AN ALGORITHM FOR FAST CALCULATION OF FLOW ENSEMBLES

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ABSTRACT. This report presents an algorithm for computing an ensemble of p solutions of the Navier-Stokes equations. The solutions are found, at each timestep, by solving a linear system with one shared coefficient matrix and p right hand sides, reducing both storage required and computational cost of the solution process. The price that must be paid is a timestep condition involving the timestep and the size of the fluctuations about the ensemble mean. Since the method is a one step method and the timestep condition involves only known quantities, it can be imposed to adapt the next timestep. The report gives a comprehensive stability analysis, an error estimate and some first tests.

1. INTRODUCTION

There are many uncertainties inherent in numerical simulation of fluid flows. Calculation of an ensemble of p solutions deals with these inherent uncertainties to increase the window of predictability (by averaging), e.g., [1], [2], [3], to estimate solution sensitivities, e.g., [4], [5] and to estimate the uncertainty in the result (by calculation of a PDF of the resulting solution), e.g., [6], [7]. Further, the bred-vectors algorithm, [1], used to select a minimal set of ensemble members capturing maximal spread of the resulting forecast itself involves repeated ensemble flow simulations. One common way to calculate these ensembles is to treat them as separate tasks, requiring computational effort and memory p-times the amount required for one simulation. If available memory is sufficient to treat the tasks in parallel, then the turnaround time is not increased, while if not then the turnaround time is multiplied by p. This report explores a new approach ((BEFE-Ensemble) below) intermediate between these two extremes which requires a negligible storage increase over one simulation (p solution vectors) and could have run time reduced over p successive simulations, depending on the block solver used and the timestep condition required for stability (Sections 3, 4). Thus the method is a new way to rebalance "the competition between high-resolution, single deterministic forecasts and ensembles" (Stensrud [8], p. 401). The motivation for the new method is that for linearly implicit methods, the linear solve is a large contributor to overall complexity and it is far cheaper in both storage and solution time to solve p linear systems with the same coefficient matrix than with p different coefficient matrices. For example, block generalized CG methods compute p residuals at each step but

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compensate in speed of convergence by producing approximations optimized over a $p \times (\#steps)$ dimensional Krylov subspace, e.g., [9], [10], [11], [12].

Accordingly, we consider a discretization of an ensemble of p solutions of the NSE requiring solution of one linear system with the same coefficient matrix and p RHS[1]. To begin, consider p Navier-Stokes equations with p slightly different initial conditions and body forces, u_j^0, f_j, on a bounded domain subject to no slip boundary conditions, for $j = 1, ..., p$:

$$\begin{aligned} u_{j,t} + u_j \cdot \nabla u_j - \nu \triangle u_j + \nabla p_j &= f_j(x,t), \text{ in } \Omega, \\ \nabla \cdot u_j &= 0, \text{ in } \Omega, \\ u_j &= 0, \text{ on } \partial\Omega, \\ u_j(x,0) &= u_j^0(x), \text{ in } \Omega. \end{aligned} \tag{1.1}$$

We denote the ensemble mean by

$$< u >^n := \frac{1}{p}\sum_{j=1}^{p} u_j^n.$$

To present the idea, suppress the spacial discretization. Using an implicit-explicit time discretization and keeping the resulting coefficient matrix independent of the ensemble member, leads to the method:

$$\begin{aligned} &\frac{u_j^{n+1} - u_j^n}{\Delta t} + < u >^n \cdot \nabla u_j^{n+1} + (u_j^n - < u >^n) \cdot \nabla u_j^n \\ &\qquad + \nabla p_j^{n+1} - \nu \Delta u_j^{n+1} = f_j^{n+1}, \\ &\nabla \cdot u_j^{n+1} = 0. \end{aligned} \tag{BEFE-Ensemble}$$

Since the resulting coefficient matrix multiplying each u_j^{n+1} is independent of j, (ensemble number), advancing one step we solve one linear system with p RHS. Naturally, if the number of ensemble members is large enough, it can be subdivided into p member sub-ensembles, balancing memory, communication and computations, and (BEFE-Ensemble) applied to each. Further, the choice of the ensemble data u_j^0 and f_j is application dependent.

The ensemble mean equation. Taking the ensemble mean of (BEFE-Ensemble), $< u >^n$ satisfies

$$\begin{aligned} &\frac{< u >^{n+1} - < u >^n}{\Delta t} + < u >^n \cdot \nabla < u >^{n+1} + \nabla < p >^{n+1} - \nu \Delta < u >^{n+1} \\ &\qquad + [< u \cdot \nabla u >^n - < u >^n \nabla < u >^n] = < f >^{n+1}, \text{ and} \\ &\nabla \cdot < u >^{n+1} = 0, \end{aligned} \tag{1.2}$$

which is a discretized variant on the usual ensemble averaged NSE.

Timestep conditions. Since (BEFE-Ensemble) involves an explicit discretization of a stretching term, a timestep restriction is necessary for long time, nonlinear stability. With an FEM spacial discretization with mesh size h, we prove in Section 3 that in both 2d and 3d (BEFE-Ensemble) is stable under

$$C\frac{\Delta t}{\nu h}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2 \leq 1. \tag{1.3}$$

[1] One easy method to do this is simply to lag the nonlinear terms and pay the price in the associated and severe Re dependent timestep restriction.

Thus, as long as the deviation of each ensemble member from the ensemble mean at each time step is not too big, the method is stable. When the deviation increases, the timestep must decrease according to (1.3). In Section 4 we give improvements of this condition. For example, in 2d we prove stability under

$$\frac{Cln(1/h)\Delta t}{\nu}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2 \leq 1 .$$

We also give a condition valid for locally refined meshes, useful in cases when local mesh widths are cut to balance locally large gradients.

In Section 5, we give an error analysis in the error in the individual ensemble member using finite element methods for spacial discretization. Analysis of the error in a PDF constructed from the approximations to the individual member's discrete approximation is an important open problem. Numerical tests confirming our theory are presented in Section 6.

2. Notation and preliminaries

Let Ω be an open, regular domain in $\mathbb{R}^d (d = 2$ or $3)$. The $L^2(\Omega)$ norm and the inner product are $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$ respectively. $H^k(\Omega)$ is the Sobolev space $W_2^k(\Omega)$, with norm $\|\cdot\|_k$. For functions $v(x,t)$ defined on $(0,T)$, we define $(1 \leq m < \infty)$

$$\|v\|_{\infty,k} := EssSup_{[0,T]}\|v(t,\cdot)\|_k, \text{ and } \|v\|_{m,k} := \left(\int_0^T \|v(t,\cdot)\|_k^m dt\right)^{1/m} .$$

The space $H^{-k}(\Omega)$ is the dual space of bounded linear functions on $H_0^k(\Omega)$. A norm for $H^{-1}(\Omega)$ is given by

$$\|f\|_{-1} = \sup_{0 \neq v \in H_0^1(\Omega)} \frac{(f,v)}{\|\nabla v\|} .$$

We base our analysis on the finite element method (FEM) for the spacial discretization. The results also extend to many other variational methods. Let X be the velocity space and Q be the pressure space:

$$X := (H_0^1(\Omega))^d, \; Q := L_0^2(\Omega).$$

For $v \in X$ the usual $H^{1/2}(\Omega)$ norm satisfies the interpolation inequality

$$\|v\|_{1/2} \leq C\sqrt{\|v\|\|\nabla v\|}.$$

The space of divergence free functions is

$$V := \{v \in X : (\nabla \cdot v, q) = 0 , \forall q \in Q\}.$$

The norm on V^* (the dual of V) is defined as

$$\|f\|_* = \sup_{0 \neq v \in V} \frac{(f,v)}{\|\nabla v\|} .$$

A weak formulation of (1.1) is: Find $u_j : [0,T] \to X$, $p_j : [0,T] \to Q$ for a.e. $t \in (0,T]$ satisfying, for $j = 1,...,p$:

$$(u_{j,t}, v) + (u_j \cdot \nabla u_j, v) + \nu(\nabla u_j, \nabla v) - (p_j, \nabla \cdot v) = (f_j, v) , \forall v \in X$$
$$u_j(x,0) = u_j^0(x) \text{ in X and } (\nabla \cdot u_j, q) = 0, \forall q \in Q.$$

We denote conforming velocity, pressure finite element spaces based on an edge to edge triangulation of Ω (with maximum triangle diameter h) by

$$X_h \subset X \ , \, Q_h \subset Q.$$

We assume that X_h and Q_h satisfy the usual discrete inf-sup condition. Taylor-Hood elements, discussed in [13], [14], are one commonly used choice of velocity-pressure finite element spaces. The discretely divergence free subspace of X_h is

$$V_h \ := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \ , \, \forall q_h \in Q_h\}.$$

We assume the mesh and finite element spaces satisfy the following standard inequalities (typical for locally quasi-uniform meshes and standard FEM spaces, see, e.g., [13]): for all $v_h \in X_h$

(Inverse Ineq) $\qquad h\|\nabla v_h\| \leq C_{(inv)} \|v_h\|,$

(Discrete Sobolev) $\qquad \|v_h\|_\infty \leq C|\ln h|^{1/2} \|\nabla v_h\|,$ in dimension $d = 2$.

Define the usual explicitly skew symmetric trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

$b^*(u, v, w)$ satisfies the bound

$$b^*(u, v, w) \leq C\|u\|_{\frac{1}{2}} \|\nabla v\| \|\nabla w\|, \text{ for all } u, v, w \in X.$$

Lemma 1. *For any $u_h, v_h, w_h \in X_h$,*

$$b^*(u_h, v_h, w_h) = \int_\Omega u_h \cdot \nabla v_h \cdot w_h \ dx + \frac{1}{2} \int_\Omega (\nabla \cdot u_h)(v_h \cdot w_h) \ dx.$$

Proof.

$$b^*(u_h, v_h, w_h) := \frac{1}{2}(u_h \cdot \nabla v_h, w_h) - \frac{1}{2}(u_h \cdot \nabla w_h, v_h).$$

Integrating by parts the second term and using $u_h|_{\partial\Omega} = 0$:

$$-(u_h \cdot \nabla w_h, v_h) = (u_h \cdot \nabla v_h, w_h) + (\nabla \cdot u_h, v_h \cdot w_h).$$

□

The fully discrete approximation we study of (1.1) is: Given $u^n_{j,h}$, find $u^{n+1}_{j,h} \in X_h$, $p^{n+1}_{j,h} \in Q_h$ satisfying

$$\begin{aligned} &\Big(\frac{u^{n+1}_{j,h} - u^n_{j,h}}{\Delta t}, v_h\Big) + b^*(< u_h >^n, u^{n+1}_{j,h}, v_h) + b^*(u^n_{j,h} - < u_h >^n, u^n_{j,h}, v_h) \\ &-(p^{n+1}_{j,h}, \nabla \cdot v_h) + \nu(\nabla u^{n+1}_{j,h}, \nabla v_h) = (f^{n+1}_j, v_h), \qquad \forall v_h \in X_h, \\ &(\nabla \cdot u^{n+1}_{j,h}, q_h) = 0, \qquad \forall q_h \in Q_h. \end{aligned} \tag{2.1}$$

C represents a positive constant independent of ν, the solution u, the time step Δt and the mesh width h. Its value may vary from situation to situation.

3. Stability of the Ensemble Method

We begin by proving unconditional, nonlinear, long time stability of (2.1) under the first timestep condition:

$$C\frac{\Delta t}{\nu h}\|\nabla(u_{j,h}^n - <u_h>^n)\|^2 \leq 1, \qquad j=1,\cdots,p\ . \tag{3.1}$$

Since (3.1) is based on known quantities and (2.1) is a 1-step method, (3.1) can be applied to adapt $\triangle t$ at every timestep to compute u_j^{n+1} stably. Improvements of (3.1) in special cases are developed in Section 4.

