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NUMERICAL ANALYSIS OF A FINITE ELEMENT, CRANK-NICOLSON DISCRETIZATION FOR MHD FLOWS AT SMALL MAGNETIC REYNOLDS NUMBERS

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Abstract. We consider the finite element method for time dependent MHD flow at small magnetic Reynolds number. We make a second (and common) simplification in the model by assuming the time scales of the electrical and magnetic components are such that the electrical field responds instantaneously to changes in the fluid motion. This report gives a comprehensive error analysis for both the semi-discrete and a fully discrete approximation. Finally, the effectiveness of the method is illustrated in several numeral experiments.

Key Words. Navier-Stokes, MHD, finite element, Crank-Nicolson

1. Introduction

Magnetohydrodynamics (MHD) is the theory of macroscopic interaction of electrically conducting fluid and electromagnetic fields. Many interesting MHD-flows involve a viscous, incompressible, electrically conducting fluid that interacts with an electromagnetic field. The governing equations for these MHD flows are the Navier-Stokes equations coupled with the pre-Maxwell equations (via the Lorentz force and Ohm's Law). The resulting system of equations (see e.g. chapter 2 in [22]) often requires an unrealistic amount of computing power and storage to properly resolve the flow details. A simplification of the usual MHD equations can be made by noting that most terrestrial applications involve small R_m; e.g. most industrial flows involving liquid metal have $R_m < 10^{-2}$. Moreover, it is customary to solve a

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quasi-static approximation when an external magnetic field is present R_m is small since the time scale of the fluid velocity is much shorter than that of the electromagnetic field [3]. We provide herein a stability and convergence analysis of a fully discrete finite element discretization for time-dependent MHD flow at a small magnetic Reynolds number and under a quasi-static approximation. Magnetic damping of jets, vortices, and turbulence are several applications, [3], [18], [21], [23].

Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). Let $R_m = UL/\eta > 0$ where U, L are the characteristic speed and length of the problem, $\eta > 0$ is the magnetic diffusivity. The dimensionless quasi-static MHD model is given by: Given time $T > 0$, body force $\mathbf{f}$, interaction parameter $N > 0$, Hartmann number $M > 0$, and setting $\Omega_T := [0,T] \times \Omega$, find velocity $\mathbf{u} : \Omega_T \to \mathbb{R}^d$, pressure $p : \Omega_T \to \mathbb{R}$, electric current density $\mathbf{j} : \Omega_T \to \mathbb{R}^d$, magnetic field $\mathbf{B} : \Omega_T \to \mathbb{R}^d$, and electric potential $\phi : \Omega_T \to \mathbb{R}$ satisfying:

$$\begin{array}{rclrcl} N^{-1}\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) & = & \mathbf{f} + M^{-2}\Delta\mathbf{u} - \nabla p + \mathbf{j}\times\mathbf{B}, & \nabla\cdot\mathbf{u} & = & 0 \\ -\nabla\phi + \mathbf{u}\times\mathbf{B} & = & \mathbf{j}, & \nabla\cdot\mathbf{j} & = & 0 \\ \nabla\times\mathbf{B} & = & R_m\mathbf{j}, & \nabla\cdot\mathbf{B} & = & 0 \end{array} \tag{1}$$

subject to boundary and initial conditions

$$\begin{array}{ll} \mathbf{u}(\mathbf{x},t) = 0, & \forall(\mathbf{x},t) \in \partial\Omega\times[0,T] \\ \phi(\mathbf{x},t) = 0, & \forall(\mathbf{x},t) \in \partial\Omega\times[0,T] \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \forall\mathbf{x}\in\Omega \end{array} \tag{2}$$

where $\mathbf{u}_0 \in H_0^1(\Omega)^d$ and $\nabla\cdot\mathbf{u}_0 = 0$. When $R_m << 1$, then $\mathbf{j}$ and $\nabla\times\mathbf{B}$ in (1)(3a) decouple. Suppose further that $\mathbf{B}$ is an applied (and known) magnetic field. Then (1) reduces to the simplified MHD (SMHD) system studied herein: Find $\mathbf{u}$, p, ϕ satisfying

$$\begin{array}{rcl} N^{-1}\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) & = & \mathbf{f} + M^{-2}\Delta\mathbf{u} - \nabla p + \mathbf{B}\times\nabla\phi + (\mathbf{u}\times\mathbf{B})\times\mathbf{B} \\ \nabla\cdot\mathbf{u} & = & 0 \\ -\Delta\phi + \nabla\cdot(\mathbf{u}\times\mathbf{B}) & = & 0. \end{array} \tag{3}$$

subject to (2). This is the time dependent version of hte model first proposed by Petrerson [19].

We provide a brief overview of previous applications and analyses of MHD flows (high and low R_m) in Section 1.1. In Section 2, we present notation and a weak formulation of (3) required in our stabilty and convergence analysis. In this report we prove stability estimates for any solution u, p, ϕ to a semi-discrete and fully discrete approximation of (3) in Propositions 3.3, 4.3 respectively. We use these estimates to prove optimal error estimates in two steps:

- Semi-discrete (finite element in space), Section 3
- Fully-discrete (finite element in space, Crank-Nicolson time-stepping), Section 4

Let $h > 0$ and $\Delta t > 0$ be a representitive measure of the spatial and time discretization. We investigate the interplay between spatial and time-stepping errors. We prove that the method is unconditionally stable and, provided that $\Delta t < CReh$, the errors satisfy to ensure

$$\mathrm{error}(\mathbf{u}, p, \phi) < \mathcal{O}(h^m + \Delta t^2) \to 0, \qquad \text{as } h, \Delta t \to 0.$$

See Theorems 3.4, 4.4 and Corollaries 3.5, 4.5.

1.1. Overview of MHD models. Applications of the MHD equations arise in astronomy and geophysics as well as numerous engineering problems including liquid metal cooling of nuclear reactors [2], [7], electromagnetic casting of metals [16], controlled thermonuclear fusion and plasma confinement [24], [8], climate change forecasting and sea water propulsion [15]. Theoretical analysis and mathematical modeling of the MHD equations can be found in [10], [3]. Existence of solutions to the continuous and a discrete MHD problem without conditions on the boundary data of u is derived in [25]. Existence and uniqueness of weak solutions to the equilibrium MHD equations is proven by Gunzburger, Meir, and Peterson in [6]. Meir and Schmidt provide an optimal convergence estimate of a finite element discretization of the equilibrium MHD equations in [17]. To the best of our knowledge, the first rigorous numerical analysis of MHD problems was conducted by Peterson [19] by considering a small R_m, steady-state, incompressible, electrically conducting fluid flow subjected to an undisturbed external magnetic field. Further developments can be found in [12], [13], [1].

2. Problem formulation

We use standard notation for Lebesgue and Sobolev spaces and their norms. Let $||\cdot||$ and $(\cdot,\cdot)$ be the L^2-norm and inner product respectively. Let $||\cdot||_{p,k} := ||\cdot||_{W_p^k(\Omega)}$ represent the $W_p^k(\Omega)$-norm and $|\cdot|_{W_p^k(\Omega)}$ the $W_p^k(\Omega)$-semi-norm. We write $H^k(\Omega) := W_2^k(\Omega)$ and $||\cdot||_k$, $|\cdot|_k$ for the corresponding norm and semi-norm. Let the context determine whether $W_p^k(\Omega)$ denotes a scalar, vector, or tensor function space. For example let $\mathbf{v} : \Omega \to \mathbb{R}^d$. Then, $\mathbf{v} \in H^1(\Omega)$ implies that $\mathbf{v} \in H^1(\Omega)^d$ and $\nabla\mathbf{v} \in H^1(\Omega)$ implies that $\nabla\mathbf{v} \in H^1(\Omega)^{d\times d}$. Write $W_q^m(0,T;W_p^k(\Omega)) = W_q^m(W_p^k)$ equipped with the standard norm. For example,

$$||\mathbf{v}||_{L^q(W_p^k)} := \begin{cases} \left(\displaystyle\int_0^T ||\mathbf{v}(\cdot,t)||_{p,k}^q dt\right)^{1/q}, & \text{if } 1 \le q < \infty \\ \text{ess sup}_{0<t<T}||\mathbf{v}(\cdot,t)||_{p,k}, & \text{if } q = \infty. \end{cases}$$

Denote the pressure, velocity, potential spaces by

$$\begin{aligned} Q &:= L_0^2(\Omega) &&= \left\{q \in L^2(\Omega) : \textstyle\int_\Omega q = 0\right\} \\ X &:= H_0^1(\Omega)^d &&= \left\{\mathbf{v} \in H^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\right\} \\ S &:= H_0^1(\Omega) &&= \left\{\psi \in H^1(\Omega) : \psi|_{\partial\Omega} = 0\right\} \end{aligned}$$

respectively. Let X^*, S^* denote the dual space of X, S respectively with norm $||\cdot||_{-1}$.

A weak formulation of (3), (2) is: Find $\mathbf{u} : [0,T] \to X$, $p : [0,T] \to Q$, and $\phi : [0,T] \to S$ for a.e. $t \in (0,T]$ satisfying

$$N^{-1}\left((\mathbf{u}_t, \mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v})\right) + M^{-2}(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v})$$
$$+ (\mathbf{u}\times\mathbf{B}, \mathbf{v}\times\mathbf{B}) - (\nabla\phi, \mathbf{v}\times\mathbf{B}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X \tag{4}$$

$$(\nabla\cdot\mathbf{u}, q) = 0, \quad \forall q \in Q \tag{5}$$

$$(\nabla\phi, \nabla\psi) - (\mathbf{u}\times\mathbf{B}, \nabla\psi) = 0, \quad \forall \psi \in S \tag{6}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega. \tag{7}$$

We obtain (4) from (3)(1a) by applying the following identities.

Lemma 2.1. *For all* $\mathbf{u}$, $\mathbf{v} \in L^2(\Omega)$, $\mathbf{B} \in L^\infty(\Omega)$, $\phi \in H^1(\Omega)$,

$$((\mathbf{u}\times\mathbf{B})\times\mathbf{B}, \mathbf{v}) = -(\mathbf{u}\times\mathbf{B}, \mathbf{v}\times\mathbf{B}), \qquad (\mathbf{B}\times\nabla\phi, \mathbf{u}) = (\mathbf{u}\times\mathbf{B}, \nabla\phi). \tag{8}$$

Proof. Follows from scalar triple product identities, see e.g. [11]. □

Herein we write $\overline{B} := ||\mathbf{B}||_{L^\infty(L^\infty)}$. Let $V := \left\{\mathbf{v} \in H_0^1(\Omega) : \nabla\cdot\mathbf{v} = 0\right\}$. We assume that (u, ϕ) is a *strong solution* of the SMHD model satisfying (4), (5), (6), (7) and $u \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;X)$, $\phi \in L^\infty(0,T;H^1(\Omega))$, $u_t \in L^2(0,T;X^*)$, and $u(x,t) \to u_0(x) \in V$ a.e. as $t \to 0$.

Restricting test functions $\mathbf{v} \in V$ reduces (4), (5), (6), (7) to: Find $\mathbf{u} : [0,T] \to V$ and $\phi : [0,T] \to S$ for a.e. $t \in (0,T]$ satisfying (6), (7), and

$$N^{-1}\left((\mathbf{u}_t, \mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v})\right) + M^{-2}(\nabla\mathbf{u}, \nabla\mathbf{v})$$
$$+ (\mathbf{u}\times\mathbf{B}, \mathbf{v}\times\mathbf{B}) - (\nabla\phi, \mathbf{v}\times\mathbf{B}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V. \tag{9}$$

Solving the problem associated with (9), (6), (7) is equivalent to (4), (5), (6), (7).

