc}(\Omega))$ and $\rho$ is bounded in $L^\infty([0,T];L^{(6m-3)/k}_{loc}(\Omega))$, the hypothesis (2.5) implies that $\rho^k\mathbf{u}$ is bounded in $L^\delta((0,T)\times\Omega')$ for some $\delta > 1$.

4.2. **Integrability of the heat flux.** In this subsection, we will need the following integrability on the temperature:

**Lemma 4.2.** *Let $\Omega$ be either the whole space $\mathbb{R}^3$ or a three-dimensional periodic box and let $T > 0$. Let $\theta$ be a function over $\Omega\times(0,T)$ such that $(\sqrt{\rho}+1)\nabla\theta^{a/2}$ and $(\sqrt{\rho}+1)\nabla\ln\theta$ belong to $L^2(\Omega\times(0,T))$ with $a \ge 2$, $\rho e \in L^\infty([0,T];L^1(\Omega))$, and $\rho^{-1/2}\nabla\mu(\rho) \in L^2(\Omega)$. Then, $\theta^{\frac{a-c+1}{2}}$ belongs to $L^2([0,T];L^6(\Omega))$ for all $0 < c \le 1$.*

*Proof.* For the proof, we refer the reader to Lemma 7.3 in [1]. □

At this stage, in order to improve the integrability of the heat flux, we need to use the following thermal energy equation (cf. equation (1.15) in [15]).

$$\partial_t(\rho\theta) + \mathrm{div}(\rho\theta\mathbf{u}) - \mathrm{div}(\kappa(\rho,\theta)\nabla\theta) = \nu|\nabla\times\mathbf{H}|^2 + \Psi : \nabla\mathbf{u} - \theta\rho\mathrm{div}\mathbf{u}. \tag{4.2}$$

As a matter of fact, we have

**Lemma 4.3.** *For any nondecreasing concave function from $R^+$ to $R$, one has*

$$\int_\Omega f'(\theta)(2\mu D(\mathbf{u}):D(\mathbf{u}) + \lambda|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2)\,dx - \int_\Omega \kappa(\rho,\theta)f''(\theta)|\nabla\theta|^2\,dx$$
$$\le \frac{d}{dt}\int_\Omega \rho f(\theta)\,dx + \int_\Omega \rho\theta f'(\theta)|\mathrm{div}\mathbf{u}|\,dx.$$

*Proof.* Multiplying (4.2) by $f'(\theta)$, one has

$$\begin{aligned}
&\int_\Omega f'(\theta)(2\mu D(\mathbf{u}):D(\mathbf{u}) + \lambda|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2)\,dx - \int_\Omega \kappa(\rho,\theta)f''(\theta)|\nabla\theta|^2\,dx\\
&\quad= \int_\Omega f'(\theta)(\theta\rho\mathrm{div}\mathbf{u} + \partial_t(\rho\theta) + \mathrm{div}(\rho\theta\mathbf{u}))\,dx\\
&\quad= \int_\Omega f'(\theta)(\theta\rho\mathrm{div}\mathbf{u} + \rho\partial_t\theta + \rho\mathbf{u}\nabla\theta)\,dx\\
&\quad= \int_\Omega f'(\theta)\theta\rho\mathrm{div}\mathbf{u} + \rho\partial_t f(\theta) + \rho\mathbf{u}\nabla f(\theta)\,dx\\
&\quad= \int_\Omega f'(\theta)\theta\rho\mathrm{div}\mathbf{u} + \partial_t(\rho f(\theta)) + \mathrm{div}(\rho\mathbf{u}f(\theta))\,dx\\
&\quad\le \int_\Omega f'(\theta)\theta\rho|\mathrm{div}\mathbf{u}|\,dx + \frac{d}{dt}\int_\Omega \rho f(\theta)\,dx,
\end{aligned}$$

here, we used twice the mass conservation equation (1.1a). □

Now, we consider $f'(\theta) = \frac{1}{\theta^c}$ for some $0 < c < 1$ in Lemma 4.3, then we have

$$\int_\Omega \frac{1}{\theta^c}(2\mu D(\mathbf{u}) : D(\mathbf{u}) + \lambda|\mathrm{div}\mathbf{u}|^2 + \nu|\nabla\times\mathbf{H}|^2)\,dx + c\int_\Omega \kappa(\rho,\theta)\frac{1}{\theta^{c+1}}|\nabla\theta|^2\,dx$$
$$\le \frac{d}{dt}\int_\Omega \rho\frac{\theta^{1-c}}{(1-c)}\,dx + \int_\Omega \rho\theta^{1-c}|\mathrm{div}\mathbf{u}|\,dx.$$

Keeping the hypothesis (2.3) in mind, we have

$$\int_\Omega (1+\rho)|\nabla(1+\theta)^{(a-c+1)/2}|^2\,dx \le \frac{d}{dt}\int_\Omega \rho\frac{\theta^{1-c}}{(1-c)}\,dx + \int_\Omega \rho\theta^{1-c}|\mathrm{div}\mathbf{u}|\,dx. \tag{4.3}$$

For the second term on the right-hand side of (4.3), one has

$$\begin{aligned}
&\left|\int_\Omega \rho\theta^{1-c}\mathrm{div}\mathbf{u}\,dx\right| \\
&\le \|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2} \\
&\quad\times(\|\rho\theta^{1-c}\chi_{\{\rho<A\}}/\sqrt{3\lambda+2\mu}\|_{L^2} + \|\rho\theta^{1-c}\chi_{\{\rho\ge A\}}/\sqrt{3\lambda+2\mu}\|_{L^2}) \\
&\le c\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2} \\
&\quad\times(\|\rho^{2/5}\theta^{1-c}\|_{L^2}\|\rho^{(6-5\beta)/10}\chi_{\{\rho<A\}}\|_{L^\infty} + A^{-m/2}\|\rho\chi_{\{\rho\ge A\}}\|_{L^6}\|\theta^{1-c}\|_{L^3}) \\
&\le c\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2} \\
&\quad\times(\|\rho\theta^{1-c}\|_{L^1}^{2/5}\|\theta^{1-c}\|_{L^6}^{3/5}A^{(6-5\beta)/10} + A^{(3-3m)/2}\|\rho\chi_{\{\rho\ge A\}}\|_{L^{6m-3}}^{m-1/2}\|\theta^{1-c}\|_{L^3}).
\end{aligned}$$