Theorem 1 (Stability of BEFE-Ensemble)**.** *Consider the method (2.1). Suppose the condition (3.1) holds. Then, for any* $N \geq 1$

$$\begin{aligned}&\frac{1}{2}\|u_{j,h}^N\|^2 + \frac{1}{4}\sum_{n=0}^{N-1}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^N\|^2 + \frac{\nu\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla u_{j,h}^{n+1}\|^2\\ &\quad\leq \sum_{n=0}^{N-1}\frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 + \frac{1}{2}\|u_{j,h}^0\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^0\|^2, \qquad j=1,...,p\ .\end{aligned}$$

Proof. Set $v_h = u_{j,h}^{n+1}$ in (2.1). This gives:

$$\begin{aligned}&\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^n\|^2 + \frac{1}{2}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2\\ &+\Delta t b^*(u_{j,h}^n - <u_h>^n, u_{j,h}^n, u_{j,h}^{n+1}) + \nu\Delta t\|\nabla u_{j,h}^{n+1}\|^2 = \Delta t(f_j^{n+1}, u_{j,h}^{n+1})\ .\end{aligned} \tag{3.2}$$

Applying Young's inequality to the right hand side gives

$$\begin{aligned}&\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^n\|^2 + \frac{1}{2}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2\\ &+\Delta t b^*(u_{j,h}^n - <u_h>^n, u_{j,h}^n, u_{j,h}^{n+1} - u_{j,h}^n) + \nu\Delta t\|\nabla u_{j,h}^{n+1}\|^2\\ &\leq \frac{\nu\Delta t}{2}\|\nabla u_{j,h}^{n+1}\|^2 + \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2\ .\end{aligned} \tag{3.3}$$

Next, we bound the trilinear term using the interpolation and inverse inequalities, as well as Lemma 1.

$$\begin{aligned}&-\Delta t b^*(u_{j,h}^n - <u_h>^n, u_{j,h}^n, u_{j,h}^{n+1} - u_{j,h}^n)\\ &\leq C\Delta t\|\nabla(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|_{\frac{1}{2}}\\ &+\frac{1}{2}C\Delta t\|\nabla\cdot(u_{j,h}^n - <u_h>^n)\|\|u_{j,h}^n\cdot(u_{j,h}^{n+1} - u_{j,h}^n)\|\\ &\leq C\Delta t\|\nabla(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|_{\frac{1}{2}}\\ &+\frac{1}{2}C\Delta t\|\nabla\cdot(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|^{\frac{1}{2}}\|\nabla(u_{j,h}^{n+1} - u_{j,h}^n)\|^{\frac{1}{2}}\\ &\leq C\Delta t\|\nabla(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|(Ch^{-\frac{1}{2}})\|u_{j,h}^{n+1} - u_{j,h}^n\|\\ &+\frac{1}{2}C\Delta t\|\nabla(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|(Ch^{-\frac{1}{2}})\|u_{j,h}^{n+1} - u_{j,h}^n\|\ .\end{aligned} \tag{3.4}$$

Using Young's inequality again gives

$$-\Delta t b^*(u_{j,h}^n - < u_h >^n, u_{j,h}^n, u_{j,h}^{n+1} - u_{j,h}^n) \tag{3.5}$$
$$\leq C\frac{\Delta t^2}{h}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2 \|\nabla u_{j,h}^n\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 \ .$$

Combining like terms, (3.3) becomes

$$\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^n\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{2}\|\nabla u_{j,h}^{n+1}\|^2 \tag{3.6}$$
$$\leq \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 + C\frac{\Delta t^2}{h}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2 \|\nabla u_{j,h}^n\|^2 \ .$$

Adding and subtracting $\frac{\nu\Delta t}{4}\|\nabla u_{j,h}^n\|^2$ gives

$$\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^n\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 - \|\nabla u_{j,h}^n\|^2\} \tag{3.7}$$
$$+\frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 + (1 - \frac{C\Delta t}{\nu h}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2)\|\nabla u_{j,h}^n\|^2\} \leq \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 \ .$$

With the restriction (3.1) assumed, we have

$$\frac{\nu\Delta t}{4}(1 - \frac{C\Delta t}{\nu h}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2)\|\nabla u_{j,h}^n\|^2 \geq 0 \ .$$

Equation (3.7) reduces to

$$\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^n\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 \tag{3.8}$$
$$+\frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 - \|\nabla u_{j,h}^n\|^2\} + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{n+1}\|^2 \leq \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 \ .$$

Summing up (3.8) from $n = 0$ to $n = N - 1$ results in

$$\frac{1}{2}\|u_{j,h}^N\|^2 + \frac{1}{4}\sum_{n=0}^{N-1}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^N\|^2 + \frac{\nu\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla u_{j,h}^{n+1}\|^2$$
$$\leq \sum_{n=0}^{N-1}\frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 + \frac{1}{2}\|u_{j,h}^0\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^0\|^2 \ . \tag{3.9}$$

This concludes the proof of stability. □

4. Sharpening the timestep condition

We have derived a global condition on the timestep that is sufficient for stability in 2d and 3d. There are many important cases where this condition is improvable:

$$\frac{C|ln(h)|\Delta t}{\nu}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2 \leq 1, \ \ 2d \ , \tag{2d C1}$$

$$\frac{C\Delta t}{\nu h^2}(\|u_{j,h}^n - < u_h >^n \|^2 + \|\nabla\cdot(u_{j,h}^n - < u_h >^n)\|^2) \leq 1, \ \ 2d \ , \tag{2d C2}$$

$$\frac{C\Delta t}{\nu h^2}\|u_{j,h}^n - < u_h >^n \|_{L^3}^2 \leq 1, \text{ 3d - no derivatives of fluctuations,} \tag{3d, L3}$$

$$\max_e \frac{C\Delta t}{\nu h_e}\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}^2 \leq 1, \ \ \text{3d-locally refined meshes.} \tag{Local}$$

4.1. **The case of 2d domains.** In 2d, embedding estimates improve and this improvement leads to an improvement of the timestep condition.

Theorem 2 (2d domains)**.** *Consider the method (2.1). Suppose the condition (2d C1) or (2d C2) holds. Then, for any $N \geq 1$*

$$\frac{1}{2}\|u_{j,h}^N\|^2 + \frac{1}{4}\sum_{n=0}^{N-1}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^N\|^2 + \frac{\nu\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla u_{j,h}^{n+1}\|^2$$
$$\leq \sum_{n=0}^{N-1}\frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 + \frac{1}{2}\|u_{j,h}^0\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^0\|^2, \qquad j = 1, ..., p\ .$$

Proof. In 2d we have

$$\begin{aligned}
&\Delta t b^*(u_{j,h}^n - <u_h>^n, u_{j,h}^n, u_{j,h}^{n+1} - u_{j,h}^n)\\
&\leq C\Delta t\|u_{j,h}^n - <u_h>^n\|_\infty\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_h^n\|\\
&+\frac{1}{2}C\Delta t\|\nabla\cdot(u_{j,h}^n - <u_h>^n)\|\|u_{j,h}^n\|_\infty\|u_{j,h}^{n+1} - u_h^n\|\\
&\leq C\sqrt{|ln(h)|}\Delta t\|\nabla(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|\\
&+\frac{1}{2}C\sqrt{|ln(h)|}\Delta t\|\nabla(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|\\
&\leq C|ln(h)|\Delta t^2\|\nabla(u_{j,h}^n - <u_h>^n)\|^2\|\nabla u_{j,h}^n\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2\ ,
\end{aligned} \tag{4.1}$$

or,

$$\begin{aligned}
&\Delta t b^*(u_{j,h}^n - <u_h>^n, u_{j,h}^n, u_{j,h}^{n+1} - u_{j,h}^n)\\
&\leq C\Delta t\|u_{j,h}^n - <u_h>^n\|_\infty\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|\\
&+\frac{1}{2}C\Delta t\|\nabla\cdot(u_{j,h}^n - <u_h>^n)\|\|u_{j,h}^n\|_\infty\|u_{j,h}^{n+1} - u_h^n\|\\
&\leq Ch^{-1}\Delta t\|u_{j,h}^n - <u_h>^n\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|\\
&+\frac{1}{2}Ch^{-1}\Delta t\|\nabla\cdot(u_{j,h}^n - <u_h>^n)\|\|\nabla u_{j,h}^n\|\|u_{j,h}^{n+1} - u_{j,h}^n\|\\
&\leq C\frac{\Delta t^2}{h^2}(\|u_{j,h}^n - <u_h>^n\|^2 + \|\nabla\cdot(u_{j,h}^n - <u_h>^n)\|^2)\|\nabla u_{j,h}^n\|^2\\
&+\frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2\ .
\end{aligned} \tag{4.2}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^n\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2\\
&+\frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 - \|\nabla u_{j,h}^n\|^2\} + \frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2\\
&+(1 - \frac{C|ln(h)|\Delta t}{\nu}\|\nabla(u_{j,h}^n - <u_h>^n)\|^2)\|\nabla u_{j,h}^n\|^2\} \leq \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2\ ,
\end{aligned} \tag{4.3}$$

or,

$$
\begin{aligned}
&\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^{n}\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^{n}\|^2 \\
&+\frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 - \|\nabla u_{j,h}^{n}\|^2\} + \frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 \\
&+(1-\frac{C\Delta t}{\nu h^2}(\|u_{j,h}^{n} - < u_h >^n\|^2 + \|\nabla\cdot(u_{j,h}^{n} - < u_h >^n)\|^2))\|\nabla u_{j,h}^{n}\|^2\} \\
&\leq \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 .
\end{aligned} \tag{4.4}
$$

□

4.2. L^3 **estimate on the fluctuating part.**

Theorem 3 (L3 estimate)**.** *Consider the method (2.1). Suppose the condition (3d, L3 norms) holds. Then, for any* $N \geq 1$

$$
\begin{aligned}
&\frac{1}{2}\|u_{j,h}^{N}\|^2 + \frac{1}{4}\sum_{n=0}^{N-1}\|u_{j,h}^{n+1} - u_{j,h}^{n}\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{N}\|^2 + \frac{\nu\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla u_{j,h}^{n+1}\|^2 \\
&\leq \sum_{n=0}^{N-1}\frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 + \frac{1}{2}\|u_{j,h}^{0}\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^{0}\|^2, \qquad j = 1, ..., p .
\end{aligned}
$$

Proof. By $H\ddot{o}lders'$ inequality, we have

$$
\begin{aligned}
&\Delta t b^*(u_{j,h}^{n} - < u_{j,h}^{n} >, u_{j,h}^{n}, u_{j,h}^{n+1} - u_{j,h}^{n}) \\
&\leq \frac{1}{2}\Delta t\|(u_{j,h}^{n} - < u_h >^n)\|_{L^3}\|\nabla u_{j,h}^{n}\|_{L^2}\|u_{j,h}^{n+1} - u_{j,h}^{n}\|_{L^6} \\
&+\frac{1}{2}\Delta t\|(u_{j,h}^{n} - < u_h >^n)\|_{L^3}\|u_{j,h}^{n}\|_{L^6}\|\nabla(u_{j,h}^{n+1} - u_{j,h}^{n})\|_{L^2}.
\end{aligned} \tag{4.5}
$$

Using the Sobolev embedding theorem and the inverse estimate on the $(u_{j,h}^{n+1} - u_{j,h}^{n})$ terms give

$$
\begin{aligned}
\|\nabla(u_{j,h}^{n+1} - u_{j,h}^{n})\|_{L^2} &\leq Ch^{-1}\|u_{j,h}^{n+1} - u_{j,h}^{n}\| \\
\|u_{j,h}^{n+1} - u_{j,h}^{n}\|_{L^6} &\leq Ch^{-1}\|u_{j,h}^{n+1} - u_{j,h}^{n}\| .
\end{aligned}
$$

Thus, for any $\epsilon > 0$,

$$
\begin{aligned}
&\Delta t b^*(u_{j,h}^{n} - < u_h >^n, u_{j,h}^{n}, u_{j,h}^{n+1} - u_{j,h}^{n}) \\
&\leq Ch^{-1}\Delta t\|(u_{j,h}^{n} - < u_h >^n)\|_{L^3}\|\nabla u_{j,h}^{n}\|\|u_{j,h}^{n+1} - u_{j,h}^{n}\| \\
&\leq \frac{\epsilon\Delta t}{2}\|u_{j,h}^{n+1} - u_{j,h}^{n}\|^2 + \frac{C\Delta t}{2\epsilon h^2}\|(u_{j,h}^{n} - < u_h >^n)\|_{L^3}^2\|\nabla u_{j,h}^{n}\|^2.
\end{aligned} \tag{4.6}
$$

We use this estimate with $\epsilon = \frac{1}{2}\Delta t^{-1}$. This gives

$$
\begin{aligned}
&\frac{1}{2}\|u_{j,h}^{n+1}\|^2 - \frac{1}{2}\|u_{j,h}^{n}\|^2 + \frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^{n}\|^2 + \frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 - \|\nabla u_{j,h}^{n}\|^2\} \\
&+\frac{\nu\Delta t}{4}\{\|\nabla u_{j,h}^{n+1}\|^2 + (1 - \frac{C\Delta t}{\nu h^2}\|u_{j,h}^{n} - < u_h >^n\|_{L^3}^2)\|\nabla u_{j,h}^{n}\|^2\} \leq \frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 .
\end{aligned} \tag{4.7}
$$

□

4.3. **Locally refined meshes:** Often meshes are locally refined in regions of sharp gradients. We show that a sufficient condition is that Δt satisfies the following for all elements e:

$$\frac{C\Delta t}{\nu h_e}\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}^2 \leq 1.$$

For the local condition we perform the same steps (locally on the element e) as in the proof of Theorem 1 noting that (i) $H\ddot{o}lders'$ inequality can be applied locally (with no dependence on $diam(e)$ therefrom), (ii) the inverse inequality holds locally, with $h_e = diam(e)$ and constant depending only on the shape (h_e/ρ_e, ρ_e is the diameter of the largest ball that can be inscribed in e) of the element, [15], and (iii) the Sobolev embedding theorem holds locally with absolute constant independent of h_e.