Let τ^h be a uniformly regular triangulation (see [5] for a precise definition) of Ω with $E \in \tau^h$ (e.g. triangles for $d = 2$ or tetrahedra for $d = 3$). Set $h = \sup_{E\in\tau^h}\{\text{diameter}(E)\}$. Let $X^h \subset X$, $Q^h \subset Q$, and $S^h \subset S$ be a conforming velocity-pressure-potential mixed finite element space. We assume that $X^h \times Q^h \times S^h$ satisfy the following:

Assumption 2.2. *There exists $C > 0$ such that*

$$\inf_{q \in Q^h} \sup_{\mathbf{v} \in X^h} \frac{(q, \nabla \cdot \mathbf{v})}{|\mathbf{v}|_1 ||q||} \geq C > 0.$$

Assumption 2.3. *If $\mathbf{u}$, p, ϕ satisfy Assumption 2.4 for some fixed $s, k, m \geq 0$, there exists $C > 0$ such that*

$$\begin{aligned} \inf_{\mathbf{v}^h \in X^h} ||\mathbf{u} - \mathbf{v}^h|| + h \inf_{\mathbf{v}^h \in X^h} ||\mathbf{u} - \mathbf{v}^h||_1 &\leq Ch^{k+1} ||\mathbf{u}||_{k+1} \\ \inf_{\psi^h \in X^h} ||\phi - \psi^h||_1 &\leq Ch^m ||\phi||_{m+1} \\ \inf_{q^h \in Q^h} ||p - q^h|| &\leq Ch^{s+1} ||p||_{s+1}. \end{aligned} \tag{10}$$

Assumption 2.4. $\mathbf{u} \in L^2(H^{k+1})$, *$p \in L^2(H^{s+1})$, and $\phi \in L^2(H^{m+1})$ for some $k \geq 0$, $m \geq 0$, $s \geq 0$.*

Assumption 2.5. *There exists a $C > 0$ such that*

$$|\mathbf{v}^h|_1 \leq Ch^{-1} ||\mathbf{v}^h||, \qquad \forall \mathbf{v}^h \in X^h. \tag{11}$$

Let $V^h = \left\{ \mathbf{v} \in X^h : \int_\Omega q \nabla \cdot \mathbf{v} = 0 \quad \forall q \in Q^h \right\}$. Note that in general $V^h \not\subset V$.

We use the explicitly skew-symmetric convective term:

$$b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2} \left((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) \right). \tag{12}$$

2.1. Fundamentals Inequalitites. Denote by $C > 0$ a generic constant independent of h, Δt, and ν. We collect some estimates from [4] and [5]. For $a, b > 0$ and any $1/p + 1/q = 1$ for $1 \leq p, q \leq \infty$ and any $\varepsilon > 0$

$$ab \leq \frac{1}{p \varepsilon^{p/q}} a^p + \frac{\varepsilon}{q} b^q, \quad \text{Young's inequality.} \tag{13}$$

For $1/r + 1/s = 1$, $1 \leq r, s \leq \infty$, and $\mathbf{u} \in L^r(\Omega)$, $\nabla \mathbf{v} \in L^s(\Omega)$

$$(\mathbf{u}, \mathbf{w}) \leq ||\mathbf{u}||_{0,r} ||\mathbf{w}||_{0,s}, \quad \text{Hölder's inequality.} \tag{14}$$

For $\mathbf{v} \in H_0^1(\Omega)$,

$$||\mathbf{v}|| \leq C|\mathbf{v}|_1, \qquad \text{Poincaré's inequality}$$

If $\mathbf{u} \in H^2(\Omega)$, then $\mathbf{u} \in L^\infty(\Omega) \cap C^0(0,T)$ and

$$||\mathbf{u}||_{0,\infty} \leq C||\mathbf{u}||_2.$$

The following estimates of the convective term are direct results of the previous inequalities. See [14] for a comprehensive compilation of associated estimates. For $\mathbf{u} \in V$ and $\mathbf{v}, \mathbf{w} \in H^1(\Omega)$,

$$c^h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}), \qquad (\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0. \tag{15}$$

If, on the other hand, $\mathbf{u} \in H_0^1(\Omega)$,

$$b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C\sqrt{||\mathbf{u}|||\mathbf{u}|_1}|\mathbf{v}|_1|\mathbf{w}|_1,$$
$$b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C|\mathbf{u}|_1|\mathbf{v}|_1\sqrt{||\mathbf{w}|||\mathbf{w}|_1}. \tag{16}$$

Moreover, if $\mathbf{v} \in H^2(\Omega)$, then

$$b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C||\mathbf{u}||||\mathbf{v}||_2|\mathbf{w}|_1, \quad b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C|\mathbf{u}|_1||\mathbf{v}||_2||\mathbf{w}||. \tag{17}$$

The elliptic projection is used in the forthcoming error analysis and is given by $P_1 : V \to V^h$ so that $\tilde{\mathbf{u}}^h := P_1(\mathbf{u})$ satisifies

$$\int_\Omega \nabla(\mathbf{u} - \tilde{\mathbf{u}}^h) : \nabla \mathbf{v} = 0, \qquad \forall \mathbf{v} \in V^h. \tag{18}$$

We similarly define the scalar elliptic projection $P_2 : S \to S^h$ so that $\tilde{\phi}^h := P_2(\phi)$ satisifies

$$\int_\Omega \nabla(\phi - \tilde{\phi}^h) : \nabla \psi = 0, \qquad \forall \psi \in S^h. \tag{19}$$

The following is a well-known property of P_1, P_2.

Lemma 2.6. *The projections $\tilde{\mathbf{u}}^h = P_1(\mathbf{u})$ and $\tilde{\phi}^h = P_2(\phi)$ defined in (18), (19) respectively are well-defined. Moreover,*

$$|\mathbf{u} - \tilde{\mathbf{u}}^h|_1 \le C \inf_{\mathbf{v}^h \in X^h} |\mathbf{u} - \mathbf{v}^h|_1, \tag{20}$$

$$|\phi - \tilde{\phi}^h|_1 \le C \inf_{\phi^h \in S^h} |\phi - \phi^h|_1. \tag{21}$$

Proof. See e.g. [5]. □

One can show through a duality argument (see e.g. p. 97 in [20]) that

$$||\mathbf{u} - \tilde{\mathbf{u}}^h||_{-1} + h||\mathbf{u} - \tilde{\mathbf{u}}^h|| \le Ch^2 \inf_{\mathbf{v}^h \in X^h} |\mathbf{u} - \mathbf{v}^h|_1^2. \tag{22}$$

3. Semi-discrete approximation

We first state the semi-discrete formulation of (4), (5), (6), (7). Suppose that $f \in X^*$, $B \in C^0([0,T], L^\infty(\Omega))$.

Problem 3.1 (Semi-discrete FE-approximation)**.** *Find $u^h : [0,T] \to X^h$, $p^h : (0,T] \to Q^h$, $\psi^h : [0,T] \to S^h$ satisfying*

$$\begin{aligned} &N^{-1}\left((\mathbf{u}_t^h, \mathbf{v}) + b^*(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v})\right) + M^{-2}(\nabla \mathbf{u}^h, \nabla \mathbf{v}) - (p^h, \nabla \cdot \mathbf{v}) \\ &\qquad + (\mathbf{u}^h \times \mathbf{B}, \mathbf{v} \times \mathbf{B}) - (\nabla \phi^h, \mathbf{v} \times \mathbf{B}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X^h \end{aligned} \tag{23}$$

$$(\nabla \cdot \mathbf{u}^h, q) = 0, \quad \forall q \in Q^h \tag{24}$$

$$(\nabla \phi^h, \nabla \psi) - (\mathbf{u}^h \times \mathbf{B}, \nabla \psi) = 0, \quad \forall \psi \in S^h \tag{25}$$

$$\mathbf{u}^h(\mathbf{x}, 0) = \mathbf{u}_0^h(\mathbf{x}) \tag{26}$$

where $\mathbf{u}_0^h(\mathbf{x}) \in X^h$ is a good approximation of $\mathbf{u}_0(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$.

Recall that the electric current density $\mathbf{j} = -\nabla\phi + \mathbf{u} \times \mathbf{B}$. This motivates the formal identification

Definition 3.2.

$$\mathbf{j}^h := -\nabla \phi^h + \mathbf{u}^h \times \mathbf{B}.$$

This definition makes sense in $L^2(\Omega)$ as we show in Proposition 3.3. We provide proofs of the a priori estimate Proposition 3.3 in Section 3.1 and of the convergence estimate Theorem 3.4 in Section 3.2.

Proposition 3.3 (Stability, semi-descritization). *Suppose that* $(\mathbf{u}^h, p^h, \phi^h)$ *satisfies (23)-(26). Then,* $\mathbf{u}^h \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\phi^h \in L^\infty(0,T;H^1(\Omega))$, *and* $\mathbf{j}^h \in L^2(0,T;L^2(\Omega))$ *so that*

$$\begin{aligned} &N^{-1}||\mathbf{u}^h||^2_{L^\infty(L^2)} + M^{-2}||\nabla \mathbf{u}^h||^2_{L^2(L^2)} \\ &\qquad + 2||\mathbf{j}^h||^2_{L^2(L^2)} \leq N^{-1}||\mathbf{u}^h_0||^2 + M^2||\mathbf{f}||^2_{L^2(H^{-1})} \end{aligned} \tag{27}$$

and

$$||\nabla \phi^h||^2_{L^\infty(L^2)} \leq \overline{B}^2 \left(||\mathbf{u}^h_0||^2 + NM^2||\mathbf{f}||^2_{L^2(H^{-1})} \right). \tag{28}$$

For the following convergence estimate, it is convenient to introduce the term

$$\begin{aligned} F^h(\mathbf{u}, p, \phi) = {} & M^{-2} \int_0^T \inf_{\mathbf{v}^h(\cdot,s) \in X^h} ||\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{v}^h)||^2_{-1} ds \\ & + N^2 \overline{B} \int_0^T \inf_{\mathbf{v}^h(\cdot,s) \in X^h} ||\mathbf{u} - \mathbf{v}^h||^2 ds \\ & + ||\mathbf{u}||^2_{L^\infty(H^1)} \int_0^T \inf_{\mathbf{v}^h(\cdot,s) \in X^h} |\mathbf{u} - \mathbf{v}^h|^2_1 ds \\ & + N^2 \int_0^T \left(\inf_{q^h(\cdot,s) \in Q^h} ||p - q^h||^2 + \inf_{\psi^h(\cdot,s) \in S^h} |\phi - \psi^h|^2_1 \right) ds. \end{aligned} \tag{29}$$

Note that $F^h \to 0$ as $h \to 0$ for smooth enough $\mathbf{u}$, p, ϕ. This is made precise in Corollary 3.5.