Thus, in view of Lemma 3.2 and Young's inequality, we have

$$\begin{aligned}
&\left|\int_\Omega \rho\theta^{1-c}\mathrm{div}\mathbf{u}\,dx\right| \\
&\le c\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}\left(\|\rho\theta^{1-c}\|_{L^1} + \|\theta^{1-c}\|_{L^6} + \|\theta^{1-c}\|_{L^3}\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2}\right) \\
&\le \varepsilon\|\sqrt{3\lambda+2\mu}\mathrm{div}\mathbf{u}\|_{L^2}^2 + c_\varepsilon\left(\|\rho\theta^{1-c}\|_{L^1}^2 + \|\theta^{1-c}\|_{L^6}^2 + \|\theta^{1-c}\|_{L^3}^2\left\|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right\|_{L^2(\Omega)}^2\right).
\end{aligned}$$

Because $\rho\theta$ and $\rho$ belong to $L^\infty([0,T];L^1(\Omega))$, we deduce that $\rho\theta^{1-c} = \rho^c(\rho\theta)^{1-c}$ belongs to $L^\infty([0,T];L^1(\Omega))$. Hence, integrating over t in the both sides of (4.3), and combining the estimates in Proposition 3.1, one has, if $0 < c \le 1$,

$$\int_0^T\int_\Omega (1+\rho)|\nabla(1+\theta)^{(a-c+1)/2}|^2\,dxdt \le C, \tag{4.4}$$

for some positive constant $C$. In particular, by Sobolev's inequality, $\theta^{a-c+1}$ belongs to $L^2([0,T];L^3(\Omega))$.

**Lemma 4.4.** *Let $\Omega$ be either the whole space $\mathbb{R}^3$ or the three-dimensional periodic box and let $T > 0$. Assume that $\theta$ satisfies the conditions in Lemma 4.2 and $\sqrt{1+\rho}\nabla\theta^{(a-c+1)/2} \in L^2([0,T];L^2(\Omega))$, $\theta^{a-c+1} \in L^2([0,T];L^3(\Omega))$, $\rho \in L^\infty([0,T];L^{6m-3}(\Omega))$. Then, one has, for some $p > 1$,*

$$\kappa(\rho,\theta)\nabla\theta \in L^p([0,T];L^p(\Omega)). \tag{4.5}$$

*Proof.* For the proof, we refer the reader to Lemma 6.2 in [1]. □

## 5. Compactness of Weak Solutions

With the *a priori* estimates and integrability lemmas obtained in the previous sections, we now study the compactness of sequences of weak solutions $\{(\rho_n, \mathbf{u}_n, \theta_n, \mathbf{H}_n)\}_{n=1}^{\infty}$ and pass to the limit in nonlinear terms.

To begin with, we will state the Aubin-Lions *compactness lemma* (see[19], Ch. IV, and [23] for more recent references) which we will use later. A simple statement goes as follows:

**Lemma 5.1** (**Aubin-Lions Lemma**). *Let $T > 0$, $p \in (1, \infty)$ and let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence of functions in $L^p([0,T]; H)$ where $H$ is a Banach space. If $\{f_n\}_{n=1}^{\infty}$ is also bounded in $L^p([0,T]; V)$, where $V$ is compactly imbedded in $H$ and $\{\partial f_n/\partial t\}_{n=1}^{\infty}$ is bounded in $L^p([0,T]; Y)$ uniformly where $H \subset Y$, then $\{f_n\}_{n=1}^{\infty}$ is relatively compact in $L^p([0,T]; H)$.*

5.1. **Compactness of the density.** From the uniform estimates derived in Lemma 3.2, we deduce that the sequence $\rho_n$ is uniformly bounded in $L^\infty([0,T]; L^{6m-3}(\Omega'))$ for all bounded subset $\Omega'$ of $\Omega$. Up to a subsequence, one may assume that $\rho_n$ converges weakly to some $\rho$ in $L^2([0,T]; L^2_{loc}(\Omega))$. In fact, we have

**Lemma 5.2.**

$$\partial_t(\rho_n^m) \text{ is bounded in } L^2([0,T]; L^{3/2}(\Omega')),$$
$$\nabla \rho_n^m \text{ is bounded in } L^\infty([0,T]; L^{3/2}(\Omega'));$$

*as a consequence, up to a subsequence, $\rho_n$ converges almost everywhere and strongly to some element $\rho$ in $C([0,T]; L^p(\Omega'))$ for all $1 \le p < 6m-3$. Moreover, the continuity equation* (1.1a) *hold in the sense of distributions.*

*Proof.* Let us consider the renormalized mass equation satisfied by $h(\rho) = \rho^m$,

$$\partial_t(h(\rho_n)) + \mathrm{div}(h(\rho_n)\mathbf{u}_n) + (m-1)\rho_n^m \mathrm{div}\mathbf{u}_n = 0.$$

The uniform bounds of $\sqrt{\rho_n}\mathbf{u}_n$ in $L^\infty([0,T]; L^2(\Omega))$ and of $\rho_n^{-1/2} h(\rho_n)$ in $L^\infty([0,T]; L^6(\Omega'))$ imply that $h(\rho_n)\mathbf{u}_n$ is bounded in $L^\infty([0,T]; L^{3/2}(\Omega'))$. On the other hand, $\rho^{m/2}\mathrm{div}\mathbf{u}_n$ is bounded in $L^2(\Omega \times (0,T))$, and the sequence $\rho^{m/2}$ is bounded in $L^\infty([0,T]; L^{6(2m-1)/m}(\Omega'))$, hence, $\rho_n^m \mathrm{div}\mathbf{u}_n$ is bounded in $L^2([0,T]; L^{3/2}(\Omega'))$. Thus, $\partial_t(h(\rho_n))$ is uniformly bounded in $L^2([0,T]; L^{3/2}(\Omega'))$.