From the stability proof we observe the following.

Lemma 2. *The conclusion of Theorem 1 (on stability) holds provided at every timestep:*

$$\begin{aligned}&\frac{1}{4}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^n\|^2\\ &+\Delta t b^*(u_{j,h}^n - < u_h >^n, u_{j,h}^n, u_{j,h}^{n+1} - u_{j,h}^n) \geq 0\ .\end{aligned}$$

The same conclusion holds if, on every element e,

$$\begin{aligned}&\int_e\{\frac{1}{4}|u_{j,h}^{n+1} - u_{j,h}^n|^2 + \frac{\nu\Delta t}{4}|\nabla u_{j,h}^n|^2\\ &+\Delta t[(u_{j,h}^n - < u_h >^n)\cdot\nabla u_{j,h}^n\cdot(u_{j,h}^{n+1} - u_{j,h}^n)\\ &+\frac{1}{2}(\nabla\cdot(u_{j,h}^n - < u_h >^n)\cdot(u_{j,h}^n\cdot(u_{j,h}^{n+1} - u_{j,h}^n)))]\}dx \geq 0\ .\end{aligned}$$

Theorem 4 (Locally refined meshes)**.** *Consider the method (2.1). Suppose the locally refined meshes condition holds. Then, for any* $N \geq 1$

$$\begin{aligned}&\frac{1}{2}\|u_{j,h}^N\|^2 + \frac{1}{4}\sum_{n=0}^{N-1}\|u_{j,h}^{n+1} - u_{j,h}^n\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^N\|^2 + \frac{\nu\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla u_{j,h}^{n+1}\|^2\\ &\leq \sum_{n=0}^{N-1}\frac{\Delta t}{2\nu}\|f_j^{n+1}\|_*^2 + \frac{1}{2}\|u_{j,h}^0\|^2 + \frac{\nu\Delta t}{4}\|\nabla u_{j,h}^0\|^2, \qquad j = 1, ..., p\ .\end{aligned}$$

Proof. We have

$$
\begin{aligned}
&|\int_e \Delta t[(u_{j,h}^n - < u_h >^n)\cdot\nabla u_{j,h}^n\cdot(u_{j,h}^{n+1}-u_{j,h}^n)\\
&+\frac{1}{2}(\nabla\cdot(u_{j,h}^n - < u_h >^n)\cdot(u_{j,h}^n\cdot(u_{j,h}^{n+1}-u_{j,h}^n)))]\,dx|\\
&\le C\Delta t\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}\|\nabla u_{j,h}^n\|_{L^2(e)}\\
&\cdot\|u_{j,h}^{n+1}-u_{j,h}^n\|_{L^2(e)}^{\frac{1}{2}}\|\nabla(u_{j,h}^{n+1}-u_{j,h}^n)\|_{L^2(e)}^{\frac{1}{2}}\\
&+\frac{1}{2}C\Delta t\|\nabla\cdot(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}\|u_{j,h}^n\cdot(u_{j,h}^{n+1}-u_{j,h}^n)\|_{L^2(e)}\\
&\le C\Delta t\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}\|\nabla u_{j,h}^n\|_{L^2(e)}\\
&\cdot\|u_{j,h}^{n+1}-u_{j,h}^n\|_{L^2(e)}^{\frac{1}{2}}\|\nabla(u_{j,h}^{n+1}-u_{j,h}^n)\|_{L^2(e)}^{\frac{1}{2}}\\
&+\frac{1}{2}C\Delta t\|\nabla\cdot(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}\|\nabla u_{j,h}^n\|_{L^2(e)}\\
&\cdot\|u_{j,h}^{n+1}-u_{j,h}^n\|_{L^2(e)}^{\frac{1}{2}}\|\nabla(u_{j,h}^{n+1}-u_{j,h}^n)\|_{L^2(e)}^{\frac{1}{2}}\\
&\le C\Delta t\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}\|\nabla u_{j,h}^n\|_{L^2(e)}(Ch_e^{-\frac{1}{2}})\|u_{j,h}^{n+1}-u_{j,h}^n\|_{L^2(e)}\\
&+\frac{1}{2}C\Delta t\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}\|\nabla u_{j,h}^n\|_{L^2(e)}(Ch_e^{-\frac{1}{2}})\|u_{j,h}^{n+1}-u_{j,h}^n\|_{L^2(e)}\ .
\end{aligned}
\tag{4.8}
$$

Using Young's inequality gives

$$
\begin{aligned}
&|\int_e \Delta t[(u_{j,h}^n - < u_h >^n)\cdot\nabla u_{j,h}^n\cdot(u_{j,h}^{n+1}-u_{j,h}^n)\\
&+\frac{1}{2}(\nabla\cdot(u_{j,h}^n - < u_h >^n)\cdot(u_{j,h}^n\cdot(u_{j,h}^{n+1}-u_{j,h}^n)))]\,dx|\\
&\le C\frac{\Delta t^2}{h_e}\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}^2\|\nabla u_{j,h}^n\|_{L^2(e)}^2+\frac{1}{4}\|u_{j,h}^{n+1}-u_{j,h}^n\|_{L^2(e)}^2\ .
\end{aligned}
\tag{4.9}
$$

Thus, under the locally refined meshes condition,

$$
\begin{aligned}
&\int_e\{\frac{1}{4}|u_{j,h}^{n+1}-u_{j,h}^n|^2+\frac{\nu\Delta t}{4}|\nabla u_{j,h}^n|^2\\
&+\Delta t[(u_{j,h}^n - < u_h >^n)\cdot\nabla u_{j,h}^n\cdot(u_{j,h}^{n+1}-u_{j,h}^n)\\
&+(u_{j,h}^n - < u_h >^n)\cdot\nabla(u_{j,h}^{n+1}-u_{j,h}^n)\cdot u_{j,h}^n]\}dx\\
&\ge\frac{\nu\Delta t}{4}(1-\frac{C\Delta t}{\nu h_e}\|\nabla(u_{j,h}^n - < u_h >^n)\|_{L^2(e)}^2)\|\nabla u_{j,h}^n\|_{L^2(e)}^2\ge 0.
\end{aligned}
\tag{4.10}
$$

Then, by Lemma 2, we obtain stability. □

5. Error Analysis for the Ensemble Method

In this section we give a detailed error analysis of the proposed method under the 3d stability condition. This analysis can be elaborated to analogous results in the cases of the other, sharpened stability conditions. Assume X_h and Q_h satisfy the usual (LBB^h) condition, then the method is equivalent to: *For* $n = 0, 1, ..., N_T$, *find* $u_{j,h}^{n+1} \in V_h$ *such that*

$$(5.1)\qquad \left(\frac{u_{j,h}^{n+1}-u_{j,h}^{n}}{\Delta t}, v_h\right) + b^*(< u_h >^n, u_{j,h}^{n+1}, v_h) + b^*(u_{j,h}^n - < u_{j,h} >^n, u_{j,h}^n, v_h)$$
$$+\nu(\nabla u_{j,h}^{n+1}, \nabla v_h) = (f_j^{n+1}, v_h)\ ,\ \forall v_h \in V_h.$$

Let $t^n = n\Delta t, n = 0, 1, 2, ..., N_T$, and $T := N_T \Delta t$. Denote $u_j^n = u_j(t^n)$, $j = 1, ..., p$. We introduce the following discrete norms:

$$|||v|||_{m,k} := \left(\sum_{n=0}^{N_T} ||v^n||_k^m \Delta t\right)^{1/m}, |||v|||_{\infty,k} = \max_{0\le n\le N_T} \|v^n\|_k.$$

To analyze the rate of convergence of the approximation we assume that the following regularity

$$u_j \in L^\infty(0,T;H^{k+1}(\Omega)) \cap H^1(0,T;H^{k+1}(\Omega)) \cap H^2(0,T;L^2(\Omega)),$$
$$p_j \in L^2(0,T;H^{s+1}(\Omega)), \text{and } f_j \in L^2(0,T;L^2(\Omega)).$$

Let $e_j^n = u_j^n - u_{j,h}^n$ be the error between the true solution and the approximation, then we have the following error estimates.

Theorem 5 (Convergence of (BEFE-Ensemble))**.** *Consider the method (5.1). Suppose that for any* $0 \le n \le N_T$*, the condition (1.3) holds*

$$C\frac{\Delta t}{\nu h}\|\nabla(u_{j,h}^n - < u_h >^n)\|^2 \le 1\ , \qquad j = 1, ..., p.$$

Then, for any $0 \le t^N \le T$*, there is a positive constant* C *independent of the mesh width and timestep such that*

$$\frac{1}{2}\|e_j^N\|^2 + \sum_{n=0}^{N-1}\frac{1}{4}\|e_j^{n+1}-e_j^n\|^2 + \frac{\nu\Delta t}{8}\|\nabla e_j^N\|^2 + C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla e_j^{n+1}\|^2$$
$$\le exp(C\frac{T}{\nu^2})\{\frac{1}{2}\|e_j^0\|^2 + \frac{\nu\Delta t}{8}\|\nabla e_j^0\|^2 + C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{\infty,0}^2|||u_j|||_{2,k+1}^2$$
$$+C\frac{\Delta t^2}{\nu}|||\nabla u_j|||_{\infty,0}^2|||\nabla u_{j,t}|||_{2,0}^2 + C\frac{h^{2k}}{\nu^2}|||u_j|||_{2,k+1}^2$$
$$(5.2)\qquad +C\frac{h^{2k+1}}{\Delta t}|||u_j|||_{2,k+1}^2 + Ch\Delta t|||\nabla u_{j,t}|||_{2,0}^2$$
$$+C\frac{h^{2s+2}}{\nu}|||p_j|||_{2,s+1}^2 + C\frac{h^{2k+2}}{\nu}|||u_{j,t}|||_{2,k+1}^2$$
$$+C\nu h^{2k}|||u_j|||_{2,k+1}^2 + \frac{C\Delta t^2}{\nu}|||u_{j,tt}|||_{2,0}^2\}\ .$$

For $k = 2, s = 1$, Taylor-Hood elements, i.e. C^0 piecewise quadratic velocity space X_h and C^0 piecewise linear pressure space Q_h, we have the following estimate.