Theorem 3.4 (Convergence of semi-descritization). *Suppose that* $(\mathbf{u}, p, \phi)$ *satisfies (4)-(7) and* $(\mathbf{u}^h, p^h, \phi^h)$ *satisfies (23)-(26). Suppose further that* $\mathbf{u} \in L^\infty(H^1)$,

$p \in L^2(L^2)$, and $\mathbf{B} \in L^\infty(L^\infty)$. Then,

$$||\mathbf{u} - \mathbf{u}^h||^2_{L^\infty(L^2)} \leq \frac{C_* M^2}{N} F^h + \inf_{0<s<T} \inf_{\mathbf{v}^h(\cdot,s)\in X^h} ||\mathbf{u}(\cdot,s) - \mathbf{v}^h(\cdot,s)||^2 \tag{30}$$

$$||\nabla(\mathbf{u} - \mathbf{u}^h)||^2_{L^2(L^2)} \leq \frac{C_* M^4}{N^2} F^h + \int_0^T \inf_{\mathbf{v}^h(\cdot,s)\in X^h} |\mathbf{u}(\cdot,s) - \mathbf{v}^h(\cdot,s)|_1^2 ds \tag{31}$$

$$\begin{aligned} ||\mathbf{j} - \mathbf{j}^h||^2_{L^2(L^2)} &\leq \frac{C_* M^2}{N^2} F^h + \overline{B}^2 \int_0^T \inf_{\mathbf{v}^h(\cdot,s)\in X^h} ||\mathbf{u}(\cdot,s) - \mathbf{v}^h(\cdot,s)||^2 ds \\ &\quad + \int_0^T \inf_{\psi^h(\cdot,s)\in S^h} |\phi(\cdot,s) - \psi^h(\cdot,s)|_1^2 ds \end{aligned} \tag{32}$$

and

$$\begin{aligned} ||\nabla(\phi - \phi^h)||^2_{L^\infty(L^2)} &\leq \frac{C_* M^2}{N} \overline{B}^2 F^h + \overline{B}^2 \inf_{0<s<T} \inf_{\mathbf{v}^h(\cdot,s)\in X^h} ||\mathbf{u}(\cdot,s) - \mathbf{v}^h(\cdot,s)||^2 \\ &\quad + \inf_{0<s<T} \inf_{\psi^h(\cdot,s)\in S^h} |\phi(\cdot,s) - \psi^h(\cdot,s)|_1^2 \end{aligned} \tag{33}$$

where $C_* > 0$ is a Gronwall constant given in (46).

Corollary 3.5. *Under the assumptions of Theorem 3.4, suppose further that, for some $k \geq 0$, $\mathbf{u} \in L^\infty(H^{k+1})$, $\mathbf{u}_t \in L^2(H^{k+1})$, $p \in L^2(H^k)$, and $\phi \in L^\infty(H^{k+1})$. Then*

$$\begin{aligned} &||\mathbf{u} - \mathbf{u}^h||^2_{L^\infty(L^2)} + M^{-2} N ||\nabla(\mathbf{u} - \mathbf{u}^h)||^2_{L^2(L^2)} \\ &\qquad + N||\mathbf{j} - \mathbf{j}^h||^2_{L^2(L^2)} + ||\nabla(\phi - \phi^h)||^2_{L^\infty(L^2)} \leq C h^{2k} \end{aligned} \tag{34}$$

where $C > 0$ is independent of $h \to 0$.

3.1. Proof of Proposition 3.3.

Set $\mathbf{v} = \mathbf{u}^h$, $\psi = \phi^h$ in (23), (25) to get

$$\frac{1}{2N} \frac{d}{dt} ||\mathbf{u}^h||^2 + M^{-2} |\mathbf{u}^h|_1^2 + (-\nabla\phi^h + \mathbf{u}^h \times \mathbf{B}, \mathbf{u}^h \times \mathbf{B}) = (\mathbf{f}, \mathbf{u}^h) \tag{35}$$

$$|\phi^h|_1^2 - (\mathbf{u}^h \times \mathbf{B}, \nabla\phi^h) = 0. \tag{36}$$

Add (35) and (36) and recall that $\mathbf{j}^h = -\nabla\phi^h + \mathbf{u}^h \times \mathbf{B}$ to get

$$\frac{1}{2N}\frac{d}{dt}||\mathbf{u}^h||^2 + M^{-2}|\mathbf{u}^h|_1^2 + ||\mathbf{j}^h||^2 = (\mathbf{f}, \mathbf{u}^h).$$

Duality of X and H^{-1} and Young's inequality provides

$$N^{-1}\frac{d}{dt}||\mathbf{u}^h||^2 + M^{-2}|\mathbf{u}^h|_1^2 + 2||\mathbf{j}^h||^2 \leq ||\mathbf{f}||_{-1}.$$

Integrate over $[0,t]$ to get (27). Next, set $\psi = \phi^h$ in (25), apply Cauchy-Schwarz and simplify to get

$$|\phi^h|_1 \leq \overline{B}||\mathbf{u}^h||.$$

Apply estimate (27) to prove (28).

3.2. Proof of Theorem 3.4. The idea is to derive an error equation for both $\mathbf{u}^h$ and ϕ^h to estimate the convergence rate. Decompose the velocity error

$$\mathbf{E}_u = \mathbf{u}^h - \mathbf{u} = \mathbf{U}^h - \eta, \qquad \mathbf{U}^h = \mathbf{u}^h - \tilde{\mathbf{u}}^h, \qquad \eta = \mathbf{u} - \tilde{\mathbf{u}}^h.$$

and the electric potential error

$$E_\phi = \phi^h - \phi = \Phi^h - \zeta, \qquad \Phi^h = \phi^h - \tilde{\phi}^h, \qquad \zeta = \phi - \tilde{\phi}^h.$$

Let $\tilde{\mathbf{u}}^h$ and $\tilde{\phi}^h$ be the elliptic projections defined in (18), (19) respectively. Fix $\tilde{q}^h \in Q^h$. Note that $(p^h, \nabla \cdot v) = 0$ for any $\mathbf{v} \in V^h$. Subtract (4) from (23) and test with $\mathbf{v} = \mathbf{U}^h$ and apply (18) to get the error equation

$$\begin{aligned}
&\frac{1}{2N}\frac{d}{dt}||\mathbf{U}^h||^2 + M^{-2}|\mathbf{U}^h|_1^2 + (-\nabla\Phi^h + \mathbf{U}^h \times \mathbf{B}, \mathbf{U}^h \times \mathbf{B}) \\
&\quad = -(p - \tilde{q}^h, \nabla \cdot \mathbf{U}^h) - N^{-1}(\eta_t, \mathbf{U}^h) - N^{-1}b^*(\mathbf{u}^h, \eta, \mathbf{U}^h) \\
&\quad - N^{-1}b^*(\mathbf{U}^h, \mathbf{u}, \mathbf{U}^h) - N^{-1}b^*(\eta, \mathbf{u}, \mathbf{U}^h) - (-\nabla\zeta + \eta \times \mathbf{B}, \mathbf{U}^h \times \mathbf{B}). \qquad (37)
\end{aligned}$$

Subtract (6) from (25) and test with and $\psi = \Phi^h$ to get

$$(-\nabla\Phi^h + \mathbf{U}^h \times \mathbf{B}, \nabla\Phi^h) = -(-\nabla\zeta + \eta \times \mathbf{B}, \nabla\Phi^h). \qquad (38)$$

Add (37) and (38) to get

$$
\begin{aligned}
&\frac{1}{2N}\frac{d}{dt}||\mathbf{U}^h||^2 + M^{-2}|\mathbf{U}^h|_1^2 + ||-\nabla\Phi^h + \mathbf{U}^h\times\mathbf{B}||^2 \\
&= -(p-\tilde{q}^h, \nabla\cdot\mathbf{U}^h) - N^{-1}(\eta_t, \mathbf{U}^h) - N^{-1}b^*(\mathbf{u}^h, \eta, \mathbf{U}^h) \\
&- N^{-1}b^*(\mathbf{U}^h, \mathbf{u}, \mathbf{U}^h) - N^{-1}b^*(\eta, \mathbf{u}, \mathbf{U}^h) \\
&- (-\nabla\zeta + \eta\times\mathbf{B}, -\nabla\Phi^h + \mathbf{U}^h\times\mathbf{B}).
\end{aligned} \tag{39}
$$

We proceed to bound each of the terms on the right-hand side of (39), absorb like-terms into the left-hand side. Throughout, fix $\varepsilon > 0$ to be some arbitrary constant to be prescribed later. First applications of Cauchy-Schwarz and Young's inequalities along with $p \in L^2(\Omega)$ give

$$
(p-\tilde{q}^h, \nabla\cdot\mathbf{U}^h) \le \frac{dM^2}{4}||p-\tilde{q}^h||^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2. \tag{40}
$$

Also,

$$
N^{-1}(\eta_t, \mathbf{U}^h) \le \frac{M^2}{4N^2}||\eta_t||_{-1}^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2. \tag{41}
$$

Recall $\mathbf{j}^h = -\nabla\phi^h + \mathbf{u}^h\times\mathbf{B}$. Write

$$
\mathbf{E}_j = \mathbf{j}^h - \mathbf{j} = \mathbf{J}^h - \chi, \qquad \mathbf{J}^h = \mathbf{j}^h - \tilde{\mathbf{j}}^h, \qquad \chi = \mathbf{j} - \tilde{\mathbf{j}}^h.
$$

Then, we have

$$
(-\nabla\zeta + \eta\times\mathbf{B}, -\nabla\Phi^h + \mathbf{U}^h\times\mathbf{B}) \le \frac{M^2}{2}||\chi||^2 + \frac{1}{2}||\mathbf{J}^h||^2. \tag{42}
$$

It remains to bound the nonlinear terms.

Lemma 3.6. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 3.4. For any* $\varepsilon > 0$ *there exists* $C > 0$ *such that*

$$\begin{aligned} &- N^{-1}b^*(\mathbf{u}^h, \eta, \mathbf{U}^h) - N^{-1}b^*(\mathbf{U}^h, \mathbf{u}, \mathbf{U}^h) - N^{-1}b^*(\eta, \mathbf{u}, \mathbf{U}^h) \\ &\quad \leq \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2 + CN^{-4}M^6\left(|\eta|_1^4 + |\mathbf{u}|_1^4\right)||\mathbf{U}^h||^2 \\ &\quad + CN^{-2}M^2||\mathbf{u}||^2_{L^\infty(H^1)}|\eta|_1^2. \end{aligned} \tag{43}$$

Proof. First, $\mathbf{u} \in L^4(H^1)$ implies

$$N^{-1}b^*(\mathbf{U}^h, \mathbf{u}, \mathbf{U}^h) \leq CN^{-4}M^6||\mathbf{U}^h||^2|\mathbf{u}|_1^4 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2$$

and $\mathbf{u} \in L^\infty(H^1)$ implies

$$N^{-1}b^*(\eta, \mathbf{u}, \mathbf{U}^h) \leq CN^{-2}M^2||\mathbf{u}||^2_{L^\infty(H^1)}|\eta|_1^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2.$$

Next, rewrite the remaining nonlinear term

$$b^*(\mathbf{u}^h, \eta, \mathbf{U}^h) = b^*(\mathbf{u}, \eta, \mathbf{U}^h) - b^*(\eta, \eta, \mathbf{U}^h) + b^*(\mathbf{U}^h, \eta, \mathbf{U}^h).$$

Then $\mathbf{u} \in L^\infty(H^1)$ implies

$$N^{-1}b^*(\mathbf{u}, \eta, \mathbf{U}^h) \leq CN^{-2}M^2||\mathbf{u}||^2_{L^\infty(H^1)}|\eta|_1^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2$$

and similarly for $b^*(\eta, \eta, \mathbf{U}^h)$. Lastly, with $\mathbf{u} \in L^4(H^1)$, we apply (16) and Young's inequality to get

$$\begin{aligned} N^{-1}b^*(\mathbf{U}^h, \eta, \mathbf{U}^h) &\leq CN^{-1}||\mathbf{U}^h||^{1/2}|\mathbf{U}^h|_1^{3/2}|\eta|_1 \\ &\leq CN^{-4}M^6||\mathbf{U}^h||^4|\eta|_1^4 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h|_1^2. \end{aligned}$$