The spatial regularity of $\rho_n^m$ can be estimated as follows. Since $\nabla(\rho_n^m) = \frac{\nabla \rho_n^m}{\sqrt{\rho_n}}\sqrt{\rho_n}$, thus, $\nabla(\rho_n^m)$ is bounded in $L^\infty([0,T]; L^{3/2}(\Omega'))$. The estimates deduced above, and thanks to Aubin-Lions Lemma, give the strong convergence of $\rho_n^m$ in $L^2([0,T]; L^2(\Omega'))$. We denote the limit of $\rho_n^m$ by $\overline{\rho^m}$. Then, since the function $h(s)$ is strictly increasing, we conclude that $\rho_n$ converges strongly to $\rho := (\overline{\rho^m})^{1/m}$ in $L^2([0,T]; L^2(\Omega'))$, and hence, by interpolation, in $L^q([0,T]; L^p(\Omega'))$ for all $1 \le p < 6m-3$ with $\frac{1}{p} = \frac{1}{q} + \frac{q-1}{(6m-3)q}$. And using the continuity equation (1.1a) again, we know actually $\rho_n$ converges strongly to $\rho := (\overline{\rho^m})^{1/m}$ in $C([0,T]; L^p(\Omega'))$ for all $1 \le p < 6m-3$.

Finally, we already know that $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is uniformly bounded in $L^{q_1}([0,T]; W^{1,q_3}(\Omega'))$. Thus, up to a subsequence, $\mathbf{u}_n$ converges weakly to some element $\mathbf{u}$ in $L^{q_1}([0,T]; W^{1,q_3}(\Omega'))$. By Sobolev's compact imbedding theorem, we also have that $\mathbf{u}_n$ is uniformly bounded in $L^{5/3}([0,T]; L^5(\Omega'))$ since $q_1 > 5/3$ and $q_3 > 15/8$. As a consequence, we pass to the limit in the mass conservation equation:

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \quad \text{in } \mathcal{D}'(\Omega \times (0,T)). \tag{5.1}$$

□

In the spirit of Lemma 5.2, we can pass to the limit in the sense of distributions for the terms $\rho_n P_e(\rho_n)$, $p_e(\rho_n)$, $\rho_n P_e(\rho_n)\mathbf{u}_n$, $p_e(\rho_n)\mathbf{u}_n$ since $\mathbf{u}_n$ converges weakly to some element $\mathbf{u}$ in $L^{q_1}([0,T];W^{1,q_3}(\Omega'))$.

5.2. **Compactness of the momentum.** In this subsection, we show the compactness of the momentum $\rho_n\mathbf{u}_n$ by Aubin-Lions Lemma. We already know from the previous subsection that $\rho_n\mathbf{u}_n$ converges weakly to $\rho\mathbf{u}$ in $L^2([0,T];L^{3/2}(\Omega'))$, due to the facts $\sqrt{\rho_n}\mathbf{u}_n \in L^\infty([0,T];L^2(\Omega))$ and $\rho_n \in L^\infty([0,T];L^3(\Omega'))$. To this end, we need to establish the uniform bound of $\partial_t(\rho_n\mathbf{u}_n)$ in some suitable functional space. Indeed, we can show that the sequence $\partial_t(\rho_n\mathbf{u}_n)$ is uniformly bounded in $L^p([0,T];H^{-s}(\Omega))$ for some $p>1$ and $s$ large enough.

From the momentum conservation equation (1.1b), we have

$$\partial_t(\rho_n\mathbf{u}_n) = -\mathrm{div}\,(\rho_n\mathbf{u}_n\otimes\mathbf{u}_n) - \nabla p_n + (\nabla\times\mathbf{H}_n)\times\mathbf{H}_n + \mathrm{div}\Psi_n.$$

For the first term on the right-hand side, $\rho_n\mathbf{u}_n\otimes\mathbf{u}_n$ is bounded in $L^{3/2}([0,T];L^{9/7}(\Omega'))$ uniformly as a product of $\rho_n^{1/3}\mathbf{u}_n$, bounded in $L^3([0,T];L^3(\Omega'))$ and $\rho_n$, bounded in $L^\infty([0,T];L^{6m-3}(\Omega'))$. For the second term, $p_n = p(\rho_n,\theta_n)$ is bounded uniformly in $L^\infty([0,T];L^1(\Omega))$, since $\rho_n\theta_n$ is uniformly bounded in $L^\infty([0,T];L^1(\Omega))$ and $p_e(\rho_n)$ is bounded in $L^\infty([0,T];L^1(\Omega))$. For the third term, by (3.15), $(\nabla\times\mathbf{H}_n)\times\mathbf{H}_n$ is bounded in $L^2([0,T];L^{3/2}(\Omega))$. As for the fourth term $\mathrm{div}\Psi_n$, from the fact $\sqrt{\mu(\rho_n)}D(\mathbf{u}_n)$ and $\sqrt{|\lambda(\rho_n)|}\mathrm{div}\mathbf{u}_n$ are bounded in $L^2(\Omega\times(0,T))$, and that $\sqrt{\mu(\rho_n)}$ and $\sqrt{|\lambda(\rho_n)|}$ are uniformly bounded in $L^\infty([0,T];L^6(\Omega'))$, we deduce that $\mathrm{div}\Psi_n$ is bounded in $L^2([0,T];L^{3/2}(\Omega'))$. Therefore, the sequence $\partial_t(\rho_n\mathbf{u}_n)$ is uniformly bounded in $L^{3/2}([0,T];W^{-1,1}(\Omega'))$.