Corollary 1. *Under the assumptions of Theorem 5, with* e_j^0 *taken to be* 0, $\Delta t/h$ *fixed to be a constant C,* (X_h, Q_h) *given by the Taylor-Hood approximation elements, we have*

$$\frac{1}{2}\|e_j^N\|^2 + \sum_{n=0}^{N-1}\frac{1}{4}\|e_j^{n+1}-e_j^n\|^2 + \frac{\nu\Delta t}{8}\|\nabla e_j^N\|^2 + C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla e_j^{n+1}\|^2 \le Ch^2\ .$$

Proof. The true solutions of the NSE u_j satisfy

$$
\begin{aligned}
(5.3)\qquad (\frac{u_j^{n+1}-u_j^n}{\Delta t}, v_h) + b^*(u_j^{n+1}, u_j^{n+1}, v_h) + \nu(\nabla u_j^{n+1}, \nabla v_h) - (p_j^{n+1}, \nabla \cdot v_h) \\
= (f_j^{n+1}, v_h) + Intp(u_j^{n+1}; v_h) \ , \qquad \text{for all } v_h \in V_h \ .
\end{aligned}
$$

where $Intp(u_j^{n+1}; v_h)$ is defined as

$$
Intp(u_j^{n+1}; v_h) = (\frac{u_j^{n+1}-u_j^n}{\Delta t} - u_{j,t}(t^{n+1}), v_h) \ .
$$

Let

$$
e_j^n = u_j^n - u_{j,h}^n = (u_j^n - I_h u_j^n) + (I_h u_j^n - u_{j,h}^n) = \eta_j^n + \xi_{j,h}^n \ , \qquad j = 1, ..., p \ .
$$

where $I_h u_j^n \in V_h$ is an interpolant of u_j^n in V_h. Denote

$$
U_j^n = u_{j,h}^n - < u_h >^n \ .
$$

Subtracting (5.1) from (5.3) gives

$$
\begin{aligned}
& (\frac{\xi_{j,h}^{n+1} - \xi_{j,h}^n}{\Delta t}, v_h) + \nu(\nabla \xi_{j,h}^{n+1}, \nabla v_h) + b^*(u_j^{n+1}, u_j^{n+1}, v_h) \\
(5.4)\qquad & -b^*(u_{j,h}^n - U_j^n, u_{j,h}^{n+1}, v_h) - b^*(U_j^n, u_{j,h}^n, v_h) - (p_j^{n+1}, \nabla \cdot v_h) \\
& = -(\frac{\eta_j^{n+1} - \eta_j^n}{\Delta t}, v_h) - \nu(\nabla \eta_j^{n+1}, \nabla v_h) + Intp(u_j^{n+1}; v_h) \ .
\end{aligned}
$$

Set $v_h = \xi_{j,h}^{n+1} \in V_h$, and rearrange the nonlinear terms, then we have

$$
\begin{aligned}
& \frac{1}{\Delta t}(\frac{1}{2}||\xi_{j,h}^{n+1}||^2 - \frac{1}{2}||\xi_{j,h}^n||^2 + \frac{1}{2}\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\|^2) + \nu||\nabla \xi_{j,h}^{n+1}||^2 \\
& = -b^*(u_j^{n+1}, u_j^{n+1}, \xi_{j,h}^{n+1}) + b^*(u_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
(5.5)\qquad & -b^*(U_j^n, u_{j,h}^{n+1} - u_{j,h}^n, \xi_{j,h}^{n+1}) + (p_j^{n+1}, \nabla \cdot \xi_{j,h}^{n+1}) \\
& -(\frac{\eta_j^{n+1} - \eta_j^n}{\Delta t}, \xi_{j,h}^{n+1}) - \nu(\nabla \eta_j^{n+1}, \nabla \xi_{j,h}^{n+1}) + Intp(u_j^{n+1}; \xi_{j,h}^{n+1}) \ .
\end{aligned}
$$

Now we bound the right hand side of the equation above. First, for the nonlinear term, adding and subtracting both $b^*(u_j^{n+1}, u_{j,h}^{n+1}, \xi_{j,h}^{n+1})$ and $b^*(U_j^n, u_j^{n+1} -$

$u_j^n, \xi_{j,h}^{n+1}$), we have

$$
\begin{aligned}
&-b^*(u_j^{n+1}, u_j^{n+1}, \xi_{j,h}^{n+1}) + b^*(u_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
&+b^*(U_j^n, u_{j,h}^{n+1} - u_{j,h}^n, \xi_{j,h}^{n+1}) \\
&= -b^*(u_j^{n+1}, e_j^{n+1}, \xi_{j,h}^{n+1}) - b^*(u_j^{n+1} - u_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
&-b^*(e_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) + b^*(U_j^n, u_{j,h}^{n+1} - u_{j,h}^n, \xi_{j,h}^{n+1}) \\
&= -b^*(u_j^{n+1}, \eta_j^{n+1}, \xi_{j,h}^{n+1}) - b^*(u_j^{n+1} - u_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
&-b^*(\eta_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) - b^*(\xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
&-b^*(U_j^n, e_j^{n+1} - e_j^n, \xi_{j,h}^{n+1}) + b^*(U_j^n, u_j^{n+1} - u_j^n, \xi_{j,h}^{n+1}) \\
&= -b^*(u_j^{n+1}, \eta_j^{n+1}, \xi_{j,h}^{n+1}) - b^*(u_j^{n+1} - u_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
&-b^*(\eta_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) - b^*(\xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) \\
&-b^*(U_j^n, \eta_j^{n+1}, \xi_{j,h}^{n+1}) + b^*(U_j^n, \eta_j^n, \xi_{j,h}^{n+1}) \\
&+b^*(U_j^n, \xi_{j,h}^n, \xi_{j,h}^{n+1}) + b^*(U_j^n, u_j^{n+1} - u_j^n, \xi_{j,h}^{n+1})\ .
\end{aligned} \tag{5.6}
$$

We estimate the nonlinear terms as follows

$$
\begin{aligned}
b^*(u_j^{n+1}, \eta_j^{n+1}, \xi_{j,h}^{n+1}) &\le C\|\nabla u_j^{n+1}\|\|\nabla \eta_j^{n+1}\|\|\nabla \xi_{j,h}^{n+1}\| \\
&\le \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla u_j^{n+1}\|^2\|\nabla \eta_j^{n+1}\|^2\ ,
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
b^*(u_j^{n+1} - u_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) &\le C\|\nabla(u_j^{n+1} - u_j^n)\|\|\nabla u_{j,h}^{n+1}\|\|\nabla \xi_{j,h}^{n+1}\| \\
&\le \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla(u_j^{n+1} - u_j^n)\|^2\|\nabla u_{j,h}^{n+1}\|^2 \\
&\le \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{C\Delta t^2}{\nu}\|\nabla \frac{u_j^{n+1} - u_j^n}{\Delta t}\|^2\|\nabla u_{j,h}^{n+1}\|^2 \\
&= \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{C\Delta t^2}{\nu}(\int_\Omega (\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (\nabla u_{j,t}) dt)^2 d\Omega)\|\nabla u_{j,h}^{n+1}\|^2 \\
&\le \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{C\Delta t^2}{\nu}(\int_\Omega (\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} |\nabla u_{j,t}|^2 dt) d\Omega)\|\nabla u_{j,h}^{n+1}\|^2 \\
&\le \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + \frac{C\Delta t}{\nu}(\int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|^2 dt)\|\nabla u_{j,h}^{n+1}\|^2\ ,
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
b^*(\eta_j^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) &\le C\|\nabla \eta_j^n\|\|\nabla u_{j,h}^{n+1}\|\|\nabla \xi_{j,h}^{n+1}\| \\
&\le \frac{\nu}{64}\|\nabla \xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla \eta_j^n\|^2\|\nabla u_{j,h}^{n+1}\|^2,
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
b^*(\xi_{j,h}^n, u_{j,h}^{n+1}, \xi_{j,h}^{n+1}) &\le C\|\nabla\xi_{j,h}^n\|^{\frac{1}{2}}\|\xi_{j,h}^n\|^{\frac{1}{2}}\|\nabla u_{j,h}^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\le C\|\nabla\xi_{j,h}^n\|^{\frac{1}{2}}\|\xi_{j,h}^n\|^{\frac{1}{2}}\|\nabla\xi_{j,h}^{n+1}\| \\
&\le C(\epsilon\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon}\|\nabla\xi_{j,h}^n\|\|\xi_{j,h}^n\|) \\
&\le C(\epsilon\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{1}{\epsilon}(\delta\|\nabla\xi_{j,h}^n\|^2 + \frac{1}{\delta}\|\xi_{j,h}^n\|) \\
&\le (\frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{\nu}{8}\|\nabla\xi_{j,h}^n\|^2) + \frac{C}{\nu^2}\|\xi_{j,h}^n\|^2,
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
b^*(U_j^n, \eta_j^{n+1}, \xi_{j,h}^{n+1}) &\le C\|\nabla U_j^n\|\|\nabla\eta_j^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\le \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla\eta_j^{n+1}\|^2\ ,
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
b^*(U_j^n, \eta_j^n, \xi_{j,h}^{n+1}) &\le C\|\nabla U_j^n\|\|\nabla\eta_j^n\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\le \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla\eta_j^n\|^2\ .
\end{aligned} \tag{5.12}
$$

The next term, $b^*(U_j^n, \xi_{j,h}^n, \xi_{j,h}^{n+1})$, is the key term in the error analysis. Note that by skew symmetry and Lemma 1

$$
\begin{aligned}
b^*(U_j^n, \xi_{j,h}^n, \xi_{j,h}^{n+1}) = b^*(U_j^n, \xi_{j,h}^n - \xi_{j,h}^{n+1}, \xi_{j,h}^{n+1}) = \\
= -(U_j^n\cdot\nabla\xi_{j,h}^{n+1}, \xi_{j,h}^n - \xi_{j,h}^{n+1}) - \frac{1}{2}(\nabla\cdot U_j^n, \left(\xi_{j,h}^n - \xi_{j,h}^{n+1}\right)\cdot\xi_{j,h}^{n+1}).
\end{aligned}
$$

Using standard estimates for each additive term (with $\varepsilon = 1/(2\triangle t)$) and an inverse inequality gives

$$
\begin{aligned}
b^*(U_j^n, \xi_{j,h}^n, \xi_{j,h}^{n+1}) &\le C\|\nabla U_j^n\|\|\nabla\xi_{j,h}^{n+1}\|\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\|_{1/2} + C\|\nabla\cdot U_j^n\|\|\xi_{j,h}^{n+1}\cdot(\xi_{j,h}^{n+1} - \xi_{j,h}^n)\| \\
&\le C\|\nabla U_j^n\|\|\nabla\xi_{j,h}^{n+1}\|\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\|_{1/2} + C\|\nabla\cdot U_j^n\|\|\nabla\xi_{j,h}^{n+1}\|\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\|_{1/2} \\
&\le C\|\nabla U_j^n\|\|\nabla\xi_{j,h}^{n+1}\|\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\|_{1/2} \le C\|\nabla U_j^n\|\|\nabla\xi_{j,h}^{n+1}\|h^{-1/2}\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\| \\
&\le \frac{1}{4\triangle t}\|\xi_{j,h}^{n+1} - \xi_{j,h}^n\|^2 + \left(C\frac{\triangle t}{h}\|\nabla U_j^n\|^2\right)\|\nabla\xi_{j,h}^{n+1}\|^2.
\end{aligned} \tag{5.13}
$$

For the next terms we have

$$
\begin{aligned}
b^*(U_j^n, u_j^{n+1} - u_j^n, \xi_{j,h}^{n+1}) &\le C\|\nabla U_j^n\|\|\nabla(u_j^{n+1} - u_j^n)\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\le \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla(u_j^{n+1} - u_j^n)\|^2 \\
&\le \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{C\Delta t}{\nu}\|\nabla U_j^n\|^2(\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|^2\, dt)\ .
\end{aligned} \tag{5.14}
$$

Next, consider the pressure term. Since $\xi_{j,h}^{n+1} \in V_h$ we have

$$
\begin{aligned}
(p_j^{n+1}, \nabla\cdot\xi_{j,h}^{n+1}) &= (p_j^{n+1} - q_{j,h}^{n+1}, \nabla\cdot\xi_{j,h}^{n+1}) \\
&\leq \|p_j^{n+1} - q_{j,h}^{n+1}\|\|\nabla\cdot\xi_{j,h}^{n+1}\| \\
&\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + C\nu^{-1}\|p_j^{n+1} - q_{j,h}^{n+1}\|^2 \ .
\end{aligned}
\tag{5.15}
$$

The other terms, are bounded as

$$
\begin{aligned}
(\frac{\eta_j^{n+1}-\eta_j^n}{\Delta t}, \xi_{j,h}^{n+1}) &\leq C\|\frac{\eta_j^{n+1}-\eta_j^n}{\Delta t}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq C\nu^{-1}\|\frac{\eta_j^{n+1}-\eta_j^n}{\Delta t}\|^2 + \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 \\
&\leq C\nu^{-1}\|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \eta_{j,t}\ dt\|^2 + \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 \\
&\leq \frac{C}{\nu\Delta t}\int_{t^n}^{t^{n+1}} \|\eta_{j,t}\|^2\ dt + \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 \ .
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
\nu(\nabla\eta_j^{n+1}, \nabla\xi_{j,h}^{n+1}) &\leq \nu\|\nabla\eta_j^{n+1}\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq C\nu\|\nabla\eta_j^{n+1}\|^2 + \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 \ .
\end{aligned}
\tag{5.17}
$$