The above estimates prove (43). ∎

Set $\varepsilon = 6$. Note that $|\eta|_1^4 \leq C|\mathbf{u}|_1^4$. Apply estimates (40)-(42) and (43) to the error equation (39), and simplify to get

$$\begin{aligned}
&\frac{d}{dt}||\mathbf{U}^h||^2 + M^{-2}N|\mathbf{U}^h|_1^2 + N||\mathbf{J}^h||^2 \\
&\qquad \leq CN^{-1}||\eta_t||_{-1}^2 + CM^2N^{-1}||\mathbf{u}||_{L^\infty(H^1)}^2|\eta|_1^2 \\
&\qquad + CM^2N||p - \tilde{q}^h||^2 + CM^2N||\chi||^2 + CN^{-3}M^6|\mathbf{u}|_1^4||\mathbf{U}^h||^2. \qquad (44)
\end{aligned}$$

Let $a(t) := CN^{-3}M^6|\mathbf{u}(\cdot,t)|_1^4$. Since $\mathbf{u} \in L^4(H^1)$, then $a(t) \in L^1(0,T)$ so that we can form its antiderivative

$$A(t) := \int_0^t a(s)ds < \infty, \qquad |\mathbf{u}(\cdot,t)|_1 \in L^4(0,T).$$

Multiplying (44) by integrating factor $\exp(-A(t))$, integrating over $[0,t]$, and multiplying through the resulting equation by $\exp(A(t))$ gives

$$\begin{aligned}
&||\mathbf{U}^h(\cdot,t)||^2 + M^{-2}N\int_0^t |\mathbf{U}^h(\cdot,s)|_1^2 ds + N\int_0^t ||\mathbf{J}^h(\cdot,s)||^2 ds \\
&\qquad \leq C_*N^{-1}\int_0^t ||\eta_t(\cdot,s)||_{-1}^2 ds + C_*M^2N^{-1}||\mathbf{u}||_{L^\infty(H^1)}^2 \int_0^t |\eta(\cdot,s)|_1^2 ds \\
&\qquad + C_*M^2N\int_0^t ||p(\cdot,s) - \tilde{q}^h(\cdot,s)||^2 ds + C_*M^2N\int_0^t ||\chi(\cdot,s)||^2 ds \qquad (45)
\end{aligned}$$

where

$$C_* := CN^{-3}M^6\int_0^t |\mathbf{u}(\cdot,s)|_1^4 ds. \qquad (46)$$

Then the triangle inequality $||\mathbf{E}_u|| \leq ||\mathbf{U}^h|| + ||\eta||$ and $||\mathbf{E}_j|| \leq ||\mathbf{J}^h|| + ||\chi||$ and elliptic projection estimates (20), (21) applied to (45) proves (30), (31), (32).

Starting now with (38), apply (19) to get

$$|\Phi^h|_1^2 = -(\mathbf{E}_u \times \mathbf{B}, \nabla\Phi^h) \leq \overline{B}||\mathbf{E}_u|||\Phi^h|_1. \qquad (47)$$

So, the triangle inequality $||E_\phi|| \leq ||\Phi^h|| + ||\zeta||$ along with estimates (20), (21) and (30) applied to (47) proves (33).

Lastly, with $\mathbf{u}$, p, ϕ satisfying Assumption 2.4, the estimates (10), (22) applied to the results in Theorem 3.4 proves the estimate in Corollary 3.5.

4. Fully-discrete approximation

The semi-discrete model Problem 3.1 reduces the SMHD equations to a stiff system of ordinary differential equations. A time-discretization is required to be chosen carefully to ensure stability and accuracy of the approximation scheme. We consider now a full-discretization via Crank-Nicolson time-stepping of the SMHD equations by following a similar development as for the semi-discrete formulation in Section 3. Let $0 = t_0 < t_1 < \ldots < t_K = T < \infty$ be a discretization of the time interval $[0, T]$ for a constant time step $\Delta t = t_n - t_{n-1}$. Write $z_n = z(t_n)$ and $z_{n+1/2} = \frac{1}{2}(z_n + z_{n+1})$.

Algorithm 4.1 (Crank-Nicolson)**.** *For each $n = 1, 2, \ldots, K-1$, find $(\mathbf{u}^h_{n+1}, p^h_{n+1}, \phi^h_{n+1}) \in X^h \times Q^h \times S^h$ satisfying*

$$
\begin{aligned}
&N^{-1}\left((\frac{\mathbf{u}^h_{n+1} - \mathbf{u}^h_n}{\Delta t}, \mathbf{v}) + b^*(\mathbf{u}^h_{n+1/2}, \mathbf{u}^h_{n+1/2}, \mathbf{v}) \right) + M^{-2}(\nabla \mathbf{u}^h_{n+1/2}, \nabla \mathbf{v}) \\
&\quad - (p^h_{n+1/2}, \nabla \cdot \mathbf{v}) + (\mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) \\
&\quad - (\nabla \phi^h_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) = (\mathbf{f}_{n+1/2}, \mathbf{v}), \quad \forall \mathbf{v} \in X^h
\end{aligned} \tag{48}
$$

$$(\nabla \cdot \mathbf{u}^h_{n+1}, q) = 0, \quad \forall q \in Q^h \tag{49}$$

$$(\nabla \phi^h_{n+1/2}, \nabla \psi) - (\mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla \psi) = 0, \quad \forall \psi \in S^h \tag{50}$$

$$\mathbf{u}^h(\mathbf{x}, 0) = \mathbf{u}^h_0(\mathbf{x}) \tag{51}$$

where $\mathbf{u}^h_0 \in V^h$ be a good approximation of $\mathbf{u}_0$.

The discrete Gronwall inequality is essential in the error analysis of Algorithm 4.1.

Lemma 4.2 (Gronwall - time step restriction). *Let $D \geq 0$ and κ_n, A_n, B_n, $C_n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A_K + \Delta t \sum_{n=0}^{K} B_n \leq \Delta t \sum_{n=0}^{K} \kappa_n A_n + \Delta t \sum_{n=0}^{K} C_n + D, \quad \forall K \geq 0.$$

Suppose that for all n, $\Delta t \kappa_n < 1$, and set $g_n = (1 - \Delta t \kappa_n)^{-1}$. Then,

$$A_K + \Delta t \sum_{n=0}^{K} B_n \leq \exp\left(\Delta t \sum_{n=0}^{K} g_n \kappa_n\right) \left[\Delta t \sum_{n=0}^{K} C_n + D\right], \quad \forall K \geq 0.$$

Proof. See pp. 369-370 in [9]. □

The estimates in (52), (53), (54) are used in Corollary 4.5: For any $n = 0, 1, \ldots, K-1$,

$$||\frac{z_{n+1} - z_n}{\Delta t}||^2 \leq C \Delta t^{-1} \int_{t_n}^{t_{n+1}} ||z_t(t)||^2 dt \tag{52}$$

$$||z_{n+1/2} - z(t_{n+1/2})||_k^2 \leq C \Delta t^3 \int_{t_n}^{t_{n+1}} ||z_{tt}(t)||_k^2 dt \tag{53}$$

$$||\frac{1}{\Delta t}(z_{n+1} - z_n) - z_t(t_{n+1/2})||^2 \leq C \Delta t^3 \int_{t_n}^{t_{n+1}} ||z_{ttt}(t)||^2 dt. \tag{54}$$

where $z \in H^1(L^2)$, $z \in H^2(H^k)$, and $z \in H^3(L^2)$ is required respectivley. Each estimate (52), (53), (54) is a result of a Taylor expansion with integral remainder.

We now show that that $\mathbf{u}_n^h$, ϕ_n^h obtained by solving (48)-(51) are stable and converge to $\mathbf{u}$, ϕ solving the SMHD equations (4)-(7).

Proposition 4.3 (Stability). *Suppose that for each $n = 1, 2, \ldots, K-1$, $(\mathbf{u}_n^h, p_n^h, \phi_n^h)$ solve (48)-(51). Then,*

$$\begin{aligned} &\max_{1 \leq n \leq K} ||\mathbf{u}_n^h||^2 + M^{-2} N \Delta t \sum_{n=1}^{K-1} |\mathbf{u}_{n+1/2}^h|_1^2 \\ &+ 2N\Delta t \sum_{n=1}^{K-1} ||\mathbf{j}_{n+1/2}^h||^2 \leq ||\mathbf{u}_0^h||^2 + M^2 N \Delta t \sum_{n=1}^{K-1} ||\mathbf{f}(\cdot, t_{n+1/2})||_{-1}^2 \end{aligned} \tag{55}$$

and

$$\max_{1\le n\le K} |\phi^h_{n+1/2}|_1^2 \le \overline{B}^2 \left(||\mathbf{u}^h_0||^2 + NM^2\Delta t \sum_{n=1}^{K-1} ||\mathbf{f}(\cdot, t_{n+1/2})||_{-1}^2 \right). \tag{56}$$

For the following convergence estimate, it is convenient to introduce the term

$$\begin{aligned} F^h_k(\mathbf{u}, p, \phi) &= M^{-2}N||\mathbf{u}^h_0 - \tilde{\mathbf{u}}^h_0||^2 + M^{-2}\int_0^{t_k} \inf_{\mathbf{v}^h(\cdot,s)\in X^h} ||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{v}^h)||_{-1}^2 ds \\ &+ \Delta t \sum_{n=0}^{k} \left(N^2\overline{B}^2 \inf_{\mathbf{v}^h_n \in X^h} ||\mathbf{u}_n - \mathbf{v}^h_n||^2 + ||\mathbf{u}||^2_{L^\infty(H^1)} \inf_{\mathbf{v}^h_n\in X^h} |\mathbf{u}_n - \mathbf{v}^h_n|_1^2 \right) \\ &+ N^2\Delta t \sum_{n=0}^{k} \left(\inf_{q^h_n \in Q^h} ||p_n - q^h_n||^2 + N^2 \Delta t \sum_{n=0}^{k} \inf_{\psi^h_n \in S^h} |\phi_n - \psi^h_n|_1^2 \right). \end{aligned} \tag{57}$$

Note that $F^h_k \to 0$ as h, $\Delta t \to 0$ for smooth enough $\mathbf{u}$, p, ϕ. This is made precise in Corollary 4.5.

Theorem 4.4 (Convergence)**.** *Suppose that* $(\mathbf{u}, p, \phi)$ *satisfies (4)-(7) and* $(\mathbf{u}^h_n, p^h_n, \phi^h_n)$ *satisfies (48)-(51). Suppose further that* $\mathbf{u} \in L^\infty(H^2) \cap C^0([0,T]; H^1)$, $\mathbf{u}_t \in C^0([0,T]; H^{-1})$, $p \in C^0([0,T]; L^2)$, $\phi \in C^0([0,T]; H^1)$, $\mathbf{B} \in L^\infty(L^\infty)$, $\mathbf{f} \in C^0([0,T]; H^{-1})$ *and that*