Next, we have

$$\begin{aligned}\partial_i(\rho_n u_{nj}) &= \rho_n\partial_i u_{nj} + u_{nj}\partial_i\rho_n \\ &= \frac{\rho_n}{\rho_n^{m/2}+\rho_n^{\beta/2}}(\rho_n^{m/2}+\rho_n^{\beta/2})\partial_i u_{nj} + \frac{1}{\mu'(\rho_n)}\sqrt{\rho_n}u_{nj}\rho_n^{-1/2}\partial_i\mu(\rho_n).\end{aligned} \tag{5.2}$$

Using the hypothesis (2.2) and the estimate $\rho_n\in L^\infty([0,T];L^{6m-3}(\Omega'))$, one can deduce that $\frac{\rho_n}{\rho_n^{m/2}+\rho_n^{\beta/2}}\in L^\infty([0,T];L^3(\Omega'))$ and $\frac{1}{\mu'(\rho_n)}\in L^\infty(\Omega\times(0,T))$. Hence, from the estimates in (3.15), we deduce that $\rho_n\mathbf{u}_n\in L^2([0,T];W^{1,1}(\Omega'))$. Thus, by Aubin-Lions Lemma, we deduce that $\rho_n\mathbf{u}_n$ converges strongly to $\rho\mathbf{u}$ in $L^{3/2}([0,T];L^p(\Omega'))$ for all $1\le p<3/2$.

As a conclusion, the product $\rho_n|\mathbf{u}_n|^2$, converges strongly to $\rho|\mathbf{u}|^2$ in $L^1(\Omega\times(0,T))$, since $\rho_n\mathbf{u}_n$ converges weakly to $\rho\mathbf{u}$ in $L^\infty([0,T];L^{3/2}(\Omega'))$, strongly to $\rho\mathbf{u}$ in $L^{3/2}([0,T];L^p(\Omega'))$ for all $1\le p<3/2$ and $\mathbf{u}_n$ is bounded uniformly in $L^{5/3}([0,T];L^5(\Omega'))$. Using the fact $\rho_n^{1/3}\mathbf{u}_n = \rho_n^{1/3}\mathbf{u}_n\chi_{\{\rho_n\le\varepsilon\}} + \rho_n^{1/2}\mathbf{u}_n\rho_n^{-1/6}\chi_{\{\rho_n>\varepsilon\}}$, $\rho_n^{1/3}\mathbf{u}_n$ is the sum of a uniformly small term in $L^1(\Omega'\times(0,T))$ and another term converging to $\rho^{1/3}\mathbf{u}\chi_{\{\rho>\varepsilon\}}$ and then we deduce that $\rho_n^{1/3}\mathbf{u}_n$ converges strongly in $L^1([0,T];L^1(\Omega'))$ to $\rho^{1/3}\mathbf{u}$. Finally, by the interpolation and the uniform bound of $\rho_n^{1/3}\mathbf{u}_n$ in $L^\delta(\Omega'\times(0,T))$ for some $\delta>3$, we conclude that $\rho_n^{1/3}\mathbf{u}_n$ converges strongly in $L^3(\Omega'\times(0,T))$ to $\rho^{1/3}\mathbf{u}$.

5.3. **Compactness of the temperature.** In this subsection, we want to derive compactness results for the energy $\mathcal{E}_n$ and the temperature $\theta_n$. The first step is to derive uniform bounds in $L^p([0,T];H^{-s}(\Omega))$ for some $p>1$ and $s$ large enough for the sequence $\partial_t(\mathcal{E}_n)$. Indeed, we can rewrite the energy conservation equation (1.1c):

$$\partial_t(\mathcal{E}_n) = -\mathrm{div}(\mathbf{u}_n(\mathcal{E}'_n+p_n)) + \mathrm{div}((\mathbf{u}_n\times\mathbf{H}_n)\times\mathbf{H}_n + \nu_n\mathbf{H}_n\times(\nabla\times\mathbf{H}_n) + \mathbf{u}_n\Psi_n + \kappa_n\nabla\theta_n).$$

For the first term of the right-hand side, we already know that $\rho_n\mathbf{u}_n|\mathbf{u}_n|^2$ is uniformly bounded in $L^q([0,T];L^q(\Omega'))$ for some $q>1$. Also, we already get the uniform bounds in

$L^q([0,T];L^q(\Omega'))$ with some $q>1$ for the terms $\rho_n^{-l}\mathbf{u}_n$, $\rho_n^k\mathbf{u}_n$ and $\kappa(\rho_n,\theta_n)\nabla\theta_n$ in Section 4. Next, the uniform bound of $\rho_n\mathbf{u}_n$ in $L^\infty([0,T];L^{3/2}(\Omega'))$ and the uniform bound of $\theta_n$ in $L^2([0,T];L^6(\Omega))$ implies that $\rho_n\mathbf{u}_n\theta_n$ is bounded in $L^q([0,T];L^q(\Omega'))$ for some $q>1$. Hence, $\rho_n\mathbf{u}_ne_n$ and $p_n\mathbf{u}_n$ are bounded in $L^q([0,T];L^q(\Omega'))$ for some $q>1$.

For the viscous flux $\mathbf{u}_n\Psi_n$, we note the facts that

$$\sqrt{\mu(\rho_n)}D(\mathbf{u}_n) \text{ and } \sqrt{|\lambda(\rho_n)|}\mathrm{div}\mathbf{u}_n \text{ are bounded in } L^2(\Omega\times(0,T)),$$

and

$$\sqrt{\mu(\rho_n)}\rho_n^{-1/3} \text{ and } \sqrt{|\lambda(\rho_n)|}\rho_n^{-1/3} \text{ are bounded in } L^\infty([0,T];L^{18(2m-1)/(3m-2)}(\Omega'))$$

due to the hypothesis (2.2) and Lemma 3.2, hence in $L^\infty([0,T];L^9(\Omega'))$, and $\rho_n^{1/3}\mathbf{u}_n$ is bounded in $L^3(\Omega\times(0,T))$. Thus, the viscous fluxes are bounded in $L^{6/5}([0,T];L^{18/17}(\Omega'))$.