Finally,

$$
\begin{aligned}
Intp(u_j^{n+1}; \xi_{j,h}^{n+1}) &= (\frac{u_j^{n+1}-u_j^n}{\Delta t} - u_{j,t}(t^{n+1}), \xi_{j,h}^{n+1}) \\
&\leq C\|\frac{u_j^{n+1}-u_j^n}{\Delta t} - u_{j,t}(t^{n+1})\|\|\nabla\xi_{j,h}^{n+1}\| \\
&\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{C}{\nu}\|\frac{u_j^{n+1}-u_j^n}{\Delta t} - u_{j,t}(t^{n+1})\|^2 \\
&\leq \frac{\nu}{64}\|\nabla\xi_{j,h}^{n+1}\|^2 + \frac{C\Delta t}{\nu}\int_{t^n}^{t^{n+1}} \|u_{j,tt}\|^2 dt \ .
\end{aligned}
\tag{5.18}
$$

Combining, we now have the following inequality:

$$
\begin{aligned}
&\frac{1}{\Delta t}(\frac{1}{2}||\xi_{j,h}^{n+1}||^2 - \frac{1}{2}||\xi_{j,h}^{n}||^2 + \frac{1}{4}\|\xi_{j,h}^{n+1} - \xi_{j,h}^{n}\|^2) + \frac{\nu}{8}(\|\nabla \xi_{j,h}^{n+1}\|^2 - \|\nabla \xi_{j,h}^{n}\|^2) \\
&\qquad + (\frac{\nu}{4} - C\frac{\triangle t}{h}\|\nabla U_j^n\|^2)\|\nabla \xi_{j,h}^{n+1}\|^2 \\
&\leq C\nu^{-1}\|\nabla u_j^{n+1}\|^2\|\nabla \eta_j^{n+1}\|^2 + \frac{C\Delta t}{\nu}(\int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|^2 dt)\|\nabla u_{j,h}^{n+1}\|^2 \\
&\quad + C\nu^{-1}\|\nabla \eta_j^n\|^2\|\nabla u_{j,h}^{n+1}\|^2 + \frac{C}{\nu^2}\|\xi_{j,h}^n\|^2 + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla \eta_j^{n+1}\|^2 \\
&\quad + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla \eta_j^n\|^2 \\
&\quad + \frac{C\Delta t}{\nu}\|\nabla U_j^n\|^2(\int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|^2 dt) + C\nu^{-1}\|p_j^{n+1} - q_{j,h}^{n+1}\|^2 \\
&\quad + \frac{C}{\nu\Delta t}\int_{t^n}^{t^{n+1}} \|\eta_{j,t}\|^2 dt + C\nu\|\nabla \eta_j^{n+1}\|^2 + \frac{C\Delta t}{\nu}\int_{t^n}^{t^{n+1}} \|u_{j,tt}\|^2 dt\ .
\end{aligned} \tag{5.19}
$$

By the timestep condition $\frac{\nu}{4} - C\frac{\triangle t}{h}\|\nabla U_j^n\|^2 \geq C\nu > 0$. Take the sum of (5.19) from n=1 to n=N-1 and multiply through by Δt

$$
\begin{aligned}
&\frac{1}{2}||\xi_{j,h}^{N}||^2 + \frac{\nu\Delta t}{8}||\nabla \xi_{j,h}^{N}||^2 + \sum_{n=0}^{N-1}\frac{1}{4}\|\xi_{j,h}^{n+1} - \xi_{j,h}^{n}\|^2 + C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla \xi_{j,h}^{n+1}\|^2 \\
&\leq \frac{1}{2}||\xi_{j,h}^{0}||^2 + \frac{\nu\Delta t}{8}||\nabla \xi_{j,h}^{0}||^2 + \Delta t\sum_{n=0}^{N-1}\frac{C}{\nu^2}\|\xi_{j,h}^n\|^2 \\
&\quad + \Delta t\sum_{n=0}^{N-1}\{C\nu^{-1}\|\nabla u_j^{n+1}\|^2\|\nabla \eta_j^{n+1}\|^2 \\
&\quad + \frac{C\Delta t}{\nu}(\int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|^2\ dt)\|\nabla u_j^{n+1}\|^2 + C\nu^{-1}\|\nabla \eta_j^n\|^2\|\nabla u_{j,h}^{n+1}\|^2 \\
&\quad + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla \eta_j^{n+1}\|^2 + C\nu^{-1}\|\nabla U_j^n\|^2\|\nabla \eta_j^n\|^2 \\
&\quad + \frac{C\Delta t}{\nu}\|\nabla U_j^n\|^2(\int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|^2 dt) + C\nu^{-1}\|p_j^{n+1} - q_{j,h}^{n+1}\|^2 \\
&\quad + \frac{C}{\nu\Delta t}\int_{t^n}^{t^{n+1}} \|\eta_{j,t}\|^2 dt + C\nu\|\nabla \eta_j^{n+1}\|^2 + \frac{C\Delta t}{\nu}\int_{t^n}^{t^{n+1}} \|u_{j,tt}\|^2\ dt\}\ .
\end{aligned} \tag{5.20}
$$

Applying interpolation inequalities gives

$$
\begin{aligned}
&\frac{1}{2}||\xi_{j,h}^{N}||^2+\sum_{n=0}^{N-1}\frac{1}{4}\|\xi_{j,h}^{n+1}-\xi_{j,h}^{n}\|^2+\frac{\nu\Delta t}{8}||\nabla\xi_{j,h}^{N}||^2+C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla\xi_{j,h}^{n+1}\|^2\\
&\leq\frac{1}{2}||\xi_{j,h}^{0}||^2+\frac{\nu\Delta t}{8}||\nabla\xi_{j,h}^{0}||^2+\Delta t\sum_{n=0}^{N-1}\frac{C}{\nu^2}\|\xi_{j,h}^{n}\|^2\\
&+C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{\infty,0}^2|||u_j|||_{2,k+1}^2+C\frac{\Delta t^2}{\nu}|||\nabla u_j|||_{\infty,0}^2|||\nabla u_{j,t}|||_{2,0}^2\\
&+C\frac{h^{2k}}{\nu^2}|||u_j|||_{2,k+1}^2+C\frac{h^{2k+1}}{\Delta t}|||u_j|||_{2,k+1}^2+Ch\Delta t|||\nabla u_{j,t}|||_{2,0}^2\\
&+C\frac{h^{2s+2}}{\nu}|||p_j|||_{2,s+1}^2+C\frac{h^{2k+2}}{\nu}|||u_{j,t}|||_{2,k+1}^2\\
&+C\nu h^{2k}|||u_j|||_{2,k+1}^2+\frac{C\Delta t^2}{\nu}|||u_{j,tt}|||_{2,0}^2\ .
\end{aligned}
\tag{5.21}
$$

The next step will be the application of the discrete Gronwall inequality (Girault and Raviart [16], p. 176).

$$
\begin{aligned}
&\frac{1}{2}\|\xi_{j,h}^{N}\|^2+\sum_{n=0}^{N-1}\frac{1}{4}\|\xi_{j,h}^{n+1}-\xi_{j,h}^{n}\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^{N}\|^2+C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla\xi_{j,h}^{n+1}\|^2\\
&\leq exp(\frac{CN\Delta t}{\nu^2})\{\frac{1}{2}\|\xi_{j,h}^{0}\|^2+\frac{\nu\Delta t}{8}||\nabla\xi_{j,h}^{0}||^2+C\frac{h^{2k}}{\nu}|||\nabla u_j|||_{\infty,0}^2|||u_j|||_{2,k+1}^2\\
&+C\frac{\Delta t^2}{\nu}|||\nabla u_j|||_{\infty,0}^2|||\nabla u_{j,t}|||_{2,0}^2+C\frac{h^{2k}}{\nu^2}|||u_j|||_{2,k+1}^2\\
&+C\frac{h^{2k+1}}{\Delta t}|||u_j|||_{2,k+1}^2+Ch\Delta t|||\nabla u_{j,t}|||_{2,0}^2\\
&+C\frac{h^{2s+2}}{\nu}|||p_j|||_{2,s+1}^2+C\frac{h^{2k+2}}{\nu}|||u_{j,t}|||_{2,k+1}^2\\
&+C\nu h^{2k}|||u_j|||_{2,k+1}^2+\frac{C\Delta t^2}{\nu}|||u_{j,tt}|||_{2,0}^2\}\ .
\end{aligned}
\tag{5.22}
$$

Recall that $e_j^n=\eta_j^n+\xi_{j,h}^n$. Use the triangle inequality on the error equation to split the error terms into terms of η_j^n and $\xi_{j,h}^n$.

$$
\begin{aligned}
&\frac{1}{2}\|e_j^N\|^2+\sum_{n=0}^{N-1}\frac{1}{4}\|e_j^{n+1}-e_j^n\|^2+\frac{\nu\Delta t}{8}\|\nabla e_j^N\|^2+C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla e_j^{n+1}\|^2\\
&\leq\frac{1}{2}\|\xi_{j,h}^{N}\|^2+\sum_{n=0}^{N-1}\frac{1}{4}\|\xi_{j,h}^{n+1}-\xi_{j,h}^{n}\|^2+\frac{\nu\Delta t}{8}\|\nabla\xi_{j,h}^{N}\|^2+C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla\xi_{j,h}^{n+1}\|^2\\
&+\frac{1}{2}\|\eta_j^N\|^2+\sum_{n=0}^{N-1}\frac{1}{4}\|\eta_j^{n+1}-\eta_j^n\|^2+\frac{\nu\Delta t}{8}\|\nabla\eta_{j,h}^N\|^2+C\Delta t\sum_{n=0}^{N-1}\nu\|\nabla\eta_j^{n+1}\|^2\ .
\end{aligned}
\tag{5.23}
$$

Applying inequality (5.21), using the previous bounds for η_j^n terms,and absorbing constants into a new constant C, we have Theorem 2. □

6. Numerical Experiments

We present numerical experiments of the algorithm (BEFE-Ensemble). Our initial tests are simple with only $p = 2$ ensemble members verifying accuracy on an academic problem and the various stability conditions on a more interesting one. For the first test, using a perturbation of the Green-Taylor vortex, [17], [18], that leads to perturbed initial conditions and boundary conditions, we confirm the predicted convergence rates. Next we study a rotating flow involving offset cylinders. Adapting the timestep we show that stability is preserved, as predicted and measured by energy, enstrophy, and aggregate angular momentum. As the Reynolds number is increased, the rate of separation of nearby trajectories in the continuous problem is expected to increase, leading to a decrease in Δt under (1.3). This is indeed observed. We use FreeFEM++ [19], with Taylor-Hood elements (continuous piecewise quadratic polynomials for the velocity and continuous linear polynomials for the pressure) in all tests.

6.1. **Convergence Experiment.** The Green-Taylor vortex is a commonly used problem for convergence rates, since the true solution is known, e.g. [20], [21], [22], [23], [24]. In $\Omega = (0,1)^2$, the exact solution of the Green-Taylor vortex is

$$
\begin{aligned}
u(x,y,t) &= -\cos(\omega\pi x)\sin(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}\,,\\
v(x,y,t) &= \sin(\omega\pi x)\cos(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}\,,\\
p(x,y,t) &= -\frac{1}{4}(\cos(2\omega\pi x)+\cos(2\omega\pi y))e^{-4\omega^2\pi^2 t/\tau}\,.
\end{aligned}
$$

Given $\tau = Re$, this is a solution of the NSE consisting of an $\omega\times\omega$ array of oppositely signed vortices that decay as $t \to \infty$. The initial condition is

$$u^0 = (-\cos(\omega\pi x)\sin(\omega\pi y), \sin(\omega\pi x)\cos(\omega\pi y))^T\ .$$

We take $\omega = 1$, $\tau = Re = 100$, $T = 1$, $h = 1/m$ and $\Delta t/h = 1/10$. Convergence rates are calculated from the error at two successive values of h in the usual manner by postulating $e(h) = Ch^\beta$ and solving for β via $\beta = \ln(e(h_1)/e(h_2))/\ln(h_1/h_2)$. The boundary condition on the problem is taken to be inhomogeneous Dirichlet: $u_h = u_{true}$, on $\partial\Omega$.