$\Delta t > 0$ *is small enough, e.g. (83). Then,*

$$\max_{1\leq n\leq K} ||\mathbf{u}_n - \mathbf{u}_n^h||^2 \leq \max_{1\leq n\leq K} \inf_{\mathbf{v}_n^h\in X^h} ||\mathbf{u}_n - \mathbf{v}_n^h||^2$$

$$+ \frac{C_* M^2}{N} F_K^h + C_* \Delta t \sum_{n=0}^{K-1} \left(M^2 N^{-1} E_{n,\Delta t}^{(1)} + N E_{n,\Delta t}^{(2)} \right) \tag{58}$$

$$\Delta t \sum_{n=0}^{K-1} |\mathbf{u}_{n+1/2} - \mathbf{u}_{n+1/2}^h|_1^2 \leq \Delta t \sum_{n=0}^{K} \inf_{\mathbf{v}_n^h\in X^h} |\mathbf{u}_n - \mathbf{v}_n^h|_1^2$$

$$+ \frac{C_* M^4}{N^2} F_K^h + C_* M^2 N^{-1} \Delta t \sum_{n=0}^{K-1} \left(M^2 N^{-1} E_{n,\Delta t}^{(1)} + N E_{n,\Delta t}^{(2)} \right) \tag{59}$$

$$\Delta t \sum_{n=0}^{K-1} ||\mathbf{j}_{n+1/2} - \mathbf{j}_{n+1/2}^h||^2 \leq \Delta t \sum_{n=0}^{K} \left(\overline{B}^2 \inf_{\mathbf{v}_n^h\in X^h} ||\mathbf{u}_n - \mathbf{v}_n^h||^2 + \inf_{\psi_n^h\in S^h} |\phi_n - \psi_n^h|_1^2 \right)$$

$$+ \frac{C_* M^2}{N^2} F_K^h + C_* \Delta t \sum_{n=0}^{K-1} \left(M^2 N^{-1} E_{n,\Delta t}^{(1)} + N E_{n,\Delta t}^{(2)} \right) \tag{60}$$

and

$$\max_{1\leq n\leq K} |\phi_n - \phi_n^h|_1^2 \leq \overline{B}^2 \max_{1\leq n\leq K} \inf_{\mathbf{v}_n^h\in X^h} ||\mathbf{u}_n - \mathbf{v}_n^h||^2 + \max_{1\leq n\leq K} \inf_{\psi_n^h\in S^h} |\phi_n - \psi_n^h|_1^2$$

$$+ \frac{C_* M^2}{N} \overline{B}^2 F_K^h + C_* \overline{B}^2 \Delta t \sum_{n=0}^{K-1} \left(M^2 N^{-1} E_{n,\Delta t}^{(1)} + N E_{n,\Delta t}^{(2)} \right) \tag{61}$$

where $C_* > 0$ *is given by (85) and* $E_{n,\Delta t}^{(1)}$, $E_{n,\Delta t}^{(2)}$ *are given by (79), (81) respectively.*

Corollary 4.5. *Under the assumptions of Theorem 3.4, suppose further that, for some* $k \geq 0$, $\mathbf{u} \in L^\infty(H^{k+1})$, $\mathbf{u}_t \in L^2(H^{k+1})$, $p \in L^2(H^k)$, *and* $\phi \in L^\infty(H^{k+1})$ *and*

$\mathbf{u}_{tt} \in L^2(H^1)$, $\mathbf{u}_{ttt} \in L^2(W^{-1,2})$, $p_{tt} \in L^2(L^2)$, $\mathbf{f}_{tt} \in L^2(W^{-1,2})$. *Then,*

$$\max_{1\le n\le K} ||\mathbf{u}_n - \mathbf{u}_n^h||^2 + \max_{1\le n\le K} |\phi_n - \phi_n^h|_1^2 + \frac{N\Delta t}{M^2}\sum_{n=1}^{K-1} |\mathbf{u}_{n+1/2} - \mathbf{u}_{n+1/2}^h|_1^2$$

$$+ N\Delta t \sum_{n=1}^{K-1} ||\mathbf{j}_{n+1/2} - \mathbf{j}_{n+1/2}^h||^2 \le C(h^{2k} + \Delta t^4) \tag{62}$$

where $C > 0$ is independent of h, $\Delta t \to 0$.

4.1. Proof of Proposition 4.3. Set $\mathbf{v} = \mathbf{u}_{n+1/2}^h$ in (48) to get

$$\frac{1}{2N\Delta t}\left(||\mathbf{u}_{n+1}^h||^2 - ||\mathbf{u}_n^h||^2\right) + M^{-2}|\mathbf{u}_{n+1/2}^h|_1^2$$

$$+ (-\nabla\phi_{n+1/2}^h + \mathbf{u}_{n+1/2}^h \times \mathbf{B}_{n+1/2}, \mathbf{u}_{n+1/2}^h \times \mathbf{B}_{n+1/2}) = (\mathbf{f}_{n+1/2}, \mathbf{u}_{n+1/2}^h). \tag{63}$$

Set $\psi = \phi_{n+1}^h$ in (50) to get

$$|\phi_{n+1/2}^h|_1^2 - (\mathbf{u}_{n+1/2}^h \times \mathbf{B}_{n+1/2}, \nabla\phi_{n+1/2}^h) = 0. \tag{64}$$

Recall that $\mathbf{j}_{n+1/2}^h = -\nabla\phi_{n+1/2}^h + \mathbf{u}_{n+1/2}^h \times \mathbf{B}_{n+1/2}$. Add (63) and (64), apply duality of X and H^{-1}, and Young's inequality to get

$$\frac{1}{2N\Delta t}\left(||\mathbf{u}_{n+1}^h||^2 - ||\mathbf{u}_n^h||^2\right) + M^{-2}|\mathbf{u}_{n+1/2}^h|_1^2 + ||\mathbf{j}_{n+1/2}^h||^2$$
$$= (\mathbf{f}_{n+1/2}, \mathbf{u}_{n+1/2}^h) \le \frac{M^2}{2}||\mathbf{f}_{n+1/2}||_{-1}^2 + \frac{1}{2M^2}|\mathbf{u}_{n+1/2}^h|_1^2.$$

Then, absorbing like-terms from right into the left-hand-side of the previous estimate, we sum from $n = 0$ to $n = K-1$ to get

$$N^{-1}||\mathbf{u}_K^h||^2 + M^{-2}\Delta t\sum_{n=0}^{K-1} |\mathbf{u}_{n+1/2}^h|_1^2 + 2\Delta t\sum_{n=0}^{K-1} ||\mathbf{j}_{n+1/2}^h||^2$$
$$\le N^{-1}||\mathbf{u}_0^h||^2 + M^2\Delta t\sum_{n=0}^{K-1} ||\mathbf{f}_{n+1/2}||_{-1}^2$$

which proves (55). Next set $\psi = \nabla\phi^h_{n+1/2}$ in (50), apply Cauchy-Schwarz and simplify to get

$$|\phi^h_{n+1/2}|_1^2 \leq ||\mathbf{u}^h_{n+1/2} \times \mathbf{B}_{n+1/2}|||\phi_{n+1}|_1.$$

Then,

$$|\phi^h_{n+1/2}|_1 \leq \overline{B} \max_{1\leq n\leq K-1} ||\mathbf{u}^h_n||$$

which, together with (55), proves (56).

4.2. Proof of Theorem 4.4, Corollary 4.5. The consistency error for the time-discretization is given by, for any $\mathbf{v} \in X$, $\psi \in S$,

$$\tau_n^{(1)}(\mathbf{v}) := N^{-1}\left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}), \mathbf{v}\right) - (p_{n+1/2} - p(\cdot, t_{n+1/2}), \nabla \cdot \mathbf{v})$$
$$+ N^{-1}b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v}) - N^{-1}b^*(\mathbf{u}(\cdot, t_{n+1/2}), \mathbf{u}(\cdot, t_{n+1/2}), \mathbf{v})$$
$$+ \left[(\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) - (\mathbf{u}(\cdot, t_{n+1/2}) \times \mathbf{B}(\cdot, t_{n+1/2}), \mathbf{v} \times \mathbf{B}(\cdot, t_{n+1/2}))\right]$$
$$- \left[(\nabla\phi_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) - (\nabla\phi(\cdot, t_{n+1/2}), \mathbf{v} \times \mathbf{B}(\cdot, t_{n+1/2}))\right]$$

(65)

$$+ M^{-2}(\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})), \nabla\mathbf{v}) + (\mathbf{f}(\cdot, t_{n+1/2}) - \mathbf{f}_{n+1/2}, \mathbf{v}),$$

$$\tau_n^{(2)}(\psi) := (\nabla(\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})), \nabla\psi)$$

(66)

$$- \left[(\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\psi) - (\mathbf{u}(\cdot, t_{n+1/2}) \times \mathbf{B}(\cdot, t_{n+1/2}), \nabla\psi)\right].$$

Using (65), rewrite (9) in a form conducive to analyzing the error between the continuous and discrete models: Find $\mathbf{u}_{n+1} \in V$, $\phi_{n+1} \in S$ satisfying

$$N^{-1}\left((\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t}, \mathbf{v}) + b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{v}) \right) + M^{-2}(\nabla \mathbf{u}_{n+1/2}, \nabla \mathbf{v}) - (p_{n+1/2}, \nabla \cdot \mathbf{v})$$

$$+ (\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2}) - (\nabla \phi_{n+1/2}, \mathbf{v} \times \mathbf{B}_{n+1/2})$$

(67)

$$= (\mathbf{f}_{n+1/2}, \mathbf{v}) + \tau_n^{(1)}(\mathbf{v}), \quad \forall \mathbf{v} \in V$$

(68)

$$(\nabla \phi_{n+1/2}, \nabla \psi) - (\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla \psi) = \tau_n^{(2)}(\psi), \quad \forall \psi \in S.$$

We first construct the error equation for velocity and electric potential. Decompose the velocity error

$$\mathbf{E}_{u,n} = \mathbf{u}_n^h - \mathbf{u}_n = \mathbf{U}_n^h - \eta_n, \qquad \mathbf{U}_n^h = \mathbf{u}_n^h - \tilde{\mathbf{u}}_n^h, \qquad \eta_n = \mathbf{u}_n - \tilde{\mathbf{u}}_n^h.$$

and the electric potential error

$$E_{\phi,n} = \phi_n^h - \phi_n = \Phi_n^h - \zeta_n, \qquad \Phi_n^h = \phi_n^h - \tilde{\phi}_n^h, \qquad \zeta_n = \phi_n - \tilde{\phi}_n^h.$$

Let $\tilde{u}_n^h$, $\tilde{\phi}_n^h$ be the elliptic projection defined in (18), (19). Fix $\tilde{q}_n^h \in Q^h$. Note that $(p^h, \nabla \cdot v) = 0$ for any $\mathbf{v} \in V^h$. Subtract (67) from (48) and set $\mathbf{v} = \mathbf{U}_{n+1/2}^h$,

$\psi = \Phi^h_{n+1/2}$ to get the error equations

$$\frac{1}{2N\Delta t}\left(||\mathbf{U}^h_{n+1}||^2 - ||\mathbf{U}^h_n||^2\right) + M^{-2}|\mathbf{U}^h_{n+1/2}|^2_1$$

$$+ (-\nabla\Phi^h_{n+1/2} + \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2})$$

$$= N^{-1}(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \mathbf{U}^h_{n+1/2}) - N^{-1}b^*(\mathbf{U}^h_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2})$$

$$+ N^{-1}b^*(\eta_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) + N^{-1}b^*(\mathbf{u}^h_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2})$$

$$+ (\eta_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) - (\nabla\zeta_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2})$$

$$+ (\tilde{q}^h_{n+1/2} - p_{n+1/2}, \nabla \cdot \mathbf{U}^h_{n+1/2}) - \tau^{(1)}_n(\mathbf{U}^h_{n+1/2}), \tag{69}$$

$$(\nabla\Phi^h_{n+1/2} - \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2})$$

$$= (\nabla\zeta_{n+1/2}, \nabla\Phi^h_{n+1/2}) - (\eta_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) - \tau^{(2)}_n(\Phi^h_{n+1/2}). \tag{70}$$

Recall the formal definition $\mathbf{j}^h = -\nabla\phi^h + \mathbf{u}^h \times \mathbf{B}$. Write $\tilde{\mathbf{j}}^h = -\nabla\tilde{\phi}^h + \tilde{\mathbf{u}}^h \times \mathbf{B}$ and

$$\mathbf{E}_{j,n} = \mathbf{j}^h_n - \mathbf{j}_n = \mathbf{J}^h_n - \chi_n, \qquad \mathbf{J}^h_n = \mathbf{j}^h_n - \tilde{\mathbf{j}}^h_n, \qquad \chi_n = \mathbf{j}_n - \tilde{\mathbf{j}}^h_n.$$