As for the terms related to the magnetic field, we have the fact $\mathbf{H}_n$ is bounded in $L^\infty([0,T];L^2(\Omega))\cap L^2([0,T],H^1(\Omega))$, hence, by interpolation, in $L^p([0,T];L^q(\Omega))$ where $p>5$, $q<30/11$, and

$$\frac{1}{q}=\frac{1}{2}-\frac{2}{3p}.$$

Thus, $(\mathbf{u}_n\times\mathbf{H}_n)\times\mathbf{H}_n$ belongs to $L^q([0,T];L^q(\Omega'))$ for some $q>1$, since $\mathbf{u}_n$ is uniformly bounded in $L^{5/3}([0,T];L^5(\Omega'))$. By the bound of the magnetic field coefficient $\nu(\rho,\theta)$, we know $\nu_n\mathbf{H}_n\times(\nabla\times\mathbf{H}_n)$ belongs to $L^{2p/(2+p)}([0,T];L^{2q/(2+q)}(\Omega'))$ for $p>5$, $2<q<30/11$, hence, in $L^q([0,T];L^q(\Omega'))$ for some $q>1$.

With the bound of $\partial_t(\mathcal{E}_n)$ in mind, we can show the strong convergence of the term $\rho_n\theta_n^2$. To this end, we will follow the argument in [1]. First, we note that the strong convergence of $\sqrt{\rho_n}\mathbf{u}_n$ to $\sqrt{\rho}\mathbf{u}$ in $L^r([0,T];L^2(\Omega'))$ and the strong convergence of $\rho_nP_e(\rho_n)$ to $\rho P_e(\rho)$ in $L^r([0,T];L^1(\Omega'))$ for all $r\in(1,\infty)$. Let us introduce

$$\mathcal{K}=\{f\in L^{|}_{loc}(\Omega)|\|\nabla f\|_{L^2(\Omega)}=1\}$$

and a sequence $\mathcal{T}_k$ of regularizing kernels given for instance by convolution operators such that the following basic properties hold:

$$\sup_{f\in\mathcal{K}}\|f-\mathcal{T}_kf\|\le\frac{C}{k},$$

and for all compact subset $\Omega'\subset\Omega$, there exists $C_{k,\Omega'}$ such that for all $f\in\mathcal{K}$,

$$\|\mathcal{T}_kf\|_{L^\infty(\Omega')}\le C_{k,\Omega'} \quad \text{and } \mathcal{T}_kf\in H^s(\Omega) \quad \text{for all } s>0.$$

Then, we can deduce that for any compact subset $\Omega'\subset\Omega$, one has

$$\begin{aligned}
\left|\int_{\Omega'\times(0,T)}(\rho_n\theta_n^2-\rho\theta^2)\,dx\,dt\right| \le{}& \frac{C}{k}(\|\rho_n\theta_n\|_{L^2(\Omega'\times(0,T))}+\|\rho\theta\|_{L^2(\Omega'\times(0,T))})\\
&+\left|\int_{\Omega'\times(0,T)}\rho\theta(\theta_n-\theta)\,dx\,dt\right|\\
&+\frac{1}{2}\|(|\mathbf{H}_n|^2-|\mathbf{H}|^2)\mathcal{T}_k\theta_n\|_{L^1(\Omega'\times(0,T))}\\
&+\frac{1}{2}\|(\rho_n|\mathbf{u}_n|^2-\rho|\mathbf{u}|^2)\mathcal{T}_k\theta_n\|_{L^1(\Omega'\times(0,T))}\\
&+\|(\rho_nP_e(\rho_n)-\rho P_e(\rho))\mathcal{T}_k\theta_n\|_{L^1(\Omega'\times(0,T))}\\
&+\left|\int_{\Omega'\times(0,T)}(\mathcal{E}_n-\mathcal{E})\mathcal{T}_k\theta_n\,dx\,t\right|.
\end{aligned}\tag{5.3}$$

Let us observe that the first term of the above right-hand side is bounded by

$$\frac{C}{k}(\|\rho_n\|_{L^\infty([0,T];L^3(\Omega'))}\|\theta_n\|_{L^2([0,T];L^6(\Omega'))} + \|\rho\|_{L^\infty([0,T];L^3(\Omega'))}\|\theta\|_{L^2([0,T];L^6(\Omega'))}).$$

Therefore, given $\varepsilon > 0$, there exists an integer $k_0$ such that the preceding term is less than $\varepsilon/4$ uniformly in $n$. Dealing with the second term is an easy task since $\theta_n$ converges weakly to $\theta$ in $L^2([0,T];L^6(\Omega'))$, so that for $n$ large enough, the second term is estimated by $\varepsilon/4$. But $\|\mathcal{T}_{k_0}\theta_n\|_{L^\infty(\Omega')}$ is uniformly bounded in $n$, whereas $\mathbf{H}_n^2$, $\rho_n|\mathbf{u}_n|^2$ and $\rho_n P_e(\rho_n)$ converges strongly respectively to $\mathbf{H}^2$, $\rho|\mathbf{u}|^2$ and $\rho P_e(\rho)$ in $L^1(\Omega' \times (0,T))$ respectively, so that the sum of the third and the fourth term is estimated by $\varepsilon/4$ for large enough $n$. For the last term with $k = k_0$, the uniform bound of $\partial_t(\mathcal{E}_n)$ in $L^p([0,T];H^{-s}(\Omega))$ where $p > 1$ implies that up to a subsequence, $\mathcal{E}_n$ converges strongly to $\mathcal{E}$ in $C([0,T];H^{-s}(\Omega))$, so that, for large enough $n$, the right-hand side of above inequality is less than $\varepsilon$. It follows that $\sqrt{\rho_n}\theta_n$ converges strongly in $L^2(\Omega \times (0,T))$.