Generation of the initial conditions. The generation of ensemble members is necessarily dependent on the application and the question asked. In the first test, we consider an ensemble of two members u_1, u_2, which are the solutions corresponding to two different initial conditions $u_1^0 = (1+\epsilon_1)u^0$, $u_2^0 = (1+\epsilon_2)u^0$ respectively. This simple choice implies u_1, u_2 have a closed form $u_{1,2} = (1+\epsilon_{1,2})u_{exact}$ so errors can be calculated. Here $\epsilon_1 = 10^{-3}$, $\epsilon_2 = -10^{-3}$ are small perturbations on the initial condition u^0. Denote $u_{exact} = (u(x,y,t), v(u,y,t))^T$ and $p_{exact} = p(x,y,t)$. Adjusting body forces and boundary conditions for each ensemble member, we have $u_1 = (1+\epsilon_1)u_{exact}$, $p_1 = (1+\epsilon_1)^2 p_{exact}$, $u_2 = (1+\epsilon_2)u_{exact}$, $p_2 = (1+\epsilon_2)^2 p_{exact}$, see [25] for explanations. From the tables we can see the convergence rate for u_1 and u_2 is first order as predicted. u_{ave} is expected to converge to $0.5*(u_1+u_2)$, which in this test is equal to u_{exact}.

m	$\|u_1 - u_{1,h}\|_{\infty,0}$	rate	$\|\nabla u_1 - \nabla u_{1,h}\|_{2,0}$	rate
$\frac{3}{2} \cdot 27$	$8.45557 \cdot 10^{-6}$	–	$2.41940 \cdot 10^{-3}$	–
$(\frac{3}{2})^2 \cdot 27$	$2.26251 \cdot 10^{-6}$	3.2515	$9.21029 \cdot 10^{-4}$	2.3819
$(\frac{3}{2})^3 \cdot 27$	$1.09082 \cdot 10^{-6}$	1.7993	$3.65861 \cdot 10^{-4}$	2.2770
$(\frac{3}{2})^4 \cdot 27$	$6.90354 \cdot 10^{-7}$	1.1283	$1.56884 \cdot 10^{-4}$	2.0883
$(\frac{3}{2})^5 \cdot 27$	$4.57036 \cdot 10^{-7}$	1.0172	$6.85081 \cdot 10^{-5}$	2.0435

TABLE 1. Errors and convergence rates for the first ensemble member

m	$\|u_2 - u_{2,h}\|_{\infty,0}$	rate	$\|\nabla u_2 - \nabla u_{2,h}\|_{2,0}$	rate
$(\frac{3}{2}) \cdot 27$	$8.42864 \cdot 10^{-6}$	–	$2.41223 \cdot 10^{-3}$	–
$(\frac{3}{2})^2 \cdot 27$	$2.25806 \cdot 10^{-6}$	3.2484	$9.18647 \cdot 10^{-4}$	2.3810
$(\frac{3}{2})^3 \cdot 27$	$1.09000 \cdot 10^{-6}$	1.7963	$3.65017 \cdot 10^{-4}$	2.2763
$(\frac{3}{2})^4 \cdot 27$	$6.89994 \cdot 10^{-7}$	1.1277	$1.56547 \cdot 10^{-4}$	2.0879
$(\frac{3}{2})^5 \cdot 27$	$4.56809 \cdot 10^{-7}$	1.0171	$6.83669 \cdot 10^{-5}$	2.0433

TABLE 2. Errors and convergence rates for the second ensemble member

m	$\|u_{exact} - u_{ave,h}\|_{\infty,0}$	rate	$\|\nabla u_{exact} - \nabla u_{ave,h}\|_{2,0}$	rate
$(\frac{3}{2}) \cdot 27$	$8.44211 \cdot 10^{-6}$	–	$2.41582 \cdot 10^{-3}$	–
$(\frac{3}{2})^2 \cdot 27$	$2.26028 \cdot 10^{-6}$	3.2500	$9.19838 \cdot 10^{-4}$	2.3815
$(\frac{3}{2})^3 \cdot 27$	$1.09041 \cdot 10^{-6}$	1.7978	$3.65439 \cdot 10^{-4}$	2.2766
$(\frac{3}{2})^4 \cdot 27$	$6.90174 \cdot 10^{-7}$	1.1280	$1.567153527 \cdot 10^{-4}$	2.0881
$(\frac{3}{2})^5 \cdot 27$	$4.56923 \cdot 10^{-7}$	1.0172	$6.84375 \cdot 10^{-5}$	2.0434

TABLE 3. Errors and convergence rates for the average of ensemble members

m	$\|\nabla p_1 - \nabla p_{1,h}\|_{2,0}$	rate	$\|\nabla p_2 - \nabla p_{2,h}\|_{2,0}$	rate
$(\frac{3}{2}) \cdot 27$	$5.93247 \cdot 10^{-2}$	–	$5.91504 \cdot 10^{-2}$	–
$(\frac{3}{2})^2 \cdot 27$	$3.97196 \cdot 10^{-2}$	0.9894	$.394309 \cdot 10^{-2}$	1.0002
$(\frac{3}{2})^3 \cdot 27$	$2.64583 \cdot 10^{-2}$	1.0020	$2.59944 \cdot 10^{-2}$	1.0276
$(\frac{3}{2})^4 \cdot 27$	$1.81013 \cdot 10^{-2}$	0.9362	$1.73880 \cdot 10^{-2}$	0.9917
$(\frac{3}{2})^5 \cdot 27$	$1.26636 \cdot 10^{-2}$	0.8811	$1.15926 \cdot 10^{-2}$	0.9999

TABLE 4. Errors and convergence rates for pressure

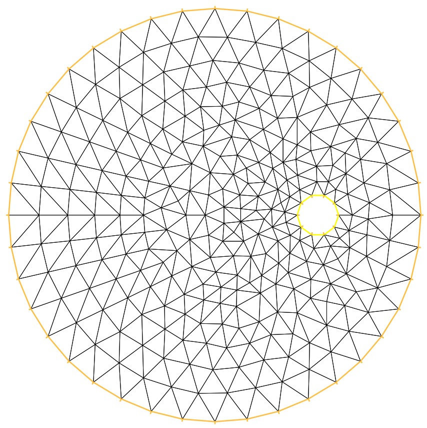
6.2. **Stability Verification.** We test the timestep condition for stability of our algorithm on a problem of flow between two offset circles, motivated by the classic problem of flow between rotating cylinders. The domain is a disk with a smaller off center obstacle inside. Let $r_1 = 1$, $r_2 = 0.1$, $c = (c_1, c_2) = (\frac{1}{2}, 0)$, then the domain is given by

$$\Omega = \{(x, y) : x^2 + y^2 \leq r_1^2 \text{ and } (x - c_1)^2 + (y - c_2)^2 \geq r_2^2\}.$$

The flow is driven by a counterclockwise rotational body force

$$f(x, y, t) = (-4y * (1 - x^2 - y^2), 4x * (1 - x^2 - y^2))^T$$

with no-slip boundary conditions suppressed on both circles. Note that the rotational force $f \equiv 0$ at the outer circle so most of the interesting structures are expected to be due to the interaction of the flow with the inner circle. The flow



rotates about $(0, 0)$ and interacts with the immersed circle $(x-c_1)^2+(y-c_2)^2 \leq r_2^2$ which induces a von *Kármán* vortex street. This vortex street rotates and itself re-interacts with the immersed circle, creating more complex structures. The mesh is parameterized by the number of mesh points (n=40) around the outer circle and the mesh points (m=10) around the immersed circle, and extended to all of Ω as a Delaunay mesh. As Re increases, this flow will be underresolved. Thus we give tests of stability but neither expect nor test accuracy.

Generation of the initial conditions. In the second test, we generate perturbations of the initial conditions that are incompressible and satisfy no-slip boundary conditions by solving steady Stokes problem on the same geometry with body forces perturbed. Let

$$f_1(x, y, t) = f(x, y, t) + \epsilon_1 * (sin(3\pi x)sin(3\pi y), cos(3\pi x)cos(3\pi y))^T,$$

$$f_2(x, y, t) = f(x, y, t) + \epsilon_2 * (sin(3\pi x)sin(3\pi y), cos(3\pi x)cos(3\pi y))^T,$$

where $\epsilon_1 = 10^{-3}$, $\epsilon_2 = -10^{-3}$. In this way, we generate $u_j^0, j = 1, 2$, satisfying the no-slip condition. The solutions of the steady Stokes problem corresponding to these two body forces are our perturbed initial conditions, which are denoted by u_1^0 and u_2^0.

We solve Navier-Stokes equations using our algorithm with these two initial conditions, which gives us u_1, u_2, and u_{ave}. We also solve the steady Stokes problem using $f(x, y, t)$ and get the initial condition u_0^0 to do a comparison. The solution of NSE with u_0^0 is denoted by u_0 (marked as 'no perturbation' in the figures).

Test 1: Taking $Re = 200$, we give plots over $0 \leq t \leq 10$ of the following quantities:

$$|\text{Angular Momentum}| = |\int_\Omega \vec{x} \times \vec{u}\, d\vec{x}|$$

$$\text{Enstrophy} = \frac{1}{2}\nu\|\nabla \times \vec{u}\|^2$$

$$\text{Energy} = \frac{1}{2}\|u\|^2$$

To ensure the algorithm is stable, we first cut Δt to enforce

$$C\frac{\Delta t}{\nu h}||\nabla(u_{j,h}^n- < u_h >^n)||^2 \leq 1, \qquad (j = 1, \cdots, p) \tag{6.1}$$

In practise, the constant C can be determined by a few pre-computations. In our test, we cut Δt to enforce the condition

$$\frac{\Delta t}{h}||\nabla(u_{j,h}^n - < u_h >^n)||^2 \leq \frac{1200}{Re}, \qquad (j = 1, 2) \tag{6.2}$$

Once this condition is violated, we update the time step with $dt^{new} = dt^{old}/2$ and keep doing this until the condition is satisfied again. Note that in this first test we cut Δt but do not increase Δt. Figure 1-3 show that the condition (6.2) is, as predicted, sufficient for stability of our algorithm for $Re = 200$. Figure 4 shows a comparison of time step evolution with respect to 3d condition (6.2) and 2d condition (2d C1) with the same constant $C = \frac{1}{1200}$. Figure 5 shows snapshots of the flow, which is complex (some complexity from the flow and some from the underresolved mesh) and seems to be pulsating. Figure 6 shows snapshots of the contours ($|Vor|/|Vor|.max > 0.1$) of vorticity.

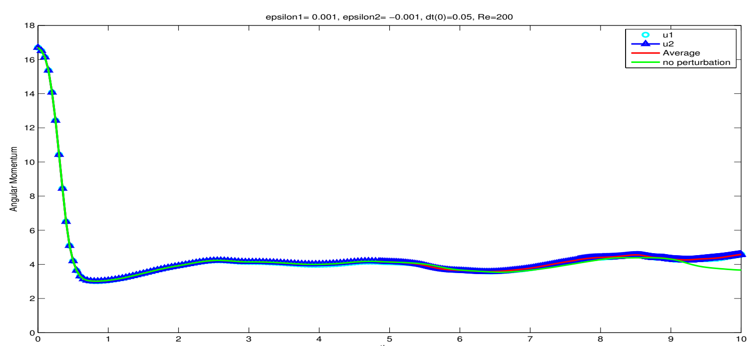


FIGURE 1. Stability: Angular Momentum, $\nu = 1/200$

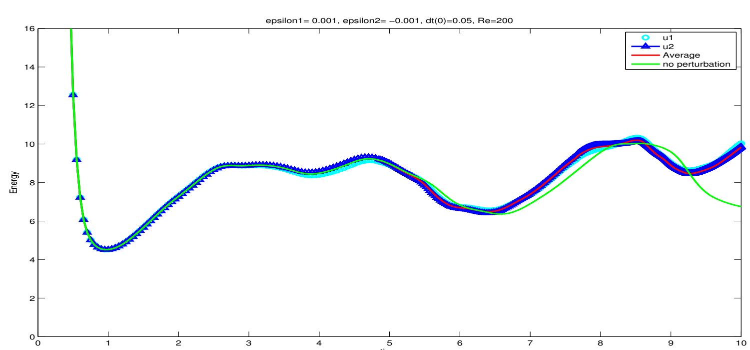


FIGURE 2. Stability: Energy with $\nu = 1/200$

Test 2: Taking $Re = 800$, we test the 2d condition (2d C1).