Add (70) and (69), combine terms to get

$$\frac{1}{2N\Delta t}\left(||\mathbf{U}^h_{n+1}||^2 - ||\mathbf{U}^h_n||^2\right) + M^{-2}|\mathbf{U}^h_{n+1/2}|^2_1 + ||\mathbf{J}^h_{n+1/2}||^2$$

$$= N^{-1}(\frac{\eta_{n+1} - \eta_n}{\Delta t}, \mathbf{U}^h_{n+1/2}) - N^{-1}b^*(\mathbf{U}^h_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2})$$

$$+ N^{-1}b^*(\eta_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) + N^{-1}b^*(\mathbf{u}^h_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2})$$

$$+ (\tilde{q}^h_{n+1/2} - p_{n+1/2}, \nabla \cdot \mathbf{U}^h_{n+1/2}) + (\chi_{n+1/2}, \mathbf{J}^h_{n+1/2})$$

$$- \tau^{(1)}_n(\mathbf{U}^h_{n+1/2}) - \tau^{(2)}_n(\Phi^h_{n+1/2}). \tag{71}$$

Note that (70) can be written

$$|\Phi^h_{n+1/2}|^2_1 = (\mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2})$$

$$(\nabla\zeta_{n+1/2}, \nabla\Phi^h_{n+1/2}) - (\eta_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) - \tau^{(2)}_n(\Phi^h_{n+1/2}). \tag{72}$$

Now add (72) to (71) to get

$$\frac{1}{2N\Delta t}\left(||\mathbf{U}^h_{n+1}||^2 - ||\mathbf{U}^h_n||^2\right) + M^{-2}|\mathbf{U}^h_{n+1/2}|_1^2 + ||\mathbf{J}^h_{n+1/2}||^2 + |\Phi^h_{n+1/2}|_1^2$$
$$= N^{-1}(\frac{\eta_{n+1}-\eta_n}{\Delta t}, \mathbf{U}^h_{n+1/2}) - N^{-1}b^*(\mathbf{U}^h_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2})$$
$$+ N^{-1}b^*(\eta_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) + N^{-1}b^*(\mathbf{u}^h_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2})$$
$$+ (\tilde{q}^h_{n+1/2} - p_{n+1/2}, \nabla\cdot\mathbf{U}^h_{n+1/2}) + (\chi_{\mathbf{n+1/2}}, \mathbf{J}^h_{n+1/2})$$
$$+ (\mathbf{U}^h_{n+1/2}\times\mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) - (\chi_{\mathbf{n+1/2}}, \nabla\Phi^h_{n+1/2})$$
$$- \tau_n^{(1)}(\mathbf{U}^h_{n+1/2}) - 2\tau_n^{(2)}(\Phi^h_{n+1/2}). \tag{73}$$

In order to prove the desired error estimation, we majorize the terms on the right-hand-side of (73), absorb like-terms into the left-hand-side, and then sum from $n = 0, \ldots, K$, and finally apply the discrete Gronwall inequality to take care of remaining terms on right-hand-side involving $||\mathbf{U}^h_n||$. Let ε, $\varepsilon_0 > 0$ be a generic Young's inequality constants to be fixed later in the analysis. First, $\mathbf{u}_t \in H^{-1}(\Omega)$ and $p \in L^2(\Omega)$ implies

$$N^{-1}(\frac{\eta_{n+1}-\eta_n}{\Delta t}, \mathbf{U}^h_{n+1/2}) \le CM^2N^{-2}||\frac{\eta_{n+1}-\eta_n}{\Delta t}||_{-1}^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2 \tag{74}$$

$$(p_{n+1/2} - \tilde{q}^h_{n+1/2}, \nabla\cdot\mathbf{U}^h_{n+1/2}) \le CM^2||p_{n+1/2} - \tilde{q}^h_{n+1/2}||^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2. \tag{75}$$

Application of Cauchy-Schwarz and Young's inequalities also provide

$$(\chi_{\mathbf{n+1/2}}, \mathbf{J}^h_{n+1/2}) + (\mathbf{U}^h_{n+1/2}\times\mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) - (\chi_{\mathbf{n+1/2}}, \nabla\Phi^h_{n+1/2})$$
$$\le C||\chi_{\mathbf{n+1/2}}||^2 + C\overline{B}^2||\mathbf{U}^h_{n+1/2}||^2 + \frac{1}{2}||\mathbf{J}^h_{n+1/2}||^2 + \frac{1}{\varepsilon_0}|\Phi^h_{n+1/2}|_1^2. \tag{76}$$

We bound the convective terms in the next lemma.

Lemma 4.6. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 4.4. For any* $\varepsilon > 0$ *and for any integer* $n \geq n_0$ *there exists* $C > 0$ *such that*

$$\begin{aligned}
&N^{-1}b^*(\mathbf{U}^h_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) \\
&\qquad - N^{-1}b^*(\mathbf{u}^h_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2}) - N^{-1}b^*(\eta_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) \\
&\leq \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2 + CM^2N^{-2}h^{-1}|\eta_{n+1/2}|_1^2||\mathbf{U}^h_{n+1/2}||^2 \\
&\qquad + CM^2N^{-2}(||\mathbf{u}||^2_{L^\infty(H^1)} + |\eta_{n+1/2}|_1^2)|\eta_{n+1/2}|_1^2. \qquad (77)
\end{aligned}$$

Proof. First, $\mathbf{u} \in H^2(\Omega)$ implies

$$\begin{aligned}
&N^{-1}b^*(\mathbf{U}^h_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) \\
&\qquad \leq CM^2N^{-2}||\mathbf{u}_{n+1/2}||_2^2||\mathbf{U}^h_{n+1/2}||^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2
\end{aligned}$$

and $\mathbf{u} \in L^\infty(H^1(\Omega))$ implies

$$\begin{aligned}
&N^{-1}b^*(\eta_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) \\
&\qquad \leq CM^2N^{-2}||\mathbf{u}||^2_{L^\infty(H^1)}|\eta_{n+1/2}|_1^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2.
\end{aligned}$$

Next, rewrite the remaining nonlinear term

$$\begin{aligned}
&b^*(\mathbf{u}^h_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2}) = b^*(\mathbf{u}_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2}) \\
&\qquad - b^*(\eta_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2}) + b^*(\mathbf{U}^h_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2}).
\end{aligned}$$

Then $\nabla \cdot \mathbf{u} = 0$ so that $\mathbf{u} \in L^\infty(L^2(\Omega))$ implies

$$\begin{aligned}
&N^{-1}b^*(\mathbf{u}_{n+1/2}, \eta_{n+1/2}, \mathbf{U}^h_{n+1/2}) = N^{-1}(\mathbf{u}_{n+1/2} \cdot \nabla\eta_{n+1/2}, \mathbf{U}^h_{n+1/2}) \\
&\leq CM^2N^{-2}||\mathbf{u}||^2_{L^\infty(L^2)}|\eta_{n+1/2}|_1^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2
\end{aligned}$$

and similarly

$$N^{-1}b^*(\eta_{n+1/2},\eta_{n+1/2},\mathbf{U}^h_{n+1/2}) \le CM^2N^{-2}|\eta_{n+1/2}|_1^4 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2$$

Lastly,

$$\begin{aligned} &N^{-1}b^*(\mathbf{U}^h_{n+1/2},\eta_{n+1/2},\mathbf{U}^h_{n+1/2}) \\ &\quad\le CN^{-1}\sqrt{||\mathbf{U}^h_{n+1/2}|||\mathbf{U}^h_{n+1/2}|_1}|\eta_{n+1/2}|_1|\mathbf{U}^h_{n+1/2}|_1 \\ &\quad\le CM^2N^{-2}h^{-1}|\eta_{n+1/2}|_1^2||\mathbf{U}^h_{n+1/2}||^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2. \end{aligned}$$

The above estimates prove (77). □

Lemma 4.7. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 4.4. Then, for any* $\varepsilon > 0$ *and any integer* $n \ge 0$

$$\begin{aligned} \tau_n^{(1)}(\mathbf{U}^h_{n+1/2}) &\le \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2 \\ &\quad + \frac{1}{2N}|\phi(\cdot,t_{n+1/2})|_1^2||\mathbf{U}^h_{n+1/2}||^2 + CM^2N^{-2}E^{(1)}_{n,\Delta t} \end{aligned} \tag{78}$$

where $E^{(1)}_{n,\Delta t} \ge 0$ *is given in (79).*

Proof. First, $\mathbf{u}_t \in H^{-1}$ implies

$$\begin{aligned} &N^{-1}(\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2}),\mathbf{U}^h_{n+1/2}) \\ &\le CM^2N^{-2}||\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})||_{-1}^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2 \end{aligned}$$

and $\mathbf{u} \in H^1(\Omega)$ implies

$$\begin{aligned} &(\nabla(\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot,t_{n+1/2})),\nabla\mathbf{U}^h_{n+1/2}) \\ &\quad\le CM^2|\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot,t_{n+1/2})|_1^2 + \frac{1}{\varepsilon M^2}|\mathbf{U}^h_{n+1/2}|_1^2. \end{aligned}$$

Similarly, $p \in L^2(\Omega)$ implies

$$(p_{n+1/2} - p(\cdot, t_{n+1/2}), \nabla \cdot \mathbf{U}^h_{n+1/2})$$
$$\leq CM^2 ||p_{n+1/2} - p(\cdot, t_{n+1/2})||^2 + \frac{1}{\varepsilon M^2} |\mathbf{U}^h_{n+1/2}|_1^2$$

and $\mathbf{f} \in H^{-1}(\Omega)$ implies

$$(\mathbf{f}(\cdot, t_{n+1/2}) - \mathbf{f}_{n+1/2}, \mathbf{U}^h_{n+1/2})$$
$$\leq CM^2 ||f(t_{n+1/2}) - f_{n+1/2}||_{-1}^2 + \frac{1}{\varepsilon M^2} |\mathbf{U}^h_{n+1/2}|_1^2.$$

We decompose the nonlinear terms so that

$$b^*(\mathbf{u}_{n+1/2}, \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2}) - (\mathbf{u}(\cdot, t_{n+1/2}) \cdot \nabla \mathbf{u}(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2})$$
$$= b^*(\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2}), \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2})$$
$$+ b^*(\mathbf{u}(\cdot, t_{n+1/2}), \mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2}).$$

Then, $\mathbf{u} \in L^\infty(H^1(\Omega))$ implies

$$N^{-1} b^*(\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2}), \mathbf{u}_{n+1/2}, \mathbf{U}^h_{n+1/2})$$
$$\leq CM^2 N^{-2} ||\mathbf{u}||^2_{L^\infty(H^1)} |\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})|_1^2 + \frac{1}{\varepsilon M^2} |\mathbf{U}^h_{n+1/2}|_1^2$$

and similarly for $b^*(\mathbf{u}(\cdot, t_{n+1/2}), \mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2})$.