On the other hand, we know the strong convergence of $\rho_n^{-1/2}$ to $\rho^{-1/2}$ in $L^2([0,T];L^p(\Omega'))$ for $p < 6$ due to the Lebesgue's dominated convergence theorem, the estimate (3.15) and the strong convergence of the density. And then, we deduce that $\theta_n$ converges to $\theta$ in $L^1([0,T];L^r_{loc}(\Omega'))$ for all $r < 3/2$. Recalling the uniform bound of $\theta_n^{a/2}$ in $L^2([0,T];L^6(\Omega'))$, we deduce that $\theta_n$ converges strongly to $\theta$ in $L^p([0,T];L^q(\Omega'))$ for all $p < a$, and $q < 3a$ with

$$\frac{1}{q} = \frac{a-p}{p(a-1)r} + \frac{p-1}{3p(a-1)},$$

for all $r < 3/2$.

5.4. **Compactness of the magnetic field.** The aim of this subsection is to show the compactness of the magnetic field. From Lemma 3.1, and the hypothesis (2.6), we deduce that $\mathbf{H}_n \in L^2([0,T];H^1(\Omega)) \cap L^\infty([0,T];L^2(\Omega))$. Thus, we can assume that $\mathbf{H}_n$ converges weakly to some element $\mathbf{H}$ with $\mathrm{div}\mathbf{H} = 0$ in $L^2([0,T];H^1(\Omega)) \cap L^\infty([0,T];L^2(\Omega))$.

On the other hand, from the equation (1.1d), we know that

$$\partial_t \mathbf{H}_n = \nabla \times (\mathbf{u}_n \times \mathbf{H}_n) - \nabla \times (\nu(\rho_n,\theta_n)\nabla \times \mathbf{H}_n). \tag{5.4}$$

For the first term on the right-hand side of (5.4), we deduce that

$$\mathbf{u}_n \times \mathbf{H}_n \in L^{5/3}([0,T];L^{10/7}(\Omega'))$$

because of $\mathbf{u}_n \in L^{5/3}([0,T];L^5(\Omega'))$ and $\mathbf{H}_n \in L^\infty([0,T];L^2(\Omega))$. For the second term on the right-hand side of (5.4), we have $\nu(\rho_n,\theta_n)\nabla \times \mathbf{H}_n \in L^2(\Omega \times (0,T))$ due to the hypothesis (2.6). Hence, $\partial_t\mathbf{H}_n$ is bounded in $L^{5/3}([0,T];W^{-1,10/7}(\Omega'))$. By Aubin-Lions Lemma, we deduce that $\mathbf{H}_n$ converges strongly to $\mathbf{H}$ in $L^{5/3}([0,T];L^p(\Omega'))$ for any $5 < p < 6$. Furthermore, due to the uniform bound of $\mathbf{H}_n$ in $L^\infty([0,T];L^2(\Omega))$, by using the interpolation, one obtain that $\mathbf{H}_n$ converges strongly to $\mathbf{H}$ in $L^p([0,T];L^q(\Omega'))$ for some $p > 5$ and some $q > 5/2$.

5.5. **Proof of Theorem 2.1.** To finish the proof of Theorem 2.1, we need to check that the limit functions $\rho$, $\mathbf{u}$, $\theta$, $\mathbf{H}$ are indeed the weak solutions, as defined in the introduction. We will complete this proof by several steps.

*Step 1: Convergence of the mass conservation equation.* Let us start with the mass conservation equation (1.1a), since $\rho_n$ converges strongly to $\rho$ in $C([0,T];L^p(\Omega'))$ for all $1 \le p < 6m-3$, and $\mathbf{u}_n$ converges weakly to $\mathbf{u}$ in $L^{q_1}([0,T];W^{1,q_3}(\Omega))$, we deduce that, by the Sobolev's compact imbedding theorem, $\rho_n\mathbf{u}_n$ converges strongly to $\rho\mathbf{u}$ in $L^r([0,T];L^1(\Omega'))$ for some $r > 1$. In particular, the mass conservation equation (1.1a) is satisfied in the sense of distributions.

*Step 2: Convergence of the momentum conservation equation.* For the momentum conservation equation (1.1b), the strong convergence of $\rho_n\mathbf{u}_n$ and $\rho_n\mathbf{u}_n \otimes \mathbf{u}_n$ in

$L^1([0,T];L^1(\Omega'))$ can ensure the passing to limit in the sense of distribution for the two corresponding term in the momentum conservation equation (1.1b). On the other hand, since $\mathbf{H}_n$ converges weakly* to $\mathbf{H}$ in $L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];H^1(\Omega))$, this implies that the nonlinear term $(\nabla\times\mathbf{H}_n)\times\mathbf{H}_n$ converges to $(\nabla\times\mathbf{H})\times\mathbf{H}$ in the sense of distributions. As a product of $\rho_n$ and $\theta_n$, which respectively converge strongly in $C([0,T];L^2(\Omega'))$ and in $L^2(\Omega\times(0,T))$, the term $\nabla(\rho_n\theta_n)$ converges to the limit $\nabla(\rho\theta)$ in the sense of distributions. The term $p_e(\rho)$ is already done in view of the hypothesis (2.5) and the strong convergence of $\rho$ in $C([0,T];L^p(\Omega'))$ for all $1\le p<6m-3$. Thus, we are left to show the convergence of the viscous flux. In fact,

$$\mu(\rho_n)D(\mathbf{u}_n)=D(\mu(\rho_n)\mathbf{u}_n)-\frac{1}{2}\left(\sqrt{\rho_n}\mathbf{u}_n\otimes\frac{\nabla\mu(\rho_n)}{\sqrt{\rho_n}}+\frac{\nabla\mu(\rho_n)}{\sqrt{\rho_n}}\otimes\sqrt{\rho_n}\mathbf{u}_n\right). \tag{5.5}$$