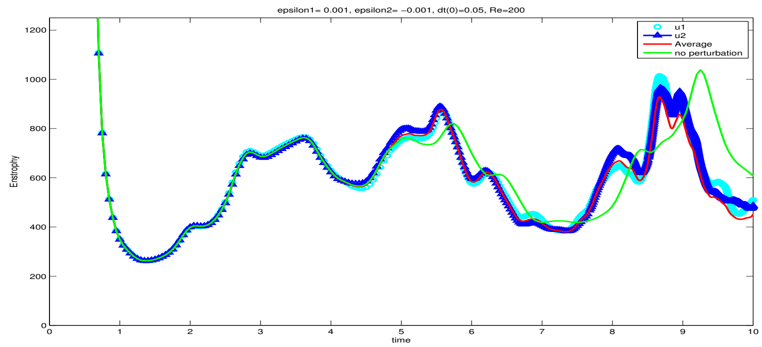


FIGURE 3. Stability: Enstrophy, $\nu = 1/200$

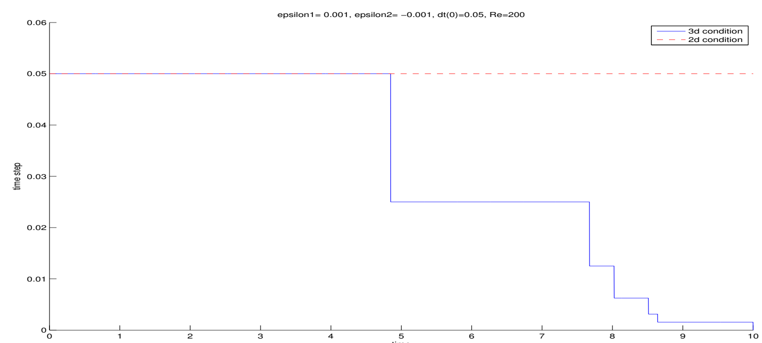


FIGURE 4. Timestep evolution, $\nu = 1/200$

Case 1: Timestep halving only.

$$|ln(h)|\Delta t||\nabla(u_{j,h}^n - < u_h >^n)||^2 \leq \frac{1200}{Re}$$

Figure 10 shows that as Re increases, Δt decreases.
Case 2: Timestep halving and doubling.

$$|ln(h)|\Delta t||\nabla(u_{j,h}^n - < u_h >^n)||^2 \leq \frac{1200}{Re}$$

$$\text{and} \qquad |ln(h)|\Delta t||\nabla(u_{j,h}^n - < u_h >^n)||^2 \geq 0.5 * \frac{1200}{Re}$$

7. CONCLUSIONS

The need for ensemble calculations arises in calculation of sensitivities by differences [4], uncertainty quantification [2], stochastic NSE simulations [26], generation of bred vectors and their use in improving forecasting skill, Kalney [1]. The most efficient way to calculate such an ensemble will vary widely depending on the application, flow, computational resources and code used. This report has presented and analyzed an algorithm for computation of an ensemble of solutions such that

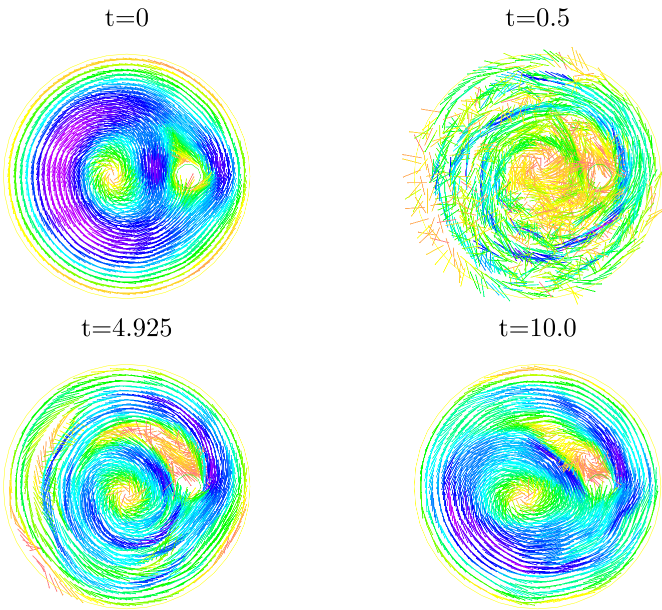


FIGURE 5. Velocity, $\nu = 1/200$

each step requires the solution of one linear system with multiple right hand sides. Stability requires a timestep condition that can easily be imposed step by step. Experimental verification of stability under the condition is given.

REFERENCES

[1] Z. TOTH AND E. KALNEY, *Ensemble forecasting at NMC: The generation of perturbations*, Bull. Amer. Meteor. Soc., 74 (1993), pp. 2317-2330.

[2] M. LEUTBECHER AND T.N. PALMER, *Ensemble forecasting*, J. Comp. Phys., 227 (2008), pp. 3515-3539.

[3] J.M. LEWIS, *Roots of ensemble forecasting*, Mon. Wea. Rev., 133 (2005), pp. 1865-1885.

[4] W.J. MARTIN AND M. XUE, *Initial condition sensitivity analysis of a mesoscale forecast using very-large ensembles*, Mon. Wea. Rev., 134 (2006), pp. 192-207.

[5] H. WAN, M.A. GIORGETTA, AND L. BONAVENTURA, *Ensemble Held-Suarez test with a spectral transform model: variability, sensitivity and convergence*, Mon. Wea. Rev., 136 (2008), pp. 1075-1092.

[6] J.D. GIRALDO AND S.G. GARCÍA GALIANO, *Building hazard maps of extreme daily rainy events from PDF ensemble, via REA method, on Senegal River Basin*, Hydrology and Earth System Sciences, 15 (2011), pp. 3605-3615.

[7] M. CARNEY, P. CUNNINGHAM, J. DOWLING AND C. LEE, *Predicting probability distributions for surf height using an ensemble of mixture density networks*, International Conference on Machine Learning, 2005.

[8] D. J. STENSRUD, *Parameterization schemes: Keys to numerical weather prediction models*, Cambridge U. Press, 2009.

[9] D.P. O'LEARY, *The block conjugate gradient algorithm and related methods*, Lin. Alg. Appl., 29 (1980), pp. 292-322.

[10] R.W. FREUND AND M. MALHOTRA, *A block QMR algorithm for non-Hermitian linear systems with multiple right-hand sides*, Lin. Alg. Appl., 254 (1997), pp. 119-157.

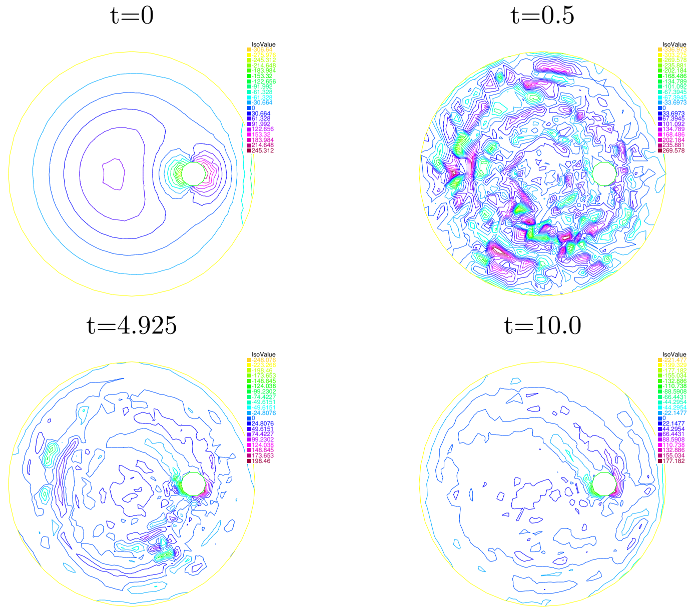


FIGURE 6. Contours of Vorticity, $\nu = 1/200$

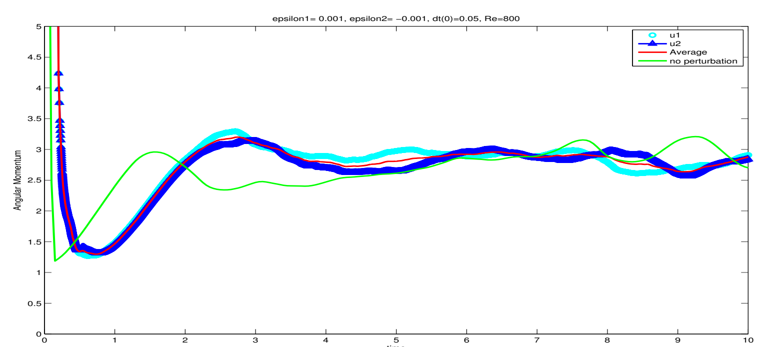


FIGURE 7. 2d-condition (timestep halving): Angular Momentum, $\nu = 1/800$

[11] E. Gallopulos and V. Simoncini, *Convergence of BLOCK GMERS and matrix polynomials*, Lin. Alg. Appl., 247 (1996), pp. 97-119.

[12] Y. T. Feng, D. R. J. Owen and D. Peric, *A block conjugate gradient method applied to linear systems with multiple right hand sides*, Comp. Meth. Appl. Mech. & Engng., 127 (1995), pp. 203-215.

[13] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 3rd edition, 2008.

[14] M.D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms*, Academic Press, 1989.

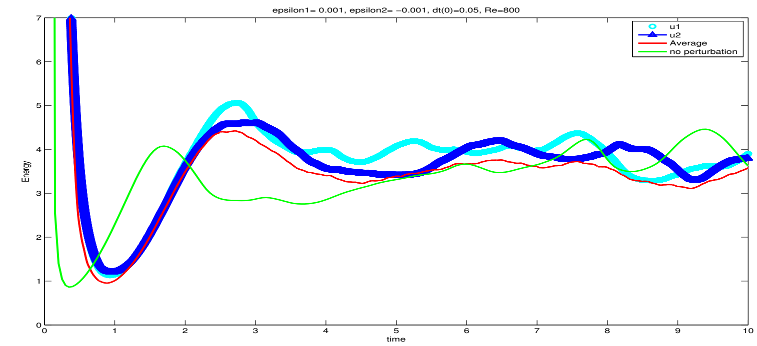


FIGURE 8. 2d-condition (timestep halving): Energy, $\nu = 1/800$

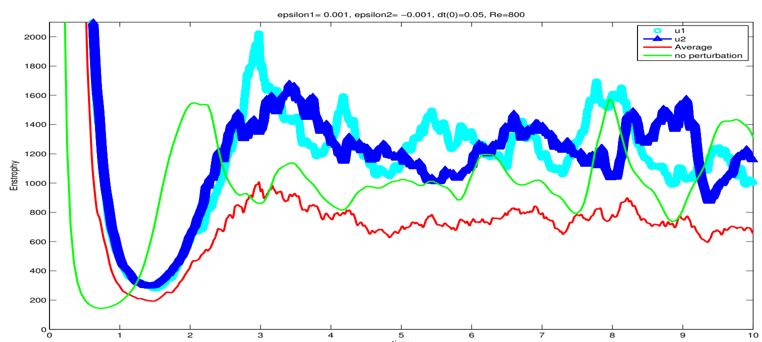


FIGURE 9. 2d-condition (timestep halving): Enstrophy, $\nu = 1/800$

[15] R. Verfürth , *On the constants in some inverse inequalities for finite element functions*, Report, Universität Bochum, May 1999.

[16] V. Girault and P. Raviart, *Finite element approximation of the Navier-Stokes equations*, Lecture Notes in Mathematics, Vol. 749, 1979.

[17] A.E. Green and G.I. Taylor, *Mechanism of the production of small eddies from larger ones*, Proc. Royal Soc. A., 158 (1937), pp. 499-521.

[18] G.I. Taylor, *On decay of vortices in a viscous fluid*, Phil. Mag., 46 (1923), pp. 671-674.

[19] F. Hecht and O. Pironneau, *FreeFEM++*, http://www.freefem.org.

[20] A.J. Chorin, *Numerical solution for the Navier-Stokes equations*, Math. Comp., 22 (1968), pp. 745-762.

[21] D. Tafti, *Comparison of some upwind-biased high-order formulations with a second order central-difference scheme for time integration of the incompressible Navier-Stokes equations*, Comput. & Fluids, 25 (1996), pp. 647-665.

[22] V. John and W. Layton, *Analysis of numerical errors in large eddy simulation*, SIAM J. Numer. Anal., 40 (2002), pp. 995-1020.