We have the electromagnetic force terms remaining. First notice that

$$(\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2})$$
$$- (\mathbf{u}(\cdot, t_{n+1/2}) \times \mathbf{B}(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2} \times \mathbf{B}(\cdot, t_{n+1/2}))$$
$$= ((\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})) \times \mathbf{B}_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2})$$
$$+ (\mathbf{u}(\cdot, t_{n+1/2}) \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})), \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2})$$
$$+ (\mathbf{u}(\cdot, t_{n+1/2}) \times \mathbf{B}(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2} \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2}))).$$

Here, $\mathbf{B} \in L^\infty(L^\infty)$ implies

$$
\begin{aligned}
&((\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})) \times \mathbf{B}_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) \\
&\qquad \leq \overline{B}^2 ||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})|| ||\mathbf{U}^h_{n+1/2}|| \\
&\qquad \leq CM^2 \overline{B}^4 ||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||^2 + \frac{1}{\varepsilon M^2} |\mathbf{U}^h_{n+1/2}|_1^2
\end{aligned}
$$

and $\mathbf{B} \in L^\infty(L^\infty)$, $\mathbf{u} \in L^\infty(L^\infty)$ together with Cauchy-Schwarz, Poincaré's, and Young's inequalities implies

$$
\begin{aligned}
&(\mathbf{u}(\cdot, t_{n+1/2}) \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})), \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) \\
&\qquad \leq CM^2 \overline{B}^2 ||\mathbf{u}||_{L^\infty(L^\infty)} ||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||^2 + \frac{1}{\varepsilon M^2} |\mathbf{U}^h_{n+1/2}|_1^2
\end{aligned}
$$

and similarly for $(\mathbf{u}(\cdot, t_{n+1/2}) \times \mathbf{B}(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2} \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})))$. Next,

$$
\begin{aligned}
&(\nabla \phi_{n+1/2}, \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) - (\nabla \phi(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2} \times \mathbf{B}(\cdot, t_{n+1/2})) \\
&\quad = (\nabla(\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})), \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) \\
&\qquad + (\nabla \phi(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2} \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2}))).
\end{aligned}
$$

Here, $\mathbf{B} \in L^\infty(L^\infty)$ and Poincaré's inequality imply that

$$
\begin{aligned}
&(\nabla(\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})), \mathbf{U}^h_{n+1/2} \times \mathbf{B}_{n+1/2}) \\
&\qquad \leq \overline{B} |\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})|_1 ||\mathbf{U}^h_{n+1/2}|| \\
&\qquad \leq CM^2 \overline{B}^2 |\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})|_1^2 + \frac{1}{\varepsilon M^2} |\mathbf{U}^h_{n+1/2}|_1^2
\end{aligned}
$$

and

$$
\begin{aligned}
&(\nabla \phi(\cdot, t_{n+1/2}), \mathbf{U}^h_{n+1/2} \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2}))) \\
&\qquad = ((\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})), \nabla \phi(\cdot, t_{n+1/2}) \times \mathbf{U}^h_{n+1/2}) \\
&\qquad \leq \frac{N}{2} ||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||^2 + \frac{1}{2N} |\phi(\cdot, t_{n+1/2})|_1^2 ||\mathbf{U}^h_{n+1/2}||^2.
\end{aligned}
$$

Lastly, set

$$
\begin{aligned}
E^{(1)}_{n,\Delta t} :=& \, ||\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{\Delta t} - \mathbf{u}_t(t_{n+1/2})||^2_{-1} \\
&+ (||\mathbf{u}||^2_{L^\infty(H^1)} + N^2)|\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})|^2_1 \\
&+ CN^2\overline{B}^4||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||^2 + CN^2\overline{B}^2|\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})|^2_1 \\
&+ N^2||p_{n+1/2} - p(\cdot, t_{n+1/2})||^2 + N^2||\mathbf{f}(\cdot, t_{n+1/2}) - \mathbf{f}_{n+1/2}||^2_{-1} \\
&+ N(N\overline{B}^2||\mathbf{u}||_{L^\infty(L^2)} + M^2N^2)||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||^2. \qquad (79)
\end{aligned}
$$

The above estimates prove (78). □

Lemma 4.8. *Let* $\mathbf{u}$ *satisfy the regularity assumptions of Theorem 4.4. Then, for any* $\varepsilon > 0$ *and any integer* $n \geq 0$

$$
\tau^{(2)}_n(\Phi^h_{n+1/2}) \leq \frac{1}{\varepsilon_0}|\Phi^h_{n+1/2}|^2_1 + CE^{(2)}_{n,\Delta t} \qquad (80)
$$

where $E^{(2)}_{n,\Delta t} \geq 0$ *is given in (81).*

Proof. Cauchy-Schwarz and Young's inequalities imply

$$
\begin{aligned}
&(\nabla(\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})), \nabla\Phi^h_{n+1/2}) \\
&\qquad \leq C|\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})|^2_1 + \frac{1}{\varepsilon_0}|\Phi^h_{n+1/2}|^2_1.
\end{aligned}
$$

Notice now that

$$
\begin{aligned}
&(\mathbf{u}_{n+1/2} \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) - (\mathbf{u}(\cdot, t_{n+1/2}) \times \mathbf{B}(\cdot, t_{n+1/2}), \nabla\Phi^h_{n+1/2}) \\
&= ((\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})) \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) \\
&\quad + (\mathbf{u}(\cdot, t_{n+1/2}) \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})), \nabla\Phi^h_{n+1/2}).
\end{aligned}
$$

Here, $\mathbf{B} \in L^\infty(L^\infty)$ implies

$$
\begin{aligned}
&((\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})) \times \mathbf{B}_{n+1/2}, \nabla\Phi^h_{n+1/2}) \\
&\qquad \leq C\overline{B}^2||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||^2 + \frac{1}{\varepsilon_0}|\Phi^h_{n+1/2}|^2_1
\end{aligned}
$$

and $\mathbf{u} \in L^\infty(L^\infty)$ together with Cauchy-Schwarz and Young's inequality imply

$$\begin{aligned}&(\mathbf{u}(\cdot, t_{n+1/2}) \times (\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})), \nabla \Phi^h_{n+1/2})\\ &\qquad \leq C||\mathbf{u}||^2_{L^\infty(L^\infty)} ||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||^2 + \frac{1}{\varepsilon_0} |\Phi^h_{n+1/2}|^2_1.\end{aligned}$$

Lastly, set

$$\begin{aligned}E^{(2)}_{n,\Delta t} &:= |\phi_{n+1/2} - \phi(\cdot, t_{n+1/2})|^2_1 + \overline{B}^2 ||\mathbf{u}_{n+1/2} - \mathbf{u}(\cdot, t_{n+1/2})||^2\\ &\qquad + ||\mathbf{u}||^2_{L^\infty(L^\infty)} ||\mathbf{B}_{n+1/2} - \mathbf{B}(\cdot, t_{n+1/2})||^2. \qquad (81)\end{aligned}$$

□

Apply estimates from (74), (75), (76), (77), (78), and (80) to (73). Set $\varepsilon = 8$ and $\varepsilon_0 = 4$ and absorb all terms including $|\mathbf{U}^h_{n+1/2}|_1$ from the right into left-hand-side of (71). The approximation (10) and $\mathbf{u} \in H^2$ imply

$$h^{-1}|\eta^n|^2_1 \leq Ch||\mathbf{u}||^2_2, \qquad |\eta^n|^2_1 \leq C||\mathbf{u}||^2_1.$$

Sum the resulting inequality on both sides from $n = 0$ to $n = K-1$. Apply the estimate (79), (81), and result of (10) from above. Assuming that $\mathbf{u} \in L^\infty(H^2)$ and multiplying by N, the result is

$$\begin{aligned}&||\mathbf{U}^h_K||^2 + M^{-2}N\Delta t \sum_{n=0}^{K-1} |\mathbf{U}^h_{n+1/2}|^2_1 + N\Delta t \sum_{n=0}^{K-1} ||\mathbf{J}^h_{n+1/2}||^2 + N\Delta t \sum_{n=0}^{K-1} |\Phi^h_{n+1/2}|^2_1\\ &\quad \leq C\Delta t \sum_{n=0}^{K-1} \left(M^2N^{-1}h||\mathbf{u}_{n+1/2}||_2 + |\phi(\cdot, t_{n+1/2})|^2_1 + N\overline{B}^2\right) ||\mathbf{U}^h_{n+1/2}||^2\\ &\quad + ||\mathbf{U}^h_0||^2 + CM^2N^{-1}\Delta t \sum_{n=0}^{K-1} \left[||\frac{\eta_{n+1} - \eta_n}{\Delta t}||^2_{-1} + ||\mathbf{u}||^2_{L^\infty(H^1)} |\eta_{n+1/2}|^2 + 1 \right]\\ &\quad + CM^2N\Delta t \sum_{n=0}^{K-1} ||\tilde{q}^h_{n+1/2} - p_{n+1/2}||^2 + CN\Delta t \sum_{n=0}^{K-1} ||\chi_{\mathbf{n+1/2}}||^2\\ &\quad + CM^2N^{-1}\Delta t \sum_{n=0}^{K-1} E^{(1)}_{n,\Delta t} + CN\Delta t \sum_{n=0}^{K-1} E^{(2)}_{n,\Delta t}. \qquad (82)\end{aligned}$$

Referring to the notation of the discrete Gronwall Lemma 4.2, let

$$\kappa_n = \left[M^2N^{-1}h||\mathbf{u}_{n+1/2}||_2 + |\phi(\cdot, t_{n+1/2})|_1^2 + N\overline{B}^2\right], \qquad \Delta t\kappa_n < 1. \tag{83}$$

Then, Lemma 4.2 applied to (82) implies

$$\begin{aligned}
||\mathbf{U}_K^h||^2 &+ \frac{N\Delta t}{M^2}\sum_{n=0}^{K-1}|\mathbf{U}_{n+1/2}^h|_1^2 + N\Delta t\sum_{n=0}^{K-1}||\mathbf{J}_{n+1/2}^h||^2 + N\Delta t\sum_{n=0}^{K-1}|\Phi_{n+1/2}^h|_1^2\\
&\leq C_*||\mathbf{U}_0^h||^2 + C_*M^2N^{-1}\Delta t\sum_{n=0}^{K-1}\left[||\frac{\eta_{n+1}-\eta_n}{\Delta t}||_{-1}^2 + ||\mathbf{u}||_{L^\infty(H^1)}^2|\eta_{n+1/2}|_1^2\right]\\
&+ C_*M^2N\Delta t\sum_{n=0}^{K-1}||\tilde{q}_{n+1/2}^h - p_{n+1/2}||^2 + C_*N\Delta t\sum_{n=0}^{K-1}||\chi_{\mathbf{n+1/2}}||^2\\
&+ C_*M^2N^{-1}\Delta t\sum_{n=0}^{K-1}E_{n,\Delta t}^{(1)} + C_*N\Delta t\sum_{n=0}^{K-1}E_{n,\Delta t}^{(2)}.
\end{aligned} \tag{84}$$

where

$$C_* := C\exp\left(\Delta t\sum_{n=0}^{K}\kappa_n g_n\right), \qquad g_n = (1-\Delta t\kappa_n)^{-1}. \tag{85}$$

Then the triangle inequality the triangle inequality $||\mathbf{E}_{u,n}|| \leq ||\mathbf{U}_n^h|| + ||\eta_n||$ and $||\mathbf{E}_{j,n}|| \leq ||\mathbf{J}_n^h|| + ||\chi_n||$ and elliptic projection estimates (20), (21) applied to (84) proves (58), (59), (60).

Returning to (70), majorize the right-hand-side with Cauchy-Schwarz and Young's inequalities in addition to (80), absorb like-terms into the left-hand-side to get

$$|\Phi_{n+1/2}^h|_1^2 \leq C\overline{B}^2||\mathbf{U}_{n+1/2}^h||^2 + C|\zeta_{n+1/2}|_1^2 + C\overline{B}^2||\eta_{n+1/2}||^2 + CE_{n,\Delta t}^{(2)}. \tag{86}$$

So, the triangle inequality $||E_{\phi,n}|| \leq ||\Phi_n^h|| + ||\zeta_n||$ along with estimates (20), (21) and (58) applied to (86) proves (61).

Lastly, with $\mathbf{u}$, p, ϕ satisfying Assumption 2.4, the estimates (10) and (52)-(54) applied to the results in Theorem 4.4 proves the estimate in Corollary 4.5.