Since $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges strongly to $\frac{\mu(\rho)}{\sqrt{\rho}}$ in $L^\infty([0,T];L^2(\Omega'))$ and $\sqrt{\rho_n}\mathbf{u}_n$ converges strongly to $\sqrt{\rho}\mathbf{u}$ in $L^2([0,T];L^2(\Omega'))$, the fist term on the right-hand side of (5.5) converges to the corresponding term in the sense of distributions. The convergence of the second term on the right-hand side of (5.5) in the sense of distributions can be shown by using the weak convergence of $\rho_n^{-1/2}\nabla\mu(\rho_n)$ to $\rho^{-1/2}\nabla\mu(\rho)$ in $L^2([0,T];L^2(\Omega'))$ and the strong convergence $\sqrt{\rho_n}\mathbf{u}_n$ in $L^2([0,T];L^2(\Omega'))$. For the bulk viscous term $\lambda(\rho_n)\mathrm{div}\mathbf{u}_n$, by the assumption (2.1), it may be written in the renormalized sense:

$$\lambda(\rho_n)\mathrm{div}\mathbf{u}_n=-2(\partial_t\mu(\rho_n)+\mathrm{div}(\mu(\rho_n)\mathbf{u}_n)),$$

which can be shown directly by the convergence of $\rho_n$ and $\mathbf{u}_n$, and hence, the convergence in the sense of distributions for the momentum conservation equation is done.

*Step 3: Convergence of the energy conservation equation.* The main difficulties in this step lie in the passage to the limit for the energy flux $\mathbf{u}(\mathcal{E}'+p)$, the heat flux $\kappa\nabla\theta$, the viscous term $\mathbf{u}\Psi$, and the nonlinear terms $(\mathbf{u}\times\mathbf{H})\times\mathbf{H}$, $\nu\mathbf{H}\times(\nabla\times\mathbf{H})$, because we already showed that $\mathcal{E}_n$ converges strongly to $\mathcal{E}$ in $C([0,T];H^{-s}(\Omega))$ for some $s>0$.

For the energy flux $\rho_n\mathbf{u}_n\theta_n$, since $\sqrt{\rho_n}\mathbf{u}_n$ and $\sqrt{\rho_n}\theta_n$ converge strongly in $L^2(\Omega'\times(0,T))$ to $\sqrt{\rho}\mathbf{u}$ and $\sqrt{\rho}\theta$ respectively, $\rho_n\mathbf{u}_n\theta_n$ converges strongly in $L^1(\Omega'\times(0,T))$ to $\rho\mathbf{u}\theta$. For the energy flux $\rho_n\mathbf{u}_n|\mathbf{u}_n|^2$, the strong convergence of $\rho_n^{-1/2}$ in $C([0,T];L^p(\Omega'))$ for all $p<6$ implies that $\rho_n^{-1/6}$ converges strongly to $\rho^{-1/6}$ in $C([0,T];L^3(\Omega'))$. Hence, the term $\rho_n^{-1/6}\sqrt{\rho_n}\mathbf{u}_n$ converges strongly to $\rho^{-1/6}\sqrt{\rho}\mathbf{u}$ in $L^2([0,T];L^{6/5}(\Omega'))$, because of the strong convergence of $\sqrt{\rho}\mathbf{u}$ in $L^2(\Omega'\times(0,T))$. And Lemma 4.1 implies that $\rho_n^{1/3}\mathbf{u}_n$ is uniformly bounded in $L^\delta(\Omega'\times(0,T))$ for some $\delta>3$. This fact, combining with the interpolation inequality, gives the strong convergence of $\rho\mathbf{u}|\mathbf{u}|^2$ in $L^1(\Omega'\times(0,T))$. The analysis at the end of Section 4 tells the strong convergence of $\rho_n\mathbf{u}_nP_e(\rho_n)$ and $\mathbf{u}_np_n$ in $L^1(\Omega'\times(0,T))$ to $\rho\mathbf{u}P_e(\rho)$ and $\mathbf{u}p$, respectively. Thus, the energy flux $\mathbf{u}_n(\mathcal{E}_n'+p_n)$ converges strongly to $\mathbf{u}(\mathcal{E}'+p)$ in $L^1(\Omega'\times(0,T))$.

The strong convergence of $\theta$ in $L^p([0,T];L^q(\Omega'))$ for $p<a$ and $q<3a$ implies that $\theta_n^{a/2}$ converges strongly in $L^2([0,T];L^3(\Omega'))$ to $\theta^{a/2}$. This fact, together the strong convergence of $\rho_n$ in $C([0,T];L^p(\Omega'))$ for $p<6m-3$, implies that $\sqrt{1+\rho_n}(1+\theta_n^{a/2})$ converges to $\sqrt{1+\rho}(1+\theta^{a/2})$ in $L^2(\Omega'\times(0,T))$. That means $\kappa^{1/2}(\rho_n,\theta_n)$ strongly converges to $\kappa^{1/2}(\rho,\theta)$ in $L^2(\Omega'\times(0,T))$. Similarly, it follows that $(1+\rho_n)^{1/2}\theta_n^{(a+c+1)/2}$ converges strongly to $(1+\rho)^{1/2}\theta^{(a+c+1)/2}$ in $L^2(\Omega'\times(0,T))$ due to the strong convergence of $\rho_n$ and $\theta_n$. Therefore, $\kappa_0(\rho_n,\theta_n)(1+\rho_n)^{1/2}(1+\theta_n)^{(a+c+1)/2}$ converges strongly to $\kappa_0(\rho,\theta)(1+\rho)^{1/2}(1+\theta)^{(a+c+1)/2}$ in $L^2(\Omega'\times(0,T))$. On the other hand, we deduce from (4.4) that $(1+\rho_n)^{1/2}\nabla(1+\theta_n)^{(a-c+1)/2}$ is uniformly bounded in $L^2(\Omega\times(0,T))$, hence weakly converges to some element $\omega$ in $L^2(\Omega\times(0,T))$. It also follows that $\nabla(1+\theta_n)^{(a-c+1)/2}$ is uniformly bounded in $L^2(\Omega\times(0,T))$, and hence weakly converges to $\nabla(1+\theta)^{(a-c+1)/2}$ in $L^2(\Omega'\times(0,T))$. Due to the strong convergence of $\rho_n$ in $L^2(\Omega' \times (0,T))$, we deduce that $\omega = (1+\rho)^{1/2}\nabla(1+\theta)^{(a-c+1)/2}$. Finally, we write