[23] D. Barbato, L.C. Berselli, and C.R. Grisanti, *Analytical and numerical results for the rational large eddy simulation model*, J. Math. Fluid Mech., 9 (2007), pp. 44-74.

[24] L.C. Berselli, *On the large eddy simulation of the Taylor-Green vortex*, J. Math. Fluid Mech., 7 (2005), pp. S164-S191.

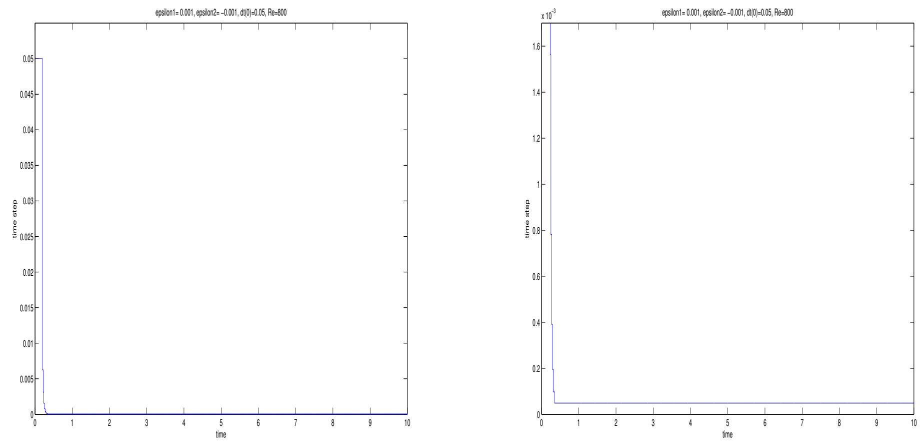


FIGURE 10. Timestep Halving: Timestep evolution (left), Zoom in (right), $\nu = 1/800$

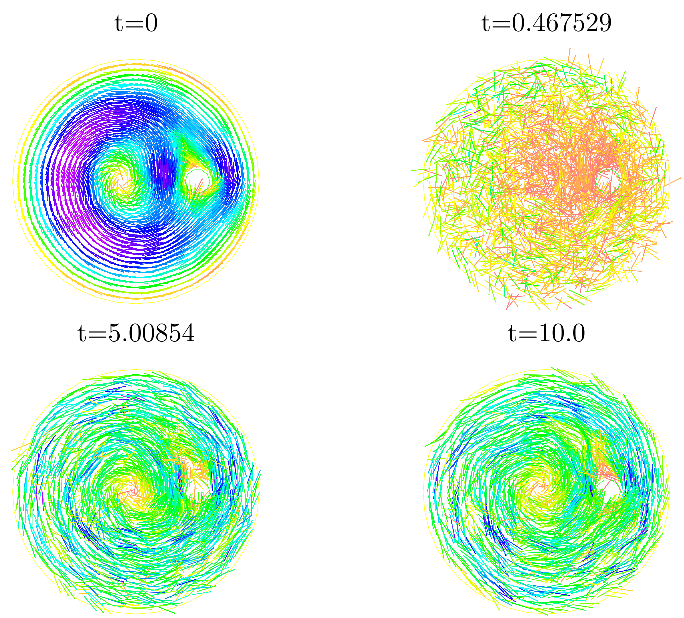


FIGURE 11. Velocity (timestep halving), $\nu = 1/800$

[25] O. Walsh, *Eddy solutions of the Navier-Stokes equations*, pp. 306-309, in: *The Navier-Stokes Equations II - Theory and Numerical Methods*, Lecture Notes in Mathematics, Volume 1530, 1992.

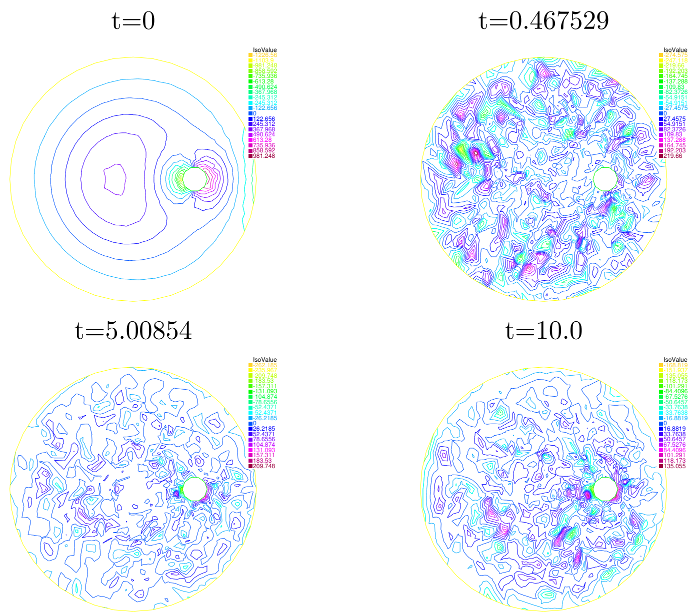


FIGURE 12. Contours of Vorticity (timestep halving), $\nu = 1/800$

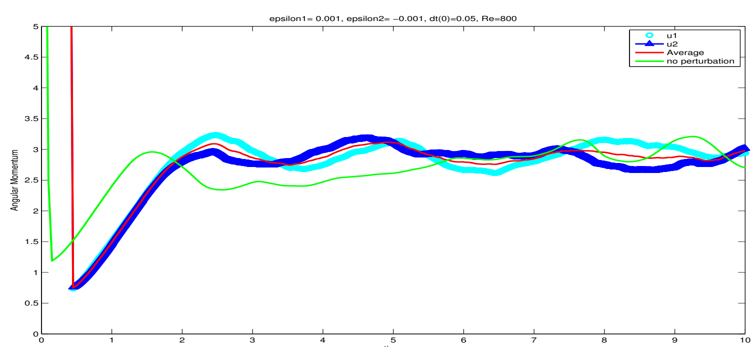


FIGURE 13. 2d-condition (timestep halving and doubling): Angular Momentum, $\nu = 1/800$

[26] L. MORICONI AND R. ROSA, *Theoretical aspects of homogeneous isotropic turbulence*, J. Braz. Soc. Mech. Sci. & Eng., 26 (2004), pp. 391-399.

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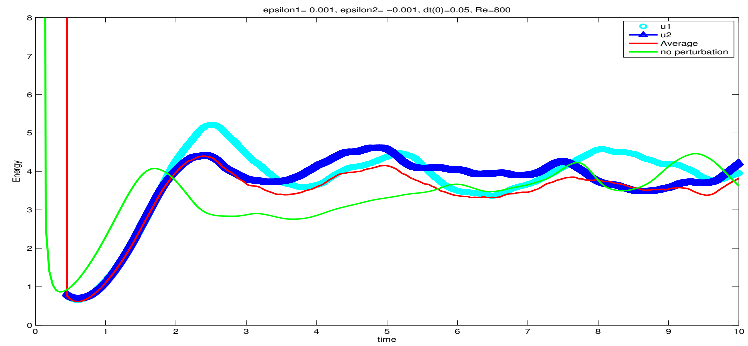

FIGURE 14. 2d-condition(timestep halving and doubling): Energy, $\nu = 1/800$

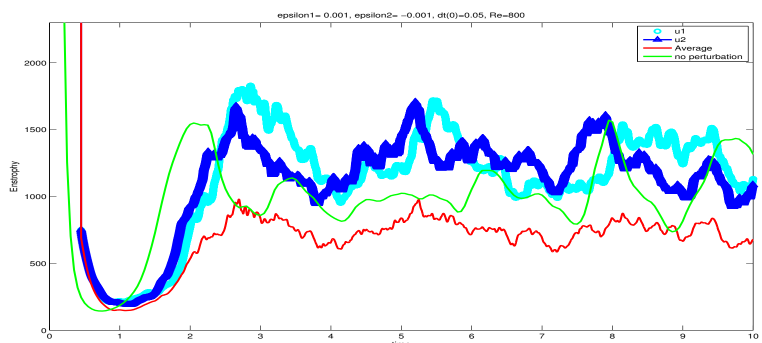

FIGURE 15. 2d-condition(timestep halving and doubling): Enstrophy, $\nu = 1/800$

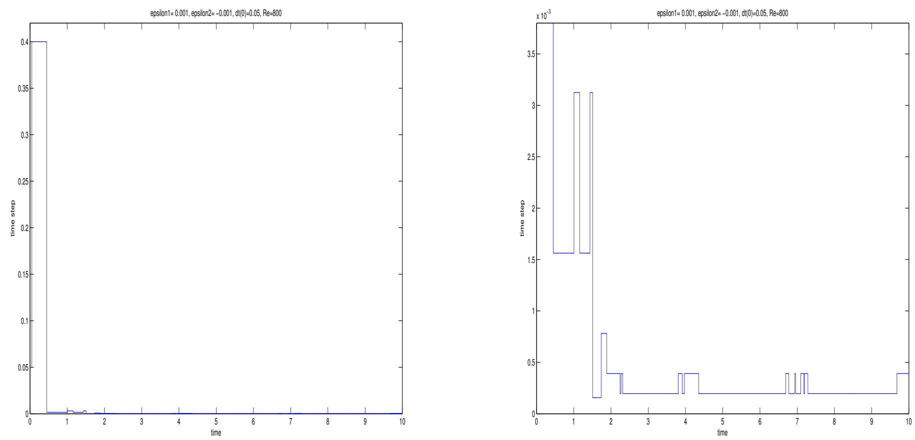

FIGURE 16. Timestep Halving and Doubling: Timestep evolution, $\nu = 1/800$,

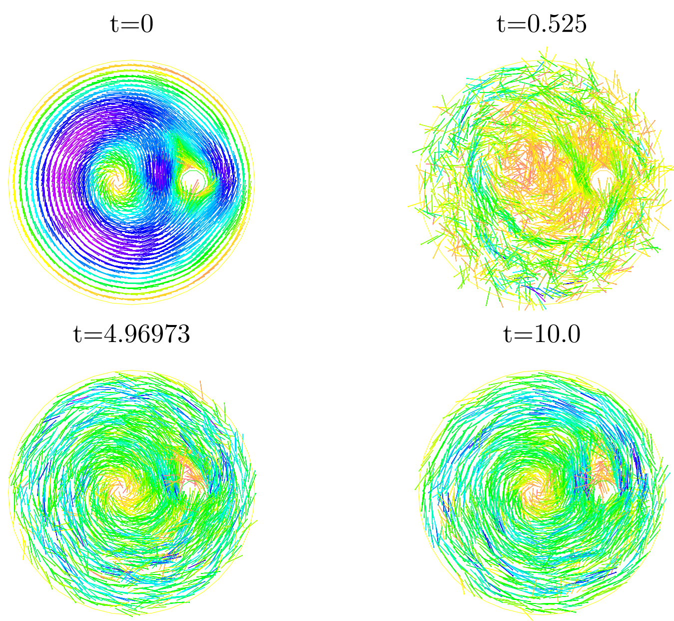

FIGURE 17. 2d-condition (timestep halving and doubling): Velocity, $\nu = 1/800$

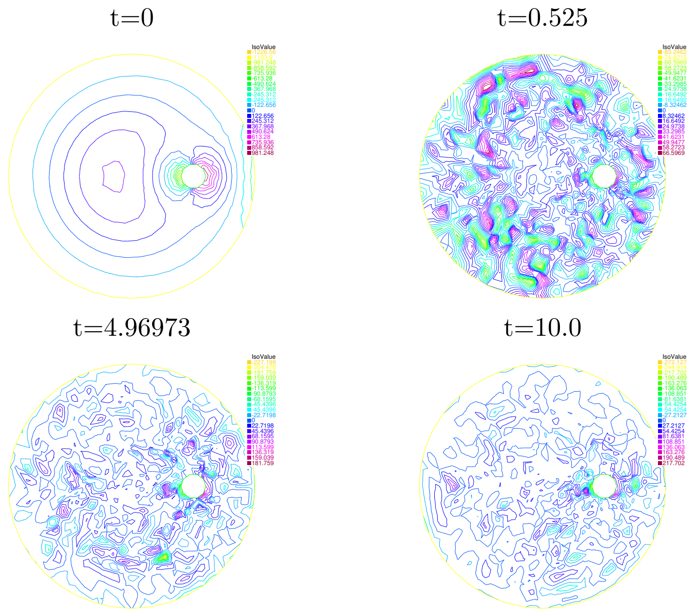

FIGURE 18. 2d-condition (timestep halving and doubling): Contours of Vorticity, $\nu = 1/800$