5. Numerical results

We consider two distinct numerical experiments in this section. First, we confirm the converge rate as h, $\Delta t \to 0$ for the fully discrete, simplified magnetohydrodynamic model (48)-(51). Second, we consider the effect of an applied magnetic field to a conducting flow in a channel past a step by illustrating the damping effect of $B \neq 0$.

We use the FreeFem++ software for each of our simulations. We utilize Taylor-Hood mixed finite elements (piecewise quadratics for velocity and piecewise linear pressure) for the discretization. We apply Newton iterations to solve the nonlinear system with a $||\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)}||_{H^1(\Omega)} < 10^{-8}$ as a stopping criterion.

Experiment 1 - Convergence Analysis: For the first experiment, $\Omega = [0,\pi]^2$, $t_0 = 0$, $T = 1$, $Re = 25$, $B = (0,0,1)$. The true solution $(\mathbf{u}, p, \phi)$ is given by

(87)
$$\begin{aligned}
\psi(x,y) &= \cos(2x)\cos(2y), & \mathbf{u}(x,y,t) &= \left(-\frac{\partial \psi(x,y)}{\partial y}, \frac{\partial \psi(x,y)}{\partial x}\right) e^{-5t} \\
p(x,y,t) &= 0, & \phi(x,y,t) &= (\psi(x,y) + x^2 - y^2)e^{-5t}.
\end{aligned}$$

We obtain $\mathbf{f}$, $\mathbf{u}|_{\partial\Omega}$ from the true solution. We solve one large coupled system for $\mathbf{u}$, p, ϕ. A uniform triangular mesh is used. We set $Re = 25$ for this experiement. Results are compiled in Tables 1 and 2 for $M = 20$, $N = M^2/Re = 16$ and $M = 200$, $N = M^2/Re = 1600$ respectively . Write $|||\cdot|||_{0,\infty} = \max_n ||\cdot||$ and $|||\cdot|||_{0,2} = (\Delta t \sum_n ||\cdot||^2)^{1/2}$. The rates of convergence suggested in Table 1 correlate with theoretical rate $\mathcal{O}(h^2 + \Delta t^2)$ predicted in Corollary 4.5 for $\mathbf{u}$ corresponding discrete norms. Note that the $\mathcal{O}(h^3)$ convergence suggested in $L^2(\Omega)$ is expected and can be shown by extension of the estimates in Theorem 4.4 via a duality argument. Similar results are reported in Table 2 for different error-measures, including the electric potential ϕ.

Experiment 2: Flow in channel over a step is a classic benchmark test. It is well-known that there exists a critical fluid Reynolds number $Re_c > 0$ so that the vortex developed in the wake of the step will detach from the step and be carried downstream for any $Re > Re_c$. In this test, we show how the SMHD

h	Δt	$\lvert\lvert\lvert \mathbf{E}_{u,n} \rvert\rvert\rvert_{0,\infty}$	rate	$\lvert\lvert\lvert \nabla\mathbf{E}_{u,n+1/2} \rvert\rvert\rvert_{0,2}$	rate
1/10	1/40	$3.727e$-2	$--$	$3.659e$-1	$--$
1/20	1/80	$4.477e$-3	3.05	$8.111e$-2	2.17
1/40	1/160	$4.498e$-4	3.32	$1.439e$-2	2.49
1/80	1/320	$4.564e$-5	3.30	$2.789e$-3	2.36
1/160	1/640	$4.805e$-6	3.24	$6.285e$-4	2.15

h	Δt	$\lvert\lvert\lvert \mathbf{E}_{j,n+1/2} \rvert\rvert\rvert_{0,2}$	rate
1/10	1/40	$5.471e$-2	$--$
1/20	1/80	$1.295e$-2	2.08
1/40	1/160	$3.326e$-3	1.96
1/80	1/320	$8.451e$-4	1.98
1/160	1/640	$2.126e$-4	1.99

TABLE 1. Convergence rate data for the first experiment, $H = 20$

h	Δt	$\lvert\lvert\lvert \mathbf{E}_{u,n+1/2} \rvert\rvert\rvert_{0,2}$	rate	$\lvert\lvert\lvert \nabla\mathbf{E}_{u,n+1/2} \rvert\rvert\rvert_{0,2}$	rate
1/10	1/100	$3.559e$-1	$--$	$1.488e$-0	$--$
1/20	1/200	$3.772e$-2	3.23	$5.546e$-1	1.42
1/40	1/400	$4.159e$-3	3.18	$2.311e$-1	1.26
1/80	1/800	$5.299e$-4	2.97	$7.135e$-2	1.69
1/160	1/1600	$5.069e$-5	3.39	$1.424e$-2	2.32

h	Δt	$\lvert\lvert\lvert \nabla\mathbf{E}_{\phi,n+1/2} \rvert\rvert\rvert_{0,2}$	rate
1/10	1/100	$3.575e$-1	$--$
1/20	1/200	$3.873e$-2	2.02
1/40	1/400	$4.890e$-3	2.98
1/80	1/800	$9.416e$-4	2.37
1/160	1/1600	$2.167e$-4	2.12

TABLE 2. Convergence rate data for the first experiment, $H = 200$

model accurately models the damping effect of the magnetic field by suppressing the shedding of these vortices.

We consider here a $[0, 40] \times [0, 10]$ channel. The step is square with width 1. The front of the step is located at $x = 5$. For velocity boundary conditions, let $\mathbf{u}|_{x=0} = 0.04y(10 - y)$ at the in-flow, do-nothing $(-M^{-2}\nabla\mathbf{u} \cdot \hat{n} + p\hat{n})|_{x=40} = 0$ at the out-flow, and no-slip $\mathbf{u} = 0$ otherwise. For the potential, let $\phi|_{y=10} = 0$ and $\nabla\phi \cdot \hat{n} = 0$ otherwise. We set $Re = M^2/N = 1800$, $M = 1000$, so that $N \approx 555.6$. We compute the SMHD solution with $\mathbf{B} = (0, 0, 1)$ as well as the NSE solution ($\mathbf{B} \equiv 0$) for comparison. We use $\Delta t = 0.005$ in both cases. Both problems are solved on a non-uniform mesh generated with the Delaunay-Vornoi algorithm, shown in Figure 1. Notice the mesh refinement along the step. The SMHD problem contained 58046 degrees of freedom. We show the streamlines for the velocity field for the corresponding NSE and SMHD solutions in Figures 2, 3 (domain restricted to [0,20] x [0,4]). Notice that as time evolves, vortices are created and separate for the NSE flow. As expected, the magnetic field suppresses this shedding for the SMHD flow though so that the vortex in the wake of the step is elongated without separation.

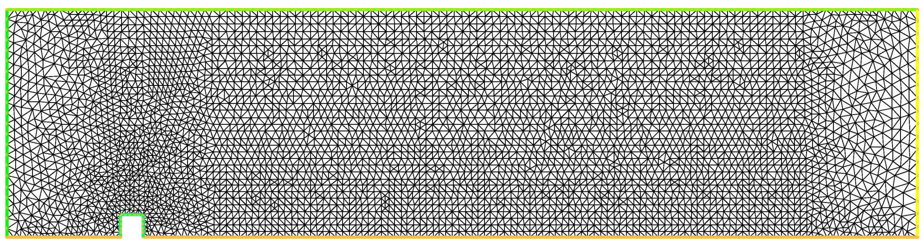

FIGURE 1. Experiment 2: Sample mesh - 8761 triangles.

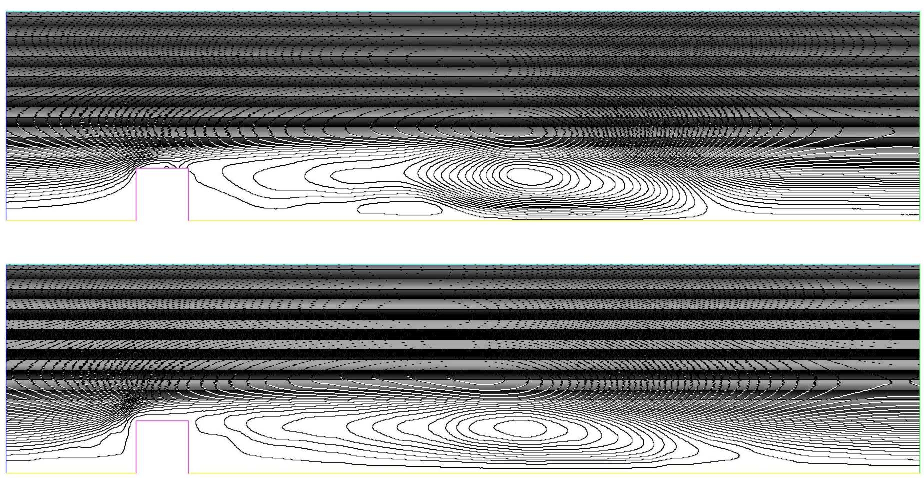

FIGURE 2. Experiment 2: Streamlines at $t = 40$, $Re = 1800$ (top) Navier-Stokes solution, $\mathbf{B} = (0, 0, 0)$, (bottom) SMHD solution, $\mathbf{B} = (0, 0, 1)$, $M = 1000$.

6. Conclusions

We introduced a finite element method for quasi-static MHD equation at small magnetic Reynolds number. We decomposed the approximation into two parts.

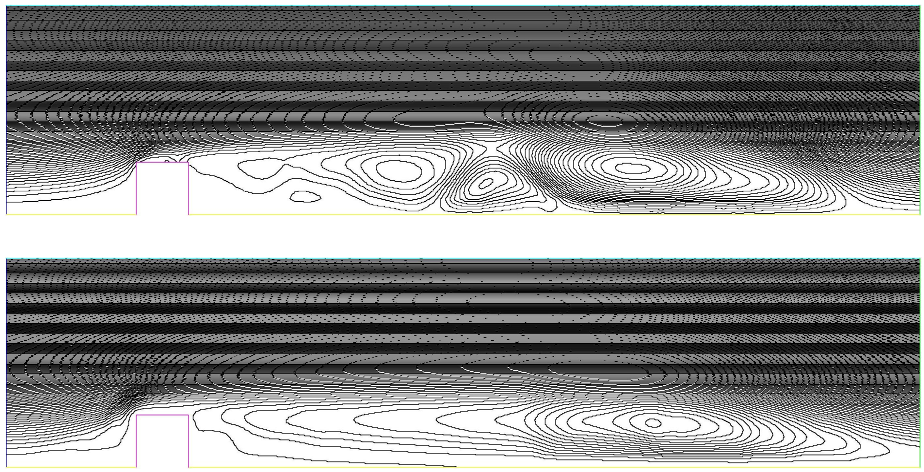

FIGURE 3. Experiment 2: Streamlines at $t = 60$, $Re = 1800$ (top) Navier-Stokes solution, $\mathbf{B} = (0, 0, 0)$, (bottom) SMHD solution, $\mathbf{B} = (0, 0, 1)$, $M = 1000$.

In the first part, we presented the stability and error analysis of semi-discrete approximation. In the second part, we presented the stability and error analysis of fully-discrete approximation, Crank-Nicolson in-time. We also conducted two numerical experiments to verify the effectiveness of the proposed model. We confirm the theoretical rate of convergence derived in this report in the first experiment. In the second experiement, we investigate how an applied magnetic field affects the dynamics of the classic flow-past-a-step problem in a channel. As expected, we show with the SMHD model that the magnetic field suppresses the shedding so that the vortex in the wake of the step is elongated without separation. For future work, note that we studied a fully coupled method between fluid velocity and electric potential. To improve speed and assuage memory requirements, we are investigating the existence and effectiveness of an *uncoupled*, approximation of SMHD. Lastly, we are investigating the integration of LES models to SMHD.

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