$$\begin{aligned}\kappa(\rho_n,\theta_n)\nabla\theta_n &= \kappa_0(\rho_n,\theta_n)(1+\rho_n)(1+\theta_n)^a\nabla\theta_n \\ &= \kappa_0(\rho_n,\theta_n)(1+\rho_n)^{1/2}(1+\theta_n)^{(a+c+1)/2}(1+\rho_n)^{1/2}\nabla(1+\theta_n)^{(a-c+1)/2}.\end{aligned}$$

This, together with the strong convergence of $\kappa_0(\rho_n,\theta_n)(1+\rho_n)^{1/2}(1+\theta_n)^{(a+c+1)/2}$ and the weak convergence of $(1+\rho_n)^{1/2}\nabla(1+\theta_n)^{(a-c+1)/2}$, implies that $\kappa(\rho_n,\theta_n)\nabla\theta_n$ converges to $\kappa(\rho,\theta)\nabla\theta$ at least in the sense of distributions.

For the viscous terms, $\sqrt{\mu(\rho_n)}D(\mathbf{u}_n)$ and $\sqrt{|\lambda(\rho_n)|}\mathrm{div}\mathbf{u}_n$ converges weakly to $\sqrt{\mu(\rho)}D(\mathbf{u})$ and $\sqrt{|\lambda(\rho)|}\mathrm{div}\mathbf{u}$ respectively in $L^2(\Omega'\times(0,T))$, because of the hypothesis (2.2), the uniform bound on $\rho_n$ in Lemma 3.2, and the uniform estimate (4.1). On the other hand, $\rho_n^{1/3}\mathbf{u}_n$ strongly converges to $\rho^{1/3}\mathbf{u}$ in $L^3(\Omega' \times (0,T))$ and $\rho_n^{-1/3}\sqrt{\mu(\rho_n)}$, as well as $\rho_n^{-1/3}\sqrt{\lambda(\rho_n)}$ converges strongly to $\rho^{-1/3}\sqrt{\mu(\rho)}$ and $\rho^{-1/3}\sqrt{\lambda(\rho)}$ respectively in $L^\infty([0,T];L^6(\Omega'))$. Hence $\Psi_n\mathbf{u}_n$ converges to $\Psi\mathbf{u}$ at least in the sense of distributions.

Finally, we deal with the convergence of two nonlinear terms: $(\mathbf{u}_n \times \mathbf{H}_n)\times\mathbf{H}_n$ and $\nu(\rho_n,\theta_n)\mathbf{H}_n \times (\nabla\times\mathbf{H}_n)$. First, since $\mathbf{H}_n$ converges strongly to $\mathbf{H}$ in $L^p([0,T];L^q(\Omega'))$ for some $p>5$ and some $q>5/2$, $\mathbf{u}_n\times\mathbf{H}_n$ weakly converges to $\mathbf{u}\times\mathbf{H}$ in $L^p([0,T];L^q(\Omega'))$ for some $p>5/4$ and $q>5/3$ because $\mathbf{u}_n$ converges weakly to $\mathbf{u}$ in $L^{5/3}([0,T];L^5(\Omega'))$. From this, we can deduce that $(\mathbf{u}_n\times\mathbf{H}_n)\times\mathbf{H}_n$ converges to $(\mathbf{u}\times\mathbf{H})\times\mathbf{H}$ in the sense of distributions. Second, the strong convergence of $\rho_n$, $\theta_n$, $\mathbf{H}_n$ and the hypothesis (2.6) imply that $\nu(\rho_n,\theta_n)\mathbf{H}_n$ converges strongly to $\nu(\rho,\theta)\mathbf{H}$ in $L^p([0,T];L^q(\Omega'))$ for some $p>5$ and some $q>5/2$. By the weak convergence of $\mathbf{H}_n$ in $L^2([0,T];H^1(\Omega))$, one deduce that $\nu(\rho_n,\theta_n)\mathbf{H}_n\times(\nabla\times\mathbf{H}_n)$ converges to $\nu(\rho,\theta)\mathbf{H}\times(\nabla\times\mathbf{H})$ at least in the sense of distributions.

Therefore, the energy conservation equation (1.1c) holds at least in the sense of distributions.

*Step 4: Convergence of the magnetic field equation.* Similar to the argument in Step 3, we can show that $\nu(\rho_n,\theta_n)\nabla\times\mathbf{H}_n$ converges weakly to $\nu(\rho,\theta)\nabla\times\mathbf{H}$ in $L^2(\Omega'\times(0,T))$. Also the strong convergence of $\mathbf{H}_n$ in $L^p([0,T];L^q(\Omega'))$ for some $p>5$ and some $q>5/2$ and the weak convergence of $\mathbf{u}_n$ in $L^{5/3}([0,T];L^5(\Omega'))$, imply that $\mathbf{u}_n\times\mathbf{H}_n$ converges to $\mathbf{u}\times\mathbf{H}$ at least in the sense of distributions. Hence, the induction equation holds at least in the sense of distributions.

The proof is complete.